CBSE Class 12

Mathematics

Previous Year Question Paper 2020

Series: HMJ/2 **Set- 2**

Code no.65/2/2

- Please check that this question paper contains **15** printed pages.
- Code number given on the right hand side of the question paper should be written on the title page of the answer-book by the candidate.
- Please check that this question paper contains **36** questions.
- Please write down the Serial Number of the question before attempting it.
- 15 minutes of time has been allotted to read this question paper. The question paper will be distributed at 10.15 a.m. From 10.15 a.m. to 10.30 a.m., the students will read the question paper only and will not write any answer on the answer script during this period.

MATHEMATICS

Time Allowed: **3** hours Maximum Marks: **80**

General Instructions:

Read the following instructions very carefully and strictly follow them:

(i) This question paper comprises **four** Sections A, B, C and D. This question paper carries **36** questions. **All** questions are compulsory.

(ii) **Section A** – Questions no. **1** to **20** comprises of **20** questions of **1** mark each.

(iii) **Section B** – Questions no. **21** to **26** comprises of **6** questions of **2** marks each.

(iv) **Section C** – Questions no. **27** to **32** comprises of **6** questions of **4** marks each.

(v) **Section D** – Questions no. **33** to **36** comprises of **4** questions of **6** marks each.

(vi) There is no overall choice in the question paper. However, an internal choice has been provided in 3 questions of one mark, 2 questions of two marks, 2 questions of four marks and 2 questions of six marks. Only one of the choices in such questions have to be attempted.

(vii) In addition to this, separate instructions are given with each section and question, wherever necessary.

(viii) Use of calculators is **not** permitted.

SECTION - A

Question numbers 1 to 20 carry 1 mark each.

Question numbers 1 to 10 are multiple choice type questions. Select the correct option.

1. The area of a triangle formed by vertices O, A and B, where $\overrightarrow{OA}=\hat{i}+2\hat{j}+3\hat{k}$ and $\overrightarrow{OB}=-3\hat{i}-2\hat{j}+\hat{k}$ is **1 Mark**

(A) $3\sqrt{5}$ sq. units

(B) $5\sqrt{5}$ sq. units

(C) $6\sqrt{5}$ sq. units

(D) 4 sq. units

Ans: Given, $\overrightarrow{OA}=\hat{i}+2\hat{j}+3\hat{k}$ and $\overrightarrow{OB}=-3\hat{i}-2\hat{j}+\hat{k}$.

We know, are of a triangle if it's vectors are given is, $\frac{1}{2}\left|\overrightarrow{A}\times\overrightarrow{B}\right|$.

Therefore, here, $\overrightarrow{OA}=\hat{i}+2\hat{j}+3\hat{k}$

And, $\overrightarrow{OB}=-3\hat{i}-2\hat{j}+\hat{k}$

Therefore, $\frac{1}{2}\left|\overrightarrow{OA}\times\overrightarrow{OB}\right|=\frac{1}{2}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 3\\ -3 & -2 & 1\end{vmatrix}$

$$\frac{1}{2}\left|\overrightarrow{OA}\times\overrightarrow{OB}\right|=\frac{1}{2}\left|\hat{i}\{(2)(1)-(-2)(3)\}-\hat{j}\{(1)(1)-(-3)(3)\}+\hat{k}\{(1)(-2)-(-3)(2)\}\right|$$

$$\Rightarrow\left|\overrightarrow{OA}\times\overrightarrow{OB}\right|=\frac{1}{2}\times\left|\hat{i}\{2+6\}-\hat{j}\{1+9\}+\hat{k}\{-2+6\}\right|$$

$$\Rightarrow\left|\overrightarrow{OA}\times\overrightarrow{OB}\right|=\frac{1}{2}\times\left|8\hat{i}-10\hat{j}+4\hat{k}\right|$$

$$\Rightarrow\left|\overrightarrow{OA}\times\overrightarrow{OB}\right|=\frac{1}{2}\sqrt{64+100+16}$$

$$\Rightarrow\left|\overrightarrow{OA}\times\overrightarrow{OB}\right|=\frac{1}{2}\sqrt{180}$$

$$\Rightarrow\left|\overrightarrow{OA}\times\overrightarrow{OB}\right|=\frac{1}{2}\times 6\sqrt{5}$$

$$\Rightarrow\left|\overrightarrow{OA}\times\overrightarrow{OB}\right|=3\sqrt{5}$$

Therefore, the area is $3\sqrt{5}$ sq. units.

Thus, the correct answer is A.

2. If $\cos\left(\sin^{-1}\frac{2}{\sqrt{5}}+\cos^{-1}x\right)=0$, then x is equal to **1 Mark**

(A) $\frac{1}{\sqrt{5}}$

(B) $-\frac{2}{\sqrt{5}}$

(C) $\frac{2}{\sqrt{5}}$

(D) 1

Ans: Given, $\cos\left(\sin^{-1}\frac{2}{\sqrt{5}}+\cos^{-1}x\right)=0$

Taking $\cos^{-1}$ on both sides, we get,

$$\Rightarrow \sin^{-1}\frac{2}{\sqrt{5}}+\cos^{-1}x=\cos^{-1}(0)$$

$\Rightarrow \sin^{-1}\frac{2}{\sqrt{5}}+\cos^{-1}x=\cos^{-1}x=\frac{\pi}{2}$

We know, $\cos^{-1}x=\frac{\pi}{2}-\sin^{-1}x$, hence,

$\Rightarrow \sin^{-1}\frac{2}{\sqrt{5}}+\frac{\pi}{2}-\sin^{-1}x=\frac{\pi}{2}$

$\Rightarrow \sin^{-1}x=\sin^{-1}\left(\frac{2}{\sqrt{5}}\right)$

$\Rightarrow x=\frac{2}{\sqrt{5}}$

Thus, the correct answer is C.

3. The interval in which the function f given by $f(x)=x^2e^{-x}$ is strictly increasing, is **1 Mark**

(A) $(-\infty,\infty)$

(B) $(-\infty,0)$

(C) $(2,\infty)$

(D) (0,2)

Ans: Given, $f(x)=x^2e^{-x}$

Now, differentiating both sides with respect to x, we get,

$\Rightarrow f'(x)=2xe^{-x}-x^2e^{-x}$

$\Rightarrow f'(x)=xe^{-x}(2-x)$

For, the function to be increasing,

$f'(x)>0$

$\Rightarrow xe^{-x}(2-x)>0$

$\Rightarrow x(2-x)>0$

$\Rightarrow x(2-x)<0$

[Since, e^{-x} can never be zero]

Using, the method of intervals, we get,

Since, $x(x-2)<0$, we will take the negative region.

Therefore, $x \in (0,2)$.

The correct option is D.

4. The function $f(x)=\frac{x-1}{x(x^2-1)}$ is discontinuous at **1 Mark**

(A) exactly one point

(B) exactly two points

(C) exactly three points

(D) no point

Ans: Given, $f(x)=\frac{x-1}{x(x^2-1)}$

We can write the function as,

$$\Rightarrow f(x)=\frac{x-1}{x(x-1)(x-1)}$$

Here, the function is discontinuous if,

$x(x-1)(x-1)=0$

x=0 or x-1=0 or x+1=0

x=0 or x=1 or x=-1

Therefore, the function is discontinuous exactly at three points.

The correct option is C.

5. The function $f:R \to [-1,1]$ defined by $f(x)=\cos x$ is **1 Mark**

(A) both one-one and onto

(B) not one-one, but onto

(C) one-one, but not onto

(D) neither one-one, nor onto

Ans: Given, $f:R \to [-1,1]$ defined by $f(x)=\cos x$.

Let, $f(x_1)=f(x_2)$

$\Rightarrow \cos x_1 = \cos x_2$

$\Rightarrow x_1 = 2n\pi \pm x_2, n \in \mathbb{Z}$

Therefore, the above equations have infinitely many solutions.

Hence, it is not a one-one function.

Also, range of cos x is [-1, 1], which is a subset of co-domain R.

Hence, the function is also not onto.

Therefore, the function is neither one-one nor onto.

Thus, the correct option is D.

6. The coordinates of the foot of the perpendicular drawn from the point (2, -3, 4) on the y-axis is **1 Mark**

(A) (2, 3, 4)

(B) (-2, -3, -4)

(C) (0, -3, 0)

(D) (2, 0, 4)

Ans: Given point is P(2, -3, 4).
Any point on y-axis is given by Q(0, k, 0), where k is any real number
So direction ratio of PQ are -2,-3,-k,4.
We know direction ratio of y-axis is given by 0,1,0.
Now since $PQ \perp$ y-axis

$\Rightarrow (0)(2)+(1)(-3-k)+(0)(4)=0$

$\Rightarrow k=-3$

Hence, coordinate of foot of perpendicular is Q(0, -3, 0).

Thus, the correct option is C.

7. The relation R in the set {1,2,3} given by R={(1,2),(2,1),(1,1)} is 1 Mark

(A) symmetric and transitive, but not reflexive

(B) reflexive and symmetric, but not transitive

(C) symmetric, but neither reflexive nor transitive

(D) an equivalence relation

Ans: The relation is not reflexive because (2,2),(3,3) are not present.

It is symmetric because, $(1,2) \in R$ and also $(2,1) \in R$, which satisfies the condition for a relation to be symmetric perfectly.

And, also, it is transitive because, $(1,2) \in R$, $(2,1) \in R$ and also $(1,1) \in R$, which satisfies the condition for a relation to be transitive perfectly.

Hence, the relation is symmetric and transitive but not reflexive.

Thus, the correct option is A.

8. The angle between the vectors $\hat{i}-\hat{j}$ and $\hat{j}-\hat{k}$ is 1 Mark

(A) $-\frac{\pi}{3}$

(B) 0

(C) $\frac{\pi}{3}$

(D) $\frac{2\pi}{3}$

Ans: Given vectors are

$\vec{a}=\hat{i}-\hat{j}$

$\vec{b}=\hat{i}-\hat{k}$

So, $\vec{a}.\vec{b}=\left(\hat{i}-\hat{j}\right).\left(\hat{i}-\hat{k}\right)$

$\vec{a}.\vec{b}=(1\times 0)+(-1\times 1)+(0\times(-1))$

$\Rightarrow \vec{a}.\vec{b}=-1$

Also, $|\vec{a}|=\sqrt{1^2+(-1)^2+0}=\sqrt{2}$

$|\vec{b}|=\sqrt{0+1^2}=\sqrt{2}$

We also know, $\vec{a}.\vec{b}=|\vec{a}|.|\vec{b}|.\cos\theta$

$\Rightarrow$ -1=$\sqrt{2}.\sqrt{2}.\cos\theta$

$\Rightarrow \cos\theta=\dfrac{-1}{2}$

$\Rightarrow \theta=\dfrac{2\pi}{3}$

The angle between the vectors is $\dfrac{2\pi}{3}$.

Thus, the correct option is D.

9. If A is a non-singular square matrix of order 3 such that A2=3A , then value of |A| is **1 Mark**

(A) -3

(B) 3

(C) 9

(D) 27

Ans: Given, A2=3A

Taking determinant on both sides,

$\Rightarrow |A^2|=|3A|$

$\Rightarrow |A^2|=3^3|A|$

$\Rightarrow |A^2|=27|A|$

$\Rightarrow |A|=27$

Therefore, the correct option is D.

10. If $\left|\vec{a}\right| = 4$ and -3 $\le \lambda \le 2$, then $\left|\lambda\vec{a}\right|$ lies in **1 Mark**

(A) [0,12]

(B) [2,3]

(C) [8,12]

(D) [-12,8]

Ans: The maximum value of λ is **2.**

So, $\left|\lambda\vec{a}\right| = \left|\lambda\right|.\left|\vec{a}\right|$

$\Rightarrow \left|\lambda\vec{a}\right| = 2.4 = 8$

The minimum value of λ is -3**.**

So, $\left|\lambda\vec{a}\right| = \left|\lambda\right|.\left|\vec{a}\right|$

$\Rightarrow \left|\lambda\vec{a}\right| = \left|-3\right|.4$

$\Rightarrow \left|\lambda\vec{a}\right| = 3.4 = 12$

So, there are no value of λ which is negative.

For, λ =0, we get,

$\left|\lambda\vec{a}\right| = \left|\lambda\right|.\left|\vec{a}\right|$

$\Rightarrow \left|\lambda\vec{a}\right| = 0.4 = 0$

Therefore, the smallest value of $\left|\lambda\vec{a}\right|$ is 0 .

Therefore, $\left|\lambda\vec{a}\right|$ lies in [0, 12].

Thus, the correct option is A**.**

Fill in the blanks in question numbers 11 to 15.

11. If the radius of the circle is increasing at the rate of 0.5 cm/s, then the rate of increase of its circumference is _____________ . **1 Mark**

Ans: Let r be the radius and C the circumference of the circle.

Then, $C=2\pi r$

It is given that $\frac{dr}{dt}=0.5$ cm/s

Now, $C=2\pi r$

Differentiating both sides w.r.t t, we get,

$\Rightarrow \frac{dr}{dt}=2\pi.\frac{dr}{dt}$

$\Rightarrow \frac{dC}{dt}=2\pi.05$

$\Rightarrow \frac{dC}{dt}=\pi$ cm/s.

Therefore, the rate in increase of the circumference is π cm/s.

12. If $\begin{vmatrix} 2x & -9 \\ -2 & x \end{vmatrix}=\begin{vmatrix} -4 & 8 \\ 1 & -2 \end{vmatrix}$, then value of x is _____________ . **1 Mark**

Ans: $\begin{vmatrix} 2x & -9 \\ -2 & x \end{vmatrix}=\begin{vmatrix} -4 & 8 \\ 1 & -2 \end{vmatrix}$

(2x)(x)-(-9)(-2)=(-4)(-2)-(8)(1)

2x2-18=8-8

2x2-18=0

2x2-18=0

2x2=18

x2=9

$x=\pm 3$

Therefore, the value of x is ± 3.

13. The corner points of the feasible region of an LPP are (0,0), (0,8), (2,7), (5,4) and (6,0). The maximum profit P=3x+2y occurs at the point ______.
1 Mark

Ans: $P_{(0,0)}=3(0)+2=0$

$P_{(0,8)}=3(0)+2(8)=16$

$P_{(2,7)}=3(2)+2(7)=20$

$P_{(5,4)}=3(5)+2(4)=23$

$P_{(6,0)}=3(6)+2(0)=18$

The maximum value is at (5,4).

14. The range of the principal value branch of the function $y\text{-}\sec^{-1}x$ is ____________ .
1 Mark

Ans: We know, $\sec^{-1}x \in [0,\pi]-\left\{\frac{\pi}{2}\right\}$.

Therefore, the range of the principal value branch of the function $y\text{-}\sec^{-1}x$ is $[0,\pi]-\left\{\frac{\pi}{2}\right\}$.

Or

The principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is ___________ . **1 Mark**

Ans: $\cos^{-1}\left(-\frac{1}{2}\right)$

We know, $\cos^{-1}\frac{2\pi}{3}=-\frac{1}{2}$

$=\cos^{-1}\left(\cos\frac{2\pi}{3}\right)$

$=\frac{2\pi}{3}$

Thus, the principle value of $\cos^{-1}\left(-\frac{1}{2}\right)$ is $=\frac{2\pi}{3}$.

15. The distance between parallel planes 2x+y-2z-6=0 and 4x+2y-4z is ____________ units. **1 Mark**

Ans: Given, 2x+y-2z-6=0 --------(1)

4x+2y-4z -------(2)

Multiplying (1) by 2, we get,

4x+2y-4z-12=0

4x+2y-4z=12

Therefore, we can write,

c1=12, c2=0

And, a=4, b=2, c=-4.

Therefore, the distance between the parallel lines is,

$$\left|\frac{c_2-c_1}{\sqrt{a^2+b^2+c^2}}\right|$$

$$=\left|\frac{0-12}{\sqrt{(4)^2+(2)^2+(-4)^2}}\right|$$

$$=\left|\frac{12}{\sqrt{16+4+16}}\right|$$

$$=\left|\frac{12}{\sqrt{36}}\right|$$

$$=\left|\frac{12}{6}\right|$$

=2 units

Or

If P(1,0,-3) is the foot of the perpendicular from the origin to the plane, then the Cartesian equation of the plane is ______________ . **1 Mark**

Ans: The given foot of the perpendicular is P(1,0,-3)**.**

The direction coefficients of the perpendicular are (1-0,0-0,-3,-0)

=(1,0,-3) .

Therefore, the equation of the plane is

$$a\left(x-x_1\right)+b\left(y-y_1\right)+c\left(z-z_1\right)=0$$

$$\Rightarrow 1\left(x-1\right)+b\left(y-0\right)-3\left(z-\left(-3\right)\right)=0$$

$$\Rightarrow x-1-3\left(z+3\right)=0$$

$$\Rightarrow x-1-3z-9=0$$

$$\Rightarrow x-3z-10=0$$

Therefore, the equation of the plane is $x-3z-10=0$.

Question numbers 16 to 20 are very short answer type questions.

16. Evaluate : $\int_{\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos^2 x dx$ **1 Mark**

Ans: Here, $f\left(x\right)=x\cos^2 x$

Now, $f\left(-x\right)=\left(-x\right)\cos^2\left(-x\right)$

$$\Rightarrow f\left(-x\right)=-x\cos^2 x = -f\left(-x\right)$$

Therefore, it is an odd function.

So, $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos^2 x dx=0$**.**

17. Find the coordinates of the point where the line $\frac{x-1}{3}=\frac{y+4}{7}=\frac{z+4}{2}$ cuts the xy-plane. **1 Mark**

Ans: If the line $\frac{x-1}{3}=\frac{y+4}{7}=\frac{z+4}{2}$ cuts the XY plane.

Then, z=0.

So, let coordinates of point be (x,y,0).

Now, $\frac{x-1}{3}=\frac{y+4}{7}=\frac{z+4}{2}=k$

Thus, x=3k+1, y=7k-4, z=2k-4

Since, z=0

2k-4=0

2k=4

k=2

Now, x=3(2)+1=7

y=7(2)-4=10

Therefore, the point is (7,10,0).

18. Find the value of k, so that the function $f(x)=\begin{cases} kx^2+5 & \text{if } x\le 1 \\ 2 & \text{if } x>1 \end{cases}$ is continuous at x=1 . **1 Mark**

Ans: For, x>1, the value of f(x), such that,

$f(x)_{x\to 1^+}=2$

For, $x\le 1$, the value of f(x), such that,

$f(x)_{x\to 1^+}=k(1)^2+5$

$f(x)_{x\to 1^+}=k+5$

For the function to be continuous

$f(x)_{x\to 1^+}=f(x)_{x\to 1^-}$

2=k+5

k=-3

19. Find the integrating factor of the differential equation

$$\mathbf{x\frac{dy}{dx}=2x^2+y}$$

1 Mark

Ans: Given, $x\frac{dy}{dx}=2x^2+y$

Dividing both sides by x, we get,

$$\frac{dy}{dx}=2x+\frac{y}{x}$$

$$\frac{dy}{dx}-\frac{y}{x}=2x$$

Therefore, $P(x)=\frac{1}{x}$

Thus, integrating factor, IF = $e^{\int=\frac{1}{x}dx}$

$$=e^{-\int=\frac{1}{x}dx}$$

$$=e^{-\log x}$$

$$=e^{\log x^{-1}}$$

$$=x^{-1}$$

$$=\frac{1}{x}$$

Thus, the integrating factor is $\frac{1}{x}$.

20. Differentiate $=\sec^2\left(x^2\right)$ with respect to x^2. 1 Mark

Ans: We need to find

$$\frac{d\left(\sec^2\left(x^2\right)\right)}{dx^2}$$

Let x2=t

So, $\frac{d\left(\sec^2(t)\right)}{dt}$

$=2.\sec t.(\sec t)'$

$=2.\sec t.\sec t.\tan t$

$=2.\sec^2 t.\tan t$

Putting t=x2

$=2.\sec^2 x^2.\tan x^2$

Or

If $y=f\left(x^2\right)$ and $f'(x)=e^{\sqrt{x}}$, then find $\frac{dy}{dx}$. **1 Mark**

Ans: Given, $y=f\left(x^2\right)$

Differentiating both sides w.r.t x, we get,

$=\frac{dy}{dx}y=f'\left(x^2\right).2x$

Also, given, $f'(x)=e^{\sqrt{x}}$.

$f'\left(x^2\right)=e^{\sqrt{x^2}}=e^x$

$\therefore \frac{dy}{dx}=2xe^x$.

SECTION - B

Question numbers 21 to 26 carry 2 marks each.

21. Find a vector r equally inclined to the three axes and whose magnitude is $3\sqrt{3}$ units. **2 Marks**

Ans: We have $\left|\vec{r}\right|=3\sqrt{3}$

Since, $\vec{r}$ is equally inclined to the three axes, direction cosines of the unit vector $\vec{r}$ will be same.

i.e., l=m=n

Now, we know that,

$l^2+m^2+n^2=1$

$\Rightarrow l^2+l^2+l^2=1$

$\Rightarrow 3l^2=1$

$\Rightarrow l^2=\frac{1}{3}$

$\Rightarrow l=\pm\frac{1}{\sqrt{3}}$

So, $\hat{r}=\pm\frac{1}{\sqrt{3}}\hat{i}\pm\frac{1}{\sqrt{3}}\hat{j}\pm\frac{1}{\sqrt{3}}\hat{k}$

$\therefore \vec{r}=\left|\vec{r}\right|.\hat{r}$

$=2\sqrt{3}\left[\pm\frac{1}{\sqrt{3}}\hat{i}\pm\frac{1}{\sqrt{3}}\hat{j}\pm\frac{1}{\sqrt{3}}\hat{k}\right]$

$=\pm 2\left[\hat{i}+\hat{j}+\hat{k}\right]$

Or

Find the angle between unit vectors a and b so that $\sqrt{3}\vec{a}$-b is also a unit vector. **2 Marks**

Ans: a and b are unit vectors and $\sqrt{3}\vec{a}$-b is also unit vector

To find: Angle between a and b

Suppose angle between a and b is θ.

$\vec{a}.\vec{b}=\left|\vec{a}\right|.\left|\vec{b}\right|.\cos\theta$ (Dot product of two vectors)

$\vec{a}.\vec{b}=\cos\theta$

As $\vec{a}$ and $\vec{b}$ are unit vector so, $\left|\vec{a}\right|=\left|\vec{b}\right|=1$.

$\sqrt{3}\vec{a}$-b is also unit vector i.e. $\left|\sqrt{3}\vec{a}-\vec{b}\right|=1$

Squaring both sides, we get,

$\left(\sqrt{3}\vec{a}-\vec{b}\right)^2=1$

$\left(\sqrt{3}\right)^2\left|\vec{a}\right|^2+\left|\vec{b}\right|^2-2.\sqrt{3}.\left|\vec{a}.\vec{b}\right|=1$

$\Rightarrow 3.1+1-2.\sqrt{3}.\cos\theta=1$

[Since, $\vec{a}.\vec{b}=\cos\theta$]

$\Rightarrow 4-2\sqrt{3}.\cos\theta=1$

$\Rightarrow 2\sqrt{3}.\cos\theta=3$

$\Rightarrow \cos\theta=\dfrac{\sqrt{3}}{2}$

$\Rightarrow \theta=\dfrac{\pi}{6}$

Therefore, the angle between the two unit vectors is $\dfrac{\pi}{6}$.

22. If $A=\begin{bmatrix}-3 & 2\\ 1 & -1\end{bmatrix}$ and $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, find scalar k so that A2+I=kA. 2 Marks

Ans: $A=\begin{bmatrix}-3 & 2\\ 1 & -1\end{bmatrix}$, $I=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

A2+I=kA

A2=A×A

$A^2=A\times A$

$$\Rightarrow A^2=\begin{bmatrix}(-3)(-3)+(2)(1) & (-3)(2)+(2)(1)\\ 1(-3)+1(-1) & 1(2)+(-1)(-1)\end{bmatrix}$$

$$\Rightarrow A^2=\begin{bmatrix}9+2 & -6-2\\ -3-1 & 2+1\end{bmatrix}$$

$$\Rightarrow A^2=\begin{bmatrix}11 & -8\\ -4 & 3\end{bmatrix}$$

A2+I=kA

$$\Rightarrow\begin{bmatrix}11 & -8\\ -4 & 3\end{bmatrix}+\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=k\begin{bmatrix}-3 & 2\\ 1 & -1\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}11+1 & -8+0\\ -4+0 & 3+1\end{bmatrix}=\begin{bmatrix}-3k & 2k\\ 1k & -1k\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}12 & -8\\ -4 & 4\end{bmatrix}=\begin{bmatrix}-3k & 2k\\ 1k & -1k\end{bmatrix}$$

Therefore, comparing the terms on both sides, we get,

k=-4.

23. If $f(x)=\sqrt{\frac{\sec x-1}{\sec x+1}}$, find $f'\left(\frac{\pi}{3}\right)$. **2 Marks**

Ans: Given, $f(x)=\sqrt{\frac{\sec x-1}{\sec x+1}}$

Using, $\sec x=\frac{1}{\cos x}$.

$$f(x)=\sqrt{\frac{\frac{1}{\cos x}-1}{\frac{1}{\cos x}+1}}$$

$$f(x)=\sqrt{\frac{1-\cos x}{1+\cos x}}$$

$$f(x)=\sqrt{\frac{2\sin^2\frac{x}{2}}{2\sin^2\frac{x}{2}}}$$

Since, $1-\cos x=2\sin^2\frac{x}{2}$ and $1+\cos x=2\cos^2\frac{x}{2}$. So, we get,

$$f(x)=\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}=\tan\frac{x}{2}$$

Differentiating both sides w.r.t x, we get,

$$f'(x)=\frac{d}{dx}\left(\tan\frac{x}{2}\right)$$

$$f'(x)=\sec^2\frac{x}{2}.\frac{1}{2}$$

$$f'(x)=\frac{1}{2}\sec^2\frac{x}{2}$$

Therefore, $f'\left(\frac{\pi}{3}\right)=\frac{1}{2}\sec^2\left(\frac{\pi}{2.3}\right)$

$$\Rightarrow f'\left(\frac{\pi}{3}\right)=\frac{1}{2}\sec^2\left(\frac{\pi}{6}\right)$$

$$\Rightarrow f'\left(\frac{\pi}{3}\right)=\frac{1}{2}\left(\frac{2}{\sqrt{3}}\right)^2$$

$$\Rightarrow f'\left(\frac{\pi}{3}\right)=\frac{2}{3}$$

Or

Find $f'(x)$ if $f(x)=(\tan x)^{\tan x}$. **2 Marks**

Ans: Given, $f(x)=(\tan x)^{\tan x}$

Let, $f(x)=(\tan x)^{\tan x}=y$

Taking log on both sides, we get,

$$\log y=\log\left(\tan x^{\tan x}\right)$$

$$\Rightarrow \log y=\tan x\log(\tan x)$$

Differentiating both sides w.r.t x , we get,

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx}=\tan x.\frac{1}{\tan x}.\sec^2x+\log(\tan x).\sec^2x$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx}=\sec^2x+\sec^2x\log(\tan x)$$

$$\Rightarrow \frac{dy}{dx}=y\left[\sec^2 x+\sec^2 x\log\left(\tan x\right)\right]$$

$$\Rightarrow \frac{dy}{dx}=y\tan x^{\tan x}\sec^2 x\left[1+\log\left(\tan x\right)\right]$$

Since, $y=\tan x^{\tan x}$.

Therefore, $f'\left(x\right)=y\tan x^{\tan x}\sec^2 x\left[1+\log\left(\tan x\right)\right]$.

24. Find the value of integral: $\int\frac{\tan^3 x}{\cos^3 x}dx$ **2 Marks**

Ans: $I=\int\frac{\tan^3 x}{\cos^3 x}dx$

$$\Rightarrow I=\int\frac{\sin^3 x}{\cos^3 x.\cos^3 x}dx$$

$$\Rightarrow I=\int\frac{\sin^3 x.\sin x}{\cos^6 x}dx$$

$$\Rightarrow I=\int\frac{\left(1-\cos^2 x\right)\sin x}{\cos^6 x}dx$$

$$\Rightarrow I=\int\frac{\sin x}{\cos^6 x}dx-\int\frac{\sin x}{\cos^4 x}dx$$

Now, let, $\cos x=t$

$-\sin x dx=dt$

$\sin x dx=-dt$

Therefore, $I=\int-\frac{1}{t^6}dt+\int\frac{1}{t^4}dt$

$$\Rightarrow I=-\int\frac{1}{t^6}dt+\int\frac{1}{t^4}dt$$

$$\Rightarrow I=-\left[\frac{t^{-6+1}}{-6+1}\right]+\left[\frac{t^{-4+1}}{-4+1}\right]+c$$

$\Rightarrow I=-\left[\frac{t^{-5}}{-5}\right]+\left[\frac{t^{-3}}{-3}\right]+c$

$\Rightarrow I=\frac{1}{5t^5}-\frac{1}{3t^3}+c$

Substituting t=cosx, we get,

$\Rightarrow I=\frac{1}{5\cos^5 x}-\frac{1}{3\cos^3 x}+c$

Therefore, $\int\frac{\tan^3 x}{\cos^3 x}dx=\frac{1}{5\cos^5 x}-\frac{1}{3\cos^3 x}+c$.

25. Show that the plane x-5y-2z=1 contains the line $\frac{x-5}{3}=y=2-z$. 2 Marks

Ans: Given:

Plane: x-5y-2z=1

In vector form, we can write the equation of plane as,

$\vec{r}.\left(\hat{i}-5\hat{j}-2\hat{k}\right)=1$

Direction ratio of the plane $\vec{P}=\left(\hat{i}-5\hat{j}-2\hat{k}\right)$

Line: $\frac{x-5}{3}=y=2-z$

In vector form, we can write the equation of line as,

$\vec{r}=\left(5\hat{i}+2\hat{k}\right)+\lambda\left(3\hat{i}+\hat{j}-\hat{k}\right)$

Direction ratio of the plane

$\vec{L}=\left(3\hat{i}+\hat{j}-\hat{k}\right)$

Now, $\vec{p}.\vec{L}=\left(\hat{i}-5\hat{j}-2\hat{k}\right).\left(3\hat{i}+\hat{j}-\hat{k}\right)$

$\vec{p}.\vec{L}=(1)(3)+(-5)(1)+(-2)(-1)$

$\Rightarrow \vec{p}.\vec{L}=3-5+2=0$

Hence this given plane contain the given line.

26. A fair dice is thrown two times. Find the probability distribution of the number of sixes. Also determine the mean of the number of sixes. 2 Marks

Ans: The dice is thrown twice.

Therefore, the sample space is

1,1	1,2	1,3	1,4	1,5	1,6
2,1	2,2	2,3	2,4	2,5	2,6
3,1	3,2	3,3	3,4	3,5	3,6
4,1	4,2	4,3	4,4	4,5	4,6
5,1	5,2	5,3	5,4	5,5	5,6
6,1	6,2	6,3	6,4	6,5	6,6

Therefore, no. of sample with 0 sixes =25

No. of sample with 1 sixes =10

no. of sample with 2 sixes =1

X	0 Sixes	1 Six	2 Sixes
p(X)	25	10	1

Now, Mean $=\sum X.P(X)$

$=0\times\frac{25}{36}+1\times\frac{10}{36}+2\times\frac{1}{36}$

$=\frac{10}{36}+\frac{2}{36}$

$=\frac{12}{36}$

$$=\frac{1}{3}$$

SECTION - C

Question numbers 27 to 32 carry 4 marks each.

27. Solve the following differential equation:

$$\left(1-e^{\frac{y}{x}}\right)dy+e^{\frac{y}{x}}\left(1-\frac{y}{x}\right)dx=0\left(x\neq 0\right).$$ **4 Marks**

Ans: Given, $\left(1-e^{\frac{y}{x}}\right)dy+e^{\frac{y}{x}}\left(1-\frac{y}{x}\right)dx=0$

$$\Rightarrow\left(1-e^{\frac{y}{x}}\right)dy=-e^{\frac{y}{x}}\left(1-\frac{y}{x}\right)dx$$

$$\Rightarrow\frac{dy}{dx}=\frac{-e^{\frac{y}{x}}\left(1-\frac{y}{x}\right)}{\left(1-e^{\frac{y}{x}}\right)}---\left(1\right)$$

Now, let, $\frac{dy}{dx}=F\left(x,y\right)$

$$\therefore\frac{dy}{dx}=F\left(x,y\right)=\frac{-e^{\frac{y}{x}}\left(1-\frac{y}{x}\right)}{\left(1-e^{\frac{y}{x}}\right)}$$

Now, $F\left(\lambda x,\lambda y\right)=\frac{-e^{\frac{\lambda y}{\lambda x}}\left(1-\frac{\lambda y}{\lambda x}\right)}{\left(1-e^{\frac{\lambda y}{\lambda x}}\right)}$

$$F(\lambda x,\lambda y)=\frac{-e^{\frac{\lambda y}{\lambda x}}\left(1-\frac{\lambda y}{\lambda x}\right)}{\left(1-e^{\frac{\lambda y}{\lambda x}}\right)}=F(x,y)$$

So, $F(\lambda x,\lambda y)=F(x,y)=\lambda^{o}F(x,y)$

Thus, F(x, y) is a homogeneous function.

Therefore, the given differential equation is a homogeneous differential equation.

Now, let, $y=xv$

$$\Rightarrow \frac{dy}{dx}=v+x.\frac{dv}{dx}$$

Now, substituting these values in (1), we get,

$$\Rightarrow \frac{dy}{dx}=\frac{-e^{\frac{y}{x}}\left(1-\frac{y}{x}\right)}{\left(1-e^{\frac{y}{x}}\right)}$$

$$\Rightarrow v+x.\frac{dy}{dx}=\frac{-e^{v}(1-v)}{\left(1+e^{v}\right)}$$

$$\Rightarrow x.\frac{dy}{dx}=\frac{-e^{v}(1-v)}{\left(1+e^{v}\right)}-v$$

$$\Rightarrow x.\frac{dy}{dx}=\frac{-e^{v}+ve^{v}}{\left(1+e^{v}\right)}-v$$

$$x.\frac{dy}{dx}=\frac{-e^{v}+ve^{v}-v\left(1+e^{v}\right)}{\left(1+e^{v}\right)}-v$$

$$\Rightarrow x.\frac{dy}{dx}=\frac{-e^{v}+ve^{v}-v-ve^{v}}{\left(1+e^{v}\right)}-v$$

$$\Rightarrow x.\frac{dy}{dx}=\frac{-e^{v}-v}{\left(1+e^{v}\right)}$$

Now, by method of substitution of differential equation, we get,

$$\Rightarrow \frac{1+e^{v}}{v+e^{v}}dv=-\frac{dx}{x}$$

Now, integrating both sides,

$$\Rightarrow \int\frac{1+e^{v}}{v+e^{v}}dv=-\int\frac{dx}{x}$$

$$\Rightarrow \int\frac{1+e^{v}}{v+e^{v}}dv=-\log|x|+\log c$$

Now, putting $v+e^{v}=t$

$$\Rightarrow \left(1+e^{v}\right)dv=dt$$

Thus, our equation becomes,

$$\Rightarrow \int\frac{dt}{t}=-\log|x|+\log c$$

$$\Rightarrow \log|t|=-\log|x|+\log c$$

Putting back, $t=v+e^{v}$, we get,

$$\Rightarrow \log\left|v+e^{v}\right|=-\log|x|+\log c$$

$$\Rightarrow \log\left|v+e^{v}\right|+\log|x|=\log c$$

$$\Rightarrow \log\left|\left(v+e^{v}\right).\right||x|=\log c$$

$\Rightarrow \log\left|\left(v+e^{v}\right).x\right|=\log c$

$\Rightarrow \log\left|vx+e^{v}x\right|=\log c$

$\Rightarrow vx+e^{v}x=c$

Putting back, $\Rightarrow y=vx \Rightarrow v=\dfrac{y}{x}$, we get,

$\Rightarrow y+e^{\frac{y}{x}}x=c$

28. A cottage industry manufactures pedestal lamps and wooden shades. Both the products require machine time as well as craftsman time in the making. The number of hour(s) required for producing 1 unit of each and the corresponding profit is given in the following table:

Item	Machine Time	Craftsman Time	Profit(in ₹)
Pedestal lamp	1.5 hours	3 hours	30
Wooden shades	3hours	1 hours	20

In a day, the factory has availability of not more than 42 hours of machine time and 24 hours of craftsman time. Assuming that all items manufactured are sold, how should the manufacturer schedule his daily production in order to maximise the profit? Formulate it as an LPP and solve it graphically. **4 Marks**

Ans: Let number of pedestal lamps $=x$

Number of wooden shades $=y$

Maximize Profit: $P=30x+20y$

According to the question:

$1.5x+3y\le 42$

$\Rightarrow \dfrac{x}{28}+\dfrac{y}{14}\le 1$

Therefore, the intercepts will be (28, 0), (0, 14).

$3x+y \le 24$

$\Rightarrow \frac{x}{8}+\frac{y}{24} \le 1$

Therefore, the intercepts will be $(8,0),(0,24)$.

$x \ge 0,\ y \ge 0$

Check profit at Corner points

At C(0,0),

P30(0)+20(0)=0

At B(0. 14),

P=30(0)+20(14)=280

At F(4,12),

P=30(4)+20(12)=360 [Max]

At D(8,0),

P=30(8)+20(0)=240

Maximum profit = Rs 360 at (number of pedestal lamps) x=4 and (Number of wooden shades) y=12.

29. Evaluate the value of integral: $\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x)\,dx$ **4 Marks**

Ans: Given, $\int_0^{\frac{\pi}{2}} \sin 2x \tan^{-1}(\sin x) dx$

$= \int_0^{\frac{\pi}{2}} \sin 2x \cos x \tan^{-1}(\sin x) dx$

Let, sin x= t

Differentiating both sides w.r.t x

Cosx dx=dt ,

x	0	$\frac{\pi}{2}$
t = sinx	Sin 0 =0	$\sin\left(\frac{\pi}{2}\right)=1$

Substituting x and dx, we get,

$=\int_0^1 2t \tan^{-1}(t) dt$

$=2\int_0^1 t \tan^{-1}(t) dt$

Now, using integration by parts, with function 1 as tan-1t and function 2 as t, we get,

$=2\left[\tan^{-1}t\int tdt-\int\left\{\frac{d\left(\tan^{-1}t\right)}{dt}\int tdt\right\}dt\right]$

$=2\left[\tan^{-1}t\int t\left\{\frac{t^2}{2}\right\}-\int\left\{\left(\frac{1}{t^2+1}\right)\left(\frac{t^2}{2}\right)\right\}dt\right]$

$=2\left[\frac{t^2}{2}\tan^{-1}t-\frac{1}{2}\int\left(\frac{t^2}{t^2+1}\right)dt\right]$

$=t^2\tan^{-1}t-\int\left(\frac{t^2}{t^2+1}\right)td$

Let $I_1=\int\frac{t^2}{t^2+1}dt$.

$$\Rightarrow I_1=\int\frac{t^2+1-1}{t^2+1}dt$$

$$\Rightarrow I_1=\int dt-\int\frac{1}{1+t^2}dt$$

$$\Rightarrow I_1=t-\tan^{-1}$$

Thus, our equation becomes,

$$=t^2\tan^{-1}t\left[t-\tan^{-1}t\right]$$

$$=t^2\tan^{-1}t-t+\tan^{-1}t$$

Now, $2\int_0^1 t\tan^{=1}(t)dt=\left[t^2\tan^{-1}t-t+\tan^{-1}t\right]_0^1$

$$2\int_0^1 t\tan^{=1}(t)dt=\left[1^2\tan^{-1}1-1-+\tan^{-1}1\right]-\left[0-0+\tan^{-1}0\right]$$

$$\Rightarrow 2\int_0^1 t\tan^{=1}(t)dt=\left[\frac{\pi}{4}-1+\frac{\pi}{4}\right]-0$$

$$\Rightarrow 2\int_0^1 t\tan^{=1}(t)dt=\frac{\pi}{2}-1$$

Therefore, $\int_0^{\frac{\pi}{2}}\sin 2x\tan^{-1}(\sin x)dx=\frac{\pi}{2}-1.$

30. Check whether the relation R in the set N of natural numbers given by R={(a,b):a is a divisor of b} is reflexive, symmetric or transitive. Also determine whether R is an equivalence relation. 4 Marks

Ans: Reflexivity:

Let there be a natural number n ,

We know that n divides n, which implies nRn.

So, Every natural number is related to itself in relation R.

Thus, relation R is reflexive .

Transitivity:

Let there be three natural numbers a,b,c and let aRb, bRc

aRb implies a divides a and bRc implies b divides c, which as combined implies that a divides c i.e. aRc.

So, Relation R is also transitive.

Symmetry:

Let there be two natural numbers a,b and let aRb,

aRb implies a divides b but it can't be assured that b necessarily divides a.

For ex, 2R4 as 2 divides 4 but 4 does not divide 2 .

Thus Relation R is not symmetric.

Hence, the relation is not an equivalence relation.

Or

Prove that $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\frac{1}{2}\sin^{-1}\frac{4}{5}$. **4 Marks**

Ans: To Prove, $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\frac{1}{2}\sin^{-1}\frac{4}{5}$

LHS= $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}$

$$=\tan^{-1}\left(\frac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}\times\frac{2}{9}}\right)$$

$$\left[\therefore \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$=\tan^{-1}\left(\frac{\frac{17}{36}}{\frac{34}{36}}\right)$$

$$=\tan^{-1}\left(\frac{1}{2}\right)$$

$$=\frac{1}{2}\sin^{-1}\left(\frac{2.\frac{1}{2}}{1+\frac{1}{4}}\right)$$

$$\left[\therefore \tan^{-1}x=\frac{1}{2}\sin^{-1}\left(\frac{2x}{1+x^2}\right)\right]$$

$$=\frac{1}{2}\sin^{-1}\left(\frac{1}{\frac{5}{4}}\right)$$

$$=\frac{1}{2}\sin^{-1}\left(\frac{4}{5}\right)$$

RHS= $=\frac{1}{2}\sin^{-1}\frac{4}{5}$

Hence, LHS = RHS.

31. Find the equation of the plane passing through the points (1,0,-2), (3,-1,0) and perpendicular to the plane 2x-y+z=8 . Also find the distance of the plane thus obtained from the origin. 4 Marks

Ans: Given points, P(1,0,-2,), Q(3,-1,0)

Given plane, 2x-y+z=8**.**

Normal vector of given plane, $\vec{n}_1=2\hat{i}-\hat{j}+\hat{k}$

Now, $\overrightarrow{PQ}=2\hat{i}-\hat{j}+2\hat{k}$

Normal vector of required plane,

$\vec{n}_2=\left(\vec{n}_1\times\overrightarrow{PQ}\right)$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -1 & 1\\ 2 & -1 & 2\end{vmatrix}$$

$=-1.\hat{i}-2\hat{j}+0\hat{k}$

$=-\hat{i}-2\hat{j}$

Required equation of plane,

-1(x-1)+(-2)(y-0)+0(z-2)=0

-x+1-2y+0=0

x+2y=1

Therefore, the required equation of the plane is, x+2y=1.

Now, distance from origin (0,0,0) is,

$$=\left|\frac{a_1.a+b_1.b+c_1.c+d}{\sqrt{a^2+b^2+c^2}}\right|$$

$$=\left|\frac{0.1+0.2+0.0+(-1)}{\sqrt{1^2 2^2+0^2}}\right|$$

$$=\left|\frac{-1}{\sqrt{1+4}}\right|$$

$= \frac{1}{\sqrt{5}}$

Therefore, distance from origin is $\frac{1}{\sqrt{5}}$ units.

32. If $\tan^{-1}\left(\frac{y}{x}\right)=\log\sqrt{x^2+y^2}$, prove that $\frac{dy}{dx}=\frac{x+y}{x-y}$. **4 Marks**

Ans: Given, $\tan^{-1}\left(\frac{y}{x}\right)=\log\sqrt{x^2+y^2}$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right)=\log\left(x^2+y^2\right)$$

Now, differentiating both sides w.r.t x,

$$\Rightarrow \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{d}{dx}\left(\frac{y}{x}\right)=\frac{1}{2}.\frac{1}{x^2+y^2}.\frac{d}{dx}\left(x^2+y^2\right)$$

$$\Rightarrow \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{x\frac{d}{dx}-y}{x^2}\right)=\frac{1}{2}.\frac{1}{x^2+y^2}.\frac{d}{dx}\left(2x+2y\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{x^2}{x^2+y^2}.\left(\frac{x\frac{dy}{dx}-y}{x^2}\right)=\frac{1}{2}.\frac{2\left(x+y\frac{dy}{dx}\right)}{x^2+y^2}$$

$$\Rightarrow x+y\frac{dy}{dx}=x\frac{dy}{dx}-y$$

$$\Rightarrow x\frac{dy}{dx}-y\frac{dy}{dx}=x+y$$

$$\Rightarrow \frac{dy}{dx}(x-y)=(x+y)$$

$$\Rightarrow \frac{dy}{dx}=\frac{x+y}{x-y}$$

Hence, proved.

Or

If $y=e^{acos^{=1}x}$, $-1<x<1$, then show that

$\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-ay^2=0$. **4 Marks**

Ans: Given curve is $y=e^{acos^{=1}x}$

Differentiating given curve,

$$y'=e^{acos^{-1}x}.\frac{d}{dx}\left(acos^{-1}x\right)$$

$$y'=e^{acos^{-1}x}.\frac{(-a)}{\sqrt{1-x^2}}$$

$$\Rightarrow y'=\frac{-ay}{\sqrt{1-x^2}}--(1)$$

$$\left[\therefore y=e^{acos^{-1}x}\right]$$

On differentiating above equation again w.r.t x, we get

$$\Rightarrow y^{n}=\frac{-a\left(-ae^{acos^{-1}x}+\frac{x.e^{acos^{-1}x}}{\sqrt{1-x^2}}\right)}{\left(1-x^2\right)}$$

$$\Rightarrow \left(1-x^2\right)y^{n}=-a\left(-ae^{acos^{-1}x}+\frac{x.e^{acos^{-1}x}}{\sqrt{1-x^2}}\right)$$

$$\Rightarrow \left(1\text{-}x^2\right)y^n = a^2e^{a\cos^{-1}x} - \frac{a.e^{a\cos^{-1}x}}{\sqrt{1\text{-}x^2}}$$

$$\Rightarrow \left(1\text{-}x^2\right)y^n = a^2y + xy'$$

[from (1)]

$$\Rightarrow \left(1\text{-}x^2\right)y^n \text{-}xy' - a^2y = 0$$

$$\Rightarrow \left(1\text{-}x^2\right)\frac{d^2y}{dx^2} \text{-}x\frac{dy}{dx} \text{-}a^2y = 0$$

Hence, proved.

SECTION - D

Question numbers 33 to 36 carry 6 marks each.

33. Amongst all open (from the top) right circular cylindrical boxes of volume 125π cm3, find the dimensions of the box which has the least surface area. 6 Marks

Ans: Given that the volume of the right circular cylindrical box $V=125\pi\ cm^3$.

Let the radius of the cylinder be h and the height be equal to h.

Volume, $V=\pi r^2 h$

$$\Rightarrow 125\pi = \pi r^2 h$$

$$\Rightarrow r^2 h = 125$$

$$\Rightarrow h = \frac{125}{r^2}$$

Surface area of the box, $S=\pi rh+\pi r^2$

$$S=\pi r\left(\frac{125}{r^2}\right)+\pi r^2$$

$S=\left(\frac{250}{r}\right)+\pi r^2$

Differentiating S w.r.t r to find the point of minima,

$\Rightarrow \frac{dS}{dr}=\frac{-250\pi}{r^2}+2\pi r$

Therefore, for the point of minima,

$\Rightarrow \frac{dS}{dr}=0$

$\Rightarrow \frac{-250\pi}{r^2}+2\pi r=0$

$\Rightarrow 2\pi r=\frac{250\pi}{r^2}$

$\Rightarrow r^3=125$

$\Rightarrow$ r = 5cm

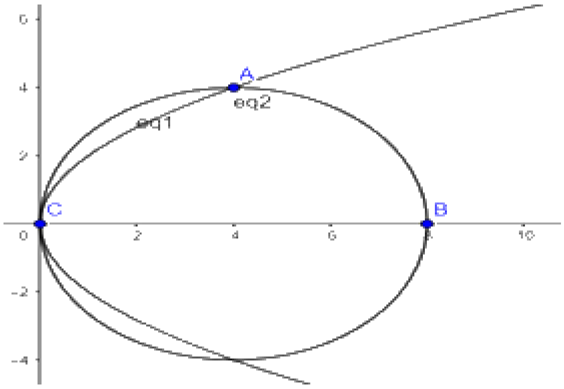
[Only positive value will be considered, as length can't have negative value]

Now, $h=\frac{125}{r^2}=\frac{125}{25}=5\,cm$

The dimension of the cylindrical box is radius, r=5 cm and height, h=5cm.

34. Using integration, find the area lying above x-axis and included between the circle $x^2+y^2=8x$ and inside the parabola $y^2=4x$. **6 Marks**

Ans:



The given equations are

$x^2+y^2=8x$ -------(1)

$y^2=4x$ -----(2)

From (1),

$x^2-8x+y^2=0$

$\Rightarrow x^2-2.5x+16+y^2=16$

$\Rightarrow (x-4)^2+y^2=(4)^2$ ------------(3)

Therefore, the equation (1) is a circle with centre (4,0) and has a radius 4.

Also, $y^2=4x$ is a parabola with vertex at origin and the axis along the x-axis opening in the positive direction.

To find the intersection points of the curves, we solve both the equation.

$\therefore x^2+4x=8x$

$x^2-4x=0$

X(x-4)=0

x=0 and x=4

When, $x=4, y=\pm 4$.

But since, it is given above the x-axis.

So, y=4.

Therefore, area, $A=\int_0^4 |y_2-y_1| dx$

$$=\int_0^4 (y_2-y_1) dx$$

$[\therefore y_2>y_1]$

$$=\int_0^4 \left[\sqrt{\left(16-(x-4)^2\right)}-2\sqrt{x}\right] dx$$

[from (2) and (3)]

$$=\int_0^4 \left[\sqrt{\left(16-(x-4)^2\right)}\right] dx-\int_0^4 -2\sqrt{x} dx$$

$$\left[\frac{(x-4)}{2}\sqrt{16-(x-4)^2}+\frac{16}{2}\sin^{-1}\left(\frac{x-4}{4}\right)\right]_0^4-\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^4$$

$$\left[\frac{(x-4)}{2}\sqrt{16-(x-4)^2}+\frac{16}{2}\sin^{-1}\left(\frac{x-4}{4}\right)\right]_0^4-\frac{4}{3}\left[x^{\frac{3}{2}}\right]_0^4$$

$$=\left[\left(\frac{(4-4)}{2}\sqrt{16-(4-4)^2}+\frac{16}{2}\sin^{-1}\left(\frac{4-4}{4}\right)\right)-\left(\frac{(0-4)}{2}\sqrt{16-(0-4)^2}+\frac{16}{2}\sin^{-1}\left(\frac{0-4}{2}\right)\right)\right]+\frac{4}{3}\left[3^{\frac{3}{2}}-0^{\frac{3}{2}}\right]$$

$$=\left[\left(\frac{(0)}{2}\sqrt{16-0}+8\sin^{-1}(0)\right)\left(-2\sqrt{16-16}+8\sin^{-1}(-1)\right)\right]-\frac{4}{3}\left[4^{\frac{3}{2}}\right]$$

$$=\left[(0+0)-\left(0-8\sin^{-1}(1)\right)\right]-\frac{4}{3}\left[4^{\frac{3}{2}}\right]$$

$$=8\frac{\pi}{2}-\frac{4}{3}.2^3$$

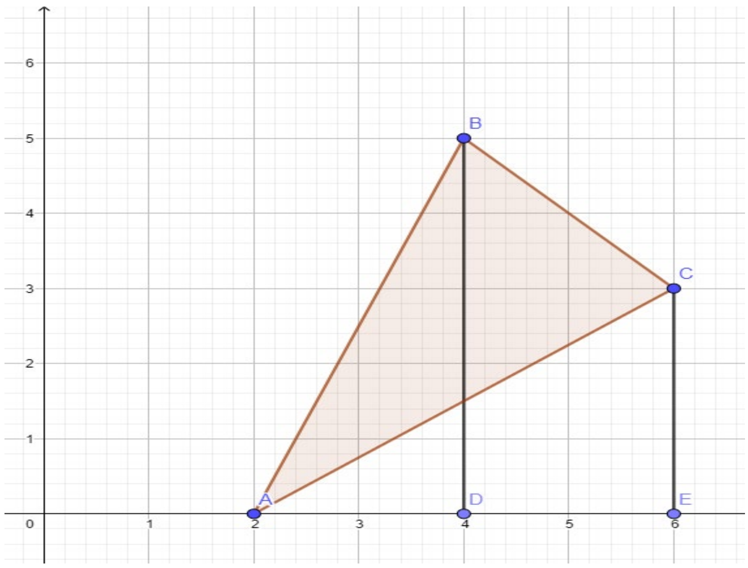
$$=4\pi-\frac{32}{3}$$

Hence, the required area of the region is $4\pi-\frac{32}{3}$ sq. units.

Or

Using the method of integration, find the area of the triangle ABC , coordinates of whose vertices are A(2,0), B(4,5) and C(6,3). **6 Marks**

Ans: The vertices of Δ ABC are A(2,0), B(4,5) and C(6,3)**.**



Equation of line segment AB is

$$y-0=\left(\frac{5-0}{4-2}\right)(x-2)$$

$$\Rightarrow y=\left(\frac{5}{2}\right)(x-2)---(1)$$

Equation of line segment BC is

$$y-5=\left(\frac{3-5}{6-4}\right)(x-4)$$

$$\Rightarrow y-5=\left(\frac{-2}{2}\right)(x-4)$$

$$\Rightarrow y-5=-x+4$$

$$\Rightarrow y=-x+4\text{-------}(2)$$

Equation of line segment CA is

$$\Rightarrow y-3=\left(\frac{0-3}{2-6}\right)(x-6)$$

$$\Rightarrow y-3=\left(\frac{-3}{4}\right)(x-6)$$

$$\Rightarrow 4(y-3)=3(x-6)$$

$$\Rightarrow 4y-12=3x-18$$

$$\Rightarrow 4y=3x-6$$

$$\Rightarrow y=\frac{3}{4}(x-2)\text{---}(3)$$

Area(Δ ABC)=Area(ABDA)+Area(BDECB)-Area(AECA)

$$=\frac{5}{2}\int_2^4(x-2)dx+\int_4^6(-x+9)dx-\int_2^6\frac{3}{4}(x-2)dx$$

$$=\frac{5}{2}\int_2^4(x-2)dx+\int_4^6(-x+9)dx-\frac{3}{4}\int_2^6(x-2)dx$$

$$=\frac{5}{2}\left[\frac{x^2}{2}-2x\right]_2^4+\left[\frac{-x^2}{2}+9x\right]_4^6-\frac{3}{4}\left[\frac{x^2}{2}-2x\right]_2^6$$

$=\frac{5}{2}[8-8-2+4]+[-18+54+8-36]-\frac{3}{4}[18-12-2+4]$

$=\frac{5}{2}[2]+[8]-\frac{3}{4}[8]$

= 5+8-6

= 7 sq. units

Therefore, the area of the triangle is 7 sq. units.

35. If $A=\begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix}$, find A^{-1} and use it to solve the following system of equations:

5x-y+4z=5

2x+3y+5z=2

5x-2y+6z=-1 **6 Marks**

Ans: Given, $A=\begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix}$.

Now, $|A|=\begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix}$

$\Rightarrow |A|=5(18+10)+1(12-25)+4(-4-15)$

$\Rightarrow |A|=140-13-76$

$\Rightarrow |A|=51$

Now, we have to find the cofactor matrix.

$=[A_{ij}]_{3\times 3}$, where, $A_{ij}=(-1)^{i+j}M_{ji}$

$$A_{11}=(-1)^{1+1}M_{11}=\begin{vmatrix}3 & 5\\ -2 & 6\end{vmatrix}=18+10=82$$

$$A_{12}=(-1)^{1+2}M_{12}=\begin{vmatrix}2 & 5\\ 5 & 6\end{vmatrix}=-(12-25)=13$$

$$A_{13}=(-1)^{1+3}M_{13}=\begin{vmatrix}2 & 3\\ 5 & -2\end{vmatrix}=-4-15=-19$$

$$A_{21}=(-1)^{2+1}M_{21}=\begin{vmatrix}-1 & 4\\ -2 & 6\end{vmatrix}=-(6+8)=-2$$

$$A_{22}=(-1)^{2+2}M_{22}=\begin{vmatrix}5 & 4\\ 5 & 6\end{vmatrix}=30-20=10$$

$$A_{23}=(-1)^{2+3}M_{23}=\begin{vmatrix}5 & -1\\ 5 & -2\end{vmatrix}=-(-10+5)=5$$

$$A_{31}=(-1)^{3+1}M_{31}=\begin{vmatrix}5 & 4\\ 2 & 5\end{vmatrix}=-5-12=-17$$

$$A_{32}=(-1)^{3+2}M_{32}=-\begin{vmatrix}5 & 4\\ 2 & 5\end{vmatrix}=-(25-8)=-17$$

$$A_{33}=(-1)^{3+3}M_{33}=\begin{vmatrix}5 & -1\\ 2 & 3\end{vmatrix}=15+2=17$$

Therefore, the cofactor matrix is,

$$\begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}=\begin{bmatrix}28 & 13 & -19\\ -2 & 10 & 5\\ -17 & -17 & 17\end{bmatrix}$$

$$\therefore \text{adj}A=\begin{bmatrix}28 & 13 & -19\\ -2 & 10 & 5\\ -17 & -17 & 17\end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{|A|}.\text{adj}A$$

$$=\frac{1}{51}\begin{bmatrix} 28 & 13 & -19 \\ -2 & 10 & 5 \\ -17 & -17 & 17 \end{bmatrix}$$

Now, given set of equations is,

5x-y+4z=5

2x+3y+5z=2

5x-2y+6z=-1

The equations can be written in matrix form as,

$$\begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$$

This is of the form AX=B, where

$$A=\begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix},X=\begin{bmatrix} x \\ y \\ z \end{bmatrix},B=\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$$

Now, multiplying AX=B by A-1, we get,

A-1(AX)=A-1B

A-1(AX)=A-1B

IX=A-1B

Now, substituting the values, we get,

$$\Rightarrow X=\frac{1}{51}\begin{bmatrix} 28 & 13 & -19 \\ -2 & 10 & 5 \\ -17 & -17 & 17 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$$

$$\Rightarrow X=\frac{1}{51}\begin{bmatrix}140-4+17\\65+20+17\\-95+10-17\end{bmatrix}$$

$$\Rightarrow X=\frac{1}{51}\begin{bmatrix}153\\102\\-102\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}3\\2\\-2\end{bmatrix}$$

Therefore, by equality of matrices.

x=3, y=2, z=-2

This is the required solution.

Or

If x,y,z are different and $\begin{vmatrix}x & x^2 & 1+x^3\\y & y^2 & 1+y^3\\z & z^2 & 1+z^3\end{vmatrix}=0$, then using properties of determinants show that 1+xyz=0. **6 Marks**

Ans: Given, $\begin{vmatrix}x & x^2 & 1+x^3\\y & y^2 & 1+y^3\\z & z^2 & 1+z^3\end{vmatrix}=0$.

Let, $\Delta=\begin{vmatrix}x & x^2 & 1+x^3\\y & y^2 & 1+y^3\\z & z^2 & 1+z^3\end{vmatrix}$

Now, expanding elements of C3 into two determinants,

$$=\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}+\begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix}$$

Taking x,y,z common from R1,R2,R3 in 2nd determinant,

$$=\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix}+xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

Replacing $C_3 \leftrightarrow C_2$ in 1st determinant,

$$=(-1)\begin{vmatrix} x & 1 & x^2 \\ y & 1 & y^2 \\ z & 1 & z^2 \end{vmatrix}+xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

Replacing $C_1 \leftrightarrow C_2$ in 1st determinant,

$$=(-1)(-1)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}+xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$=\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}+xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$=\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}(1+xyz)$$

Using $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$,

$$=\begin{vmatrix} 1 & x & x^2 \\ 1\text{-}1 & y\text{-}x & y^2\text{-}x^2 \\ 1\text{-}1 & z\text{-}x & z^2\text{-}x^2 \end{vmatrix}(1+xyz)$$

$$=\begin{vmatrix} 1 & x & x^2 \\ 1-1 & y-x & (y-x)(y+x) \\ 1-1 & z-x & (z-x)(z+x) \end{vmatrix}(1+xyz)$$

Taking common factor (y-x) from R2 and (z-x) from R3,

$$=\begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & (y+x) \\ 0 & 1 & (z+x) \end{vmatrix}(1+xyz)(y-x)(z-x)$$

Expanding determinant through C1, we get,

$$\left[1\left\{(z+x)-(y-x)\right\}\right](1+xyz)(y-x)(z-x)$$

$$=[z-y](1+xyz)(y-x)(z-x)$$

$$=(1+xyz)(y-x)(z-x)(z-y)$$

Given, x,y,z are different.

Therefore, (x-y) $\neq$ (z-x) $\neq$ (z-y) $\neq 0$

Given, $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$

$$(1+xyz)(y-x)(z-x)(z-y)=0$$

Since, (x-y) $\neq$ (z-x) $\neq$ (z-y) $\neq 0$

Therefore, 1+xyz=0

Hence, proved.

36. A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn randomly one-by-one without replacement and

are found to be both kings. Find the probability of the lost card being a king.
6 Marks

Ans: Let E1 be the event that the card is a king.

And, E2 be the event that the card is not a king.

Let A denote the lost card.

Out of 52 cards 4 are king and 48 are non-king.

Probability that the card is a king, $P(E_1)=\frac{4}{52}=\frac{1}{13}$

Probability that the card is not a king, $P(E_2)=\frac{48}{52}=\frac{12}{13}$

Two cards can be drawn out of 4 king in ${}^4C_2$ ways and 2 kings can be drawn out of 51 cards in ${}^{51}C_2$ ways.

Probability of getting two kings out of the remaining cards if the lost card is a king,

$$P\left(\frac{A}{E_1}\right)=\frac{{}^3C_2}{{}^{51}C_1}$$

$$\Rightarrow P\left(\frac{A}{E_1}\right)=\frac{3}{\frac{51.50}{2}}$$

$$\Rightarrow P\left(\frac{A}{E_1}\right)=\frac{1}{425}$$

Probability of getting two kings out of the remaining cards if the lost card is not a king,

$$\Rightarrow P\left(\frac{A}{E_2}\right)=\frac{{}^4C_2}{{}^{51}C_2}$$

$$\Rightarrow P\left(\frac{A}{E_2}\right)=\frac{\frac{4.3}{2}}{\frac{51.50}{2}}$$

$$\Rightarrow P\left(\frac{A}{E_2}\right)=\frac{2}{425}$$

Therefore, probability of getting two cards when on lost card is king, is,

$$P\left(\frac{E_1}{A}\right)=\frac{\frac{1}{13}.\frac{1}{425}}{\frac{1}{13}.\frac{1}{425}+\frac{12}{13}.\frac{2}{425}}$$

$$P\left(\frac{E_1}{A}\right)=\frac{\frac{1}{13}.\frac{1}{425}}{\frac{1}{13}.\frac{1}{425}+\frac{12}{13}.\frac{2}{425}}$$

$$P\left(\frac{E_1}{A}\right)=\frac{1}{25}$$

Probability that the lost card is a king is $\frac{1}{25}$.

CBSE Question Paper 2019
Class 12 Mathematics

Time allowed: 3 hours
Maximum Marks: 100

General Instructions:

(i) ***All*** *questions are compulsory.*
(ii) *This question paper contains* ***29*** *questions divided into four sections A, B, C and D. Section A comprises of 4 questions of* ***one mark*** *each, Section B comprises of* ***8*** *questions of* ***two marks*** *each, Section C comprises of 11 questions of* ***four marks*** *each and Section D comprises of* ***6*** *questions of* ***six marks*** *each.*
(iii) *All questions in Section* ***A*** *are to be answered in one word, one sentence or as per the exact requirement of the question.*
(iv) *There is no overall choice. However, internal choice has been provided in 1 question of Section* ***A****, 3 questions of Section* ***B****, 3 questions of Section* ***C*** *and 3 questions of Section* ***D****. You have to attempt only* ***one*** *of the alternatives in all such questions.*
(v) *Use of calculators is not permitted. You may ask logarithmic tables, if required.*

SECTION-A

1. If A is a square matrix of order 3 with $|A| = 4$ **then write the value of** $|-2A|$

Solution. Since, order of the matrix, $n = 3$

$|A| = 4$

$|-2A| = (-2)^n |A|$

$|-2A| = (-2)^3 \times 4$

$|-2A| = -32$

Therefore, the value of $|-2A|$ is -32

2. If $\mathbf{y = sin^{-1}x + cos^{-1}x, find \frac{dy}{dx}}$

Solution.

$$y = \sin^{-1} x + \cos^{-1} x$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(\sin^{-1} x + \cos^{-1} x\right)$$

$$= \frac{d}{dx}\left(\sin^{-1} x\right) + \frac{d}{dx}\left(\cos^{-1} x\right)$$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}}$$

$$= 0$$

Therefore, $\frac{dy}{dx} = 0$

3. Write the order and degree of the differential equation $\left(\mathbf{\frac{d^4y}{dx^4}}\right)^2 = \left[\mathbf{x} + \left(\mathbf{\frac{dy}{dx}}\right)^2\right]^3$

Solution. Since,

$$\left(\frac{d^4 y}{dx^4}\right)^2 = \left[x + \left(\frac{dy}{dx}\right)^2\right]^3$$

$$\left(\frac{d^4 y}{dx^4}\right)^2 = x^3 + \left(\frac{dy}{dx}\right)^6 + 3x^2\left(\frac{dy}{dx}\right)^2 + 3x\left(\frac{dy}{dx}\right)^4$$

The highest power raised to $\frac{d^4 y}{dx^4}$ is 2 and degree of the differential equation is 2

4. If the line has the direction ratios -18,12,-4, then what are its direction cosines?

OR

Find the Cartesian equation of the line which passes through the point (-2,4,-5) is parallel to the line $\mathbf{\frac{x+3}{3} = \frac{4-y}{5} = \frac{z+8}{6}}$

Solution.

The direction ratios of the lines are $-18,12,-4$

Direction cosines of the lines are $-\frac{18}{\sqrt{18^2+12^2+4^2}},\frac{12}{\sqrt{18^2+12^2+4^2}},-\frac{4}{\sqrt{18^2+12^2+4^2}}$

Hence, direction cosine of line are $-\frac{9}{11},\frac{6}{11},-\frac{2}{11}$

OR

The cartesion equation of the line which passes through the point $(-2,4,-5)$ and is parallel to the line $\frac{x+3}{3}=\frac{y-4}{-5}=\frac{z+8}{6}$ is $\frac{x+2}{3}=\frac{y-4}{-5}=\frac{z+5}{6}$

SECTION - B

5. If * is defined on the set R of all real number by * : a * b = $\sqrt{a^2+b^2}$ find the identity element if exist in R with respect to *

Solution. As per the question

Let b be the identity element then

$a*b=b*a=a$

$a*b=\sqrt{(a)^2+(b)^2}=a$

$\Rightarrow (a)^2+(b)^2=(a)^2$

$\Rightarrow b=0$

Similarly,

$b*a=\sqrt{(b)^2+(a)^2}=a$

$\Rightarrow (b)^2+(a)^2=(a)^2$

$\Rightarrow b=0$

Therefore, 0 is the identity element

6. If A = $\begin{bmatrix} 0 & 2 \\ 3 & -4 \end{bmatrix}$ and kA = $\begin{bmatrix} 0 & 3a \\ 2b & 24 \end{bmatrix}$ then find the value of k,a and b

Solution. Given,

$$kA = \begin{bmatrix} 0 & 3a \\ 2b & 24 \end{bmatrix} \quad (i)$$

$$A = \begin{bmatrix} 0 & 2 \\ 3 & -4 \end{bmatrix}, \text{implies } kA = \begin{bmatrix} 0 & 2k \\ 3k & -4k \end{bmatrix} \quad (ii)$$

$$\begin{bmatrix} 0 & 2k \\ 3k & -4k \end{bmatrix} = \begin{bmatrix} 0 & 3\text{a} \\ 2\text{b} & 24 \end{bmatrix}$$

$$-4\text{k} = 24 \Rightarrow \text{k} = -6$$

$$3a = 2k \Rightarrow a = -4$$

$$2b = 3k \Rightarrow b = -9$$

7. Find $\int \frac{\sin x - \cos x}{\sqrt{1+\sin 2x}} dx, 0 < x < \pi / 2$

Solution. According to question,

$$let\ I = \int \frac{\sin x - \cos x}{\sqrt{1+\sin 2x}} dx, 0 < x < \frac{\pi}{2}$$

$$I = \int \frac{\sin x - \cos x}{\sqrt{\sin^2 x + \cos^2 x + 2\sin x \cdot \cos x}} dx$$

$$= \int \frac{\sin x - \cos x}{\sqrt{(\sin x + \cos x)^2}} dx$$

$$= \int \frac{\sin x - \cos x}{\sin x + \cos x} dx$$

$$let \sin x + \cos x = t$$

$$\Rightarrow (\cos x - \sin x) dx = dt$$

$$I = \int \frac{-1}{t} dt$$

$$= -\ln t + C$$

$$= \ln\left(\frac{1}{t}\right) + C$$

$$\Rightarrow I = \ln\left(\frac{1}{\sin x + \cos x}\right) + C$$

8. Find $\int \frac{\sin(x-a)}{\sin(x+a)} dx$

OR

Find $\int (\log x)^2 dx$

Solution

$$Let\ I=\int \frac{\sin(x-a)}{\sin(x+a)}dx$$

$$\Rightarrow I=\int \frac{\sin[(x+a)-2a]}{\sin(x+a)}dx$$

$$=\int \frac{\sin(x+a)\cdot\cos(2a)-\cos(x+a)\cdot\sin(2a)}{\sin(x+a)}dx$$

$$=\int \cos(2a)dx-\int \cot(x+a)\cdot\sin(2a)dx$$

$$=x\cdot\cos(2a)-\log|\sin(x+a)|\cdot\sin(2a)+C$$

OR

$$Let\ I=\int (\log x)^2\ dx$$

$$\Rightarrow I=\int 1\cdot(\log x)^2\ dx$$

$$\Rightarrow I=x\cdot(\log x)^2-\int \frac{2x\log x}{x}dx$$

$$\Rightarrow I=x\cdot(\log x)^2-I_1+c_1 \qquad .....(\text{i})$$

$$I_1=\int 2\cdot\log x dx$$

$$\Rightarrow I_1=2x\cdot\log x-2\int \frac{x}{x}dx$$

$$\Rightarrow I_1=2x\cdot\log x-2x+c_2 \qquad .....(ii)$$

$$I=x\cdot(\log x)^2-2x\cdot\log x+2x+c_1-c_2$$

$$I=x\cdot(\log x)^2-2x\cdot\log x+2x+C \qquad (where\,C=c_1-c_2)$$

9. From the differential equation representing the family of curves $y^2=m(a^2-x^2)$ by eliminating the arbitrary constant m and a

Solution

The equation $y^2 = m\left(a^2 - x^2\right)$ where m and a are arbitrary constants

$y^2 = m\left(a^2 - x^2\right)$ .....(i)

$2y\frac{dy}{dx} = -2mx$ .....(ii)

$\Rightarrow -2m = 2\frac{y}{x}\frac{dy}{dx}$

$2\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = -2m$ .....(iii)

$2\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = 2\frac{y}{x}\frac{dy}{dx}$

$y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 - \left(\frac{y}{x}\right)\frac{dy}{dx} = 0$

therefore the required differential equation is $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 - \left(\frac{y}{x}\right)\frac{dy}{dx} = 0$

10. Find the unit vector perpendicular to both the vectors $\vec{a}$ and $\vec{b}$, where $\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$

OR

Show that the vectors $\hat{i} - 2\hat{j} + 3\hat{k}$, $-2\hat{i} + 3\hat{j} - 4\hat{k}$ and $\hat{i} - 3\hat{j} + 5\hat{k}$ are coplanner

Solution

$\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ *and* $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$

let $\vec{n}$ be the vector perpendicular to $\vec{a}$ and $\vec{b}$

$\vec{n} = \vec{a} \times \vec{b}$

$$\vec{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -7 & 7 \\ 3 & -2 & 2 \end{vmatrix} = 19\hat{j} + 19\hat{k}$$

$$\hat{n} = \frac{19\hat{j} + 19\hat{k}}{\sqrt{19^2 + 19^2}} = \frac{1}{\sqrt{2}}\left(\hat{j} + \hat{k}\right)$$

OR

let $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$

$\vec{b} = -2\hat{i} + 3\hat{j} - 4\hat{k}$

$\vec{c} = \hat{i} - 3\hat{j} + 5\hat{k}$

$$\left[\vec{a}\,\vec{b}\,\vec{c}\right] = \begin{vmatrix} 1 & -2 & 3 \\ -2 & 3 & -4 \\ 1 & -3 & 5 \end{vmatrix}$$

$$= 1(15-12) + 2(-10+4) + 3(6-3)$$

$$= 3 - 12 + 9$$

$$= 0$$

therefore, $\vec{a}, \vec{b}, \vec{c}$ are coplanar

11. Mother, father and son line up at random for a family photo. If A and B are two events given by A = Son on one end, B = Father in the middle, find P(B/A).

Solution

If mother (M), father (F), and son (S) line up for the family picture, then the sample space will be

S = {MFS, MSF, FMS, FSM, SMF, SFM} = A= {MFS, FMS, SMF, SFM}

$$P(A \cap B) = \frac{2}{6} = \frac{1}{3}$$

$$P(B) = \frac{2}{6} = \frac{1}{3}$$

$$P(A) = \frac{4}{6} = \frac{2}{3}$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{2}$$

12. Let X be a random variable which assumes values x1, x2, x3, x4 such that 2P(X = x1) = 3P(X = x2) = P(X = x3) = 5P(X = x4). Find the probability distribution of X.

OR

A coin is tossed 5 times. Find the probability of getting (i) at least 4 heads, and (ii) at most 4 heads.

Solution

$Let\ \mathrm{P}(X = x_3) = x$

$\mathrm{P}(X = x_1) = \frac{x}{2}$

$\mathrm{P}(X = x_2) = \frac{x}{3}$

$\mathrm{P}(X = x_4) = \frac{x}{5}$

$\sum_{i=1}^{4} \mathrm{P}(x_i) = 1$

$\mathrm{P}(x_1) + \mathrm{P}(x_2) + \mathrm{P}(x_3) + \mathrm{P}(x_4) = 1$

$\frac{x}{2} + \frac{x}{3} + x + \frac{x}{5} = 1$

$x = \frac{30}{61}$

$\mathrm{P}(X = x_1) = \frac{15}{61}; \mathrm{P}(X = x_2) = \frac{10}{61}; \mathrm{P}(X = x_3) = \frac{30}{61}; \mathrm{P}(X = x_4) = \frac{6}{61}$

So, the probability distribution function will be

X	1	2	3	4
$\mathrm{P}(X = x_i)$	$\frac{15}{61}$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

OR

Total number of probability of tossing a coin 5 times is 32

(i) Probability of getting atleast 4 heads

$P(X=4)+P(X=5)$

${}^5C_4\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^4+{}^5C_5\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^5$

$={}^5C_4\left(\frac{1}{2}\right)^5+{}^5C_5\left(\frac{1}{2}\right)^5$

$=\frac{6}{32}=\frac{3}{16}$

(ii) probability of getting at most 4 head

$P(X=1)+P(X=2)+P(X=3)+P(X=4)$

${}^5C_1\left(\frac{1}{2}\right)^5+{}^5C_2\left(\frac{1}{2}\right)^5+{}^5C_3\left(\frac{1}{2}\right)^5+{}^5C_4\left(\frac{1}{2}\right)^5$

$=\left(\frac{1}{2}\right)^5[5+10+10+5]$

$=\frac{15}{16}$

SECTION – C

14. If $\tan^{-1}x-\cot^{-1}x=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right), x>0$ **then find the value of x and hence find the value of** $\sec^{-1}\left(\frac{2}{x}\right)$

Solution

$$\tan^{-1}x-\cot^{-1}x=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right),x>0$$

$$\Rightarrow \tan^{-1}x-\tan^{-1}\left(\frac{1}{x}\right)=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \qquad \left[\because\ \cot^{-1}x=\tan^{-1}\left(\frac{1}{x}\right),\ x>0\right]$$

$$\Rightarrow \tan^{-1}\left(\frac{x-\frac{1}{x}}{1+x\cdot\frac{1}{x}}\right)=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\Rightarrow \frac{x^2-1}{2x}=\frac{1}{\sqrt{3}}$$

$$\Rightarrow \sqrt{3}x^2-2x-\sqrt{3}=0$$

$$\Rightarrow \sqrt{3}x^2-3x+x-\sqrt{3}=0$$

$$\Rightarrow \sqrt{3}x\left(x-\sqrt{3}\right)+1\left(x-\sqrt{3}\right)=0$$

$$\Rightarrow \left(x-\sqrt{3}\right)\left(\sqrt{3}x+1\right)=0$$

$$\Rightarrow x=-\frac{1}{\sqrt{3}},\sqrt{3}$$

$$\because\ x>0,\ x=\sqrt{3}$$

$$\Rightarrow \sec^{-1}\left(\frac{2}{x}\right)=\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$$

$$\Rightarrow \sec^{-1}\left(\frac{2}{x}\right)=\sec^{-1}\left(\sec\frac{\pi}{6}\right)$$

$$\Rightarrow \sec^{-1}\left(\frac{2}{x}\right)=\frac{\pi}{6}$$

15. Using properties of determinant prove that

$$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}=4abc$$

Solution

$$Let\,\Delta=\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

$R_1 \to R_1 - R_2 - R_3$

$$\Delta=\begin{vmatrix} 0 & -2c & -2b \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

Expending R_1

$$\Delta = 0\begin{vmatrix} c+a & b \\ c & a+b \end{vmatrix} - (-2c)\begin{vmatrix} b & b \\ c & a+b \end{vmatrix} + (-2b)\begin{vmatrix} b & c+a \\ c & c \end{vmatrix}$$

$$= 2c\left(ab+b^2-bc\right)-2b\left(bc-c^2-ac\right)$$

$$= 2abc+2cb^2-2bc^2-2b^2c+2bc^2+2abc$$

$$= 4abc$$

16. If $\left(\sin x\right)^y = x+y,\ find\ \dfrac{dy}{dx}$

Solution

$$\left(\sin x\right)^y = x+y$$

$$\log\left(\sin x\right)^y = \log\left(x+y\right)$$

$$\Rightarrow y\log\left(\sin x\right) = \log\left(x+y\right) \qquad \text{.....(i)}$$

$$\log\left(\sin x\right)\cdot\frac{dy}{dx} + y\cdot\frac{d}{dx}\left[\log\left(\sin x\right)\right] = \frac{d}{dx}\left[\log\left(x+y\right)\right]$$

$$\Rightarrow \log\left(\sin x\right)\cdot\frac{dy}{dx} + y\cdot\frac{\cos x}{\sin x} = \frac{1}{\left(x+y\right)}\cdot\left(1+\frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx}\left[\log\left(\sin x\right) - \frac{1}{\left(x+y\right)}\right] = \frac{1}{\left(x+y\right)} - y\cdot\cot x$$

$$\Rightarrow \frac{dy}{dx} = \frac{1-\left(xy+y^2\right)\cdot\cot x}{\left(x+y\right)\cdot\log\left(\sin x\right)-1}$$

17. If $y=\left(\sec^{-1}x\right)^2, x>0\, show that\ x^2\left(x^2-1\right)\dfrac{d^2y}{dx^2}+\left(2x^3-x\right)\dfrac{dy}{dx}-2=0$

Solution

$y=\left(\sec^{-1}x\right)^{2}, x>0$

$$\Rightarrow \frac{dy}{dx}=2\sec^{-1}x\cdot\frac{d\left(\sec^{-1}x\right)}{dx}$$

$$\Rightarrow \frac{dy}{dx}=2\sec^{-1}x\cdot\frac{1}{x\sqrt{x^{2}-1}} \qquad .....(i)$$

$$\Rightarrow \frac{d^{2}y}{dx^{2}}=2\left[\frac{1}{x^{2}\left(x^{2}-1\right)}\right]+2\sec^{-1}x\left[\frac{-\sqrt{x^{2}-1}-x\left(\frac{2x}{2\sqrt{x^{2}-1}}\right)}{x^{2}\left(x^{2}-1\right)}\right]$$

$$\Rightarrow \frac{d^{2}y}{dx^{2}}=2\left[\frac{1}{x^{2}\left(x^{2}-1\right)}\right]+2\sec^{-1}x\cdot\frac{1}{x\sqrt{x^{2}-1}}\left[\frac{x\left(1-2x^{2}\right)}{x^{2}\left(x^{2}-1\right)}\right] \qquad .....(ii)$$

$$\frac{d^{2}y}{dx^{2}}=2\left[\frac{1}{x^{2}\left(x^{2}-1\right)}\right]+\frac{dy}{dx}\left[\frac{x\left(1-2x^{2}\right)}{x^{2}\left(x^{2}-1\right)}\right]$$

$$\Rightarrow x^{2}\left(x^{2}-1\right)\frac{d^{2}y}{dx^{2}}+\left(2x^{3}-x\right)\cdot\frac{dy}{dx}-2=0$$

18. Find the equation of a tangent and the normal to the curve $y=\frac{(x-7)}{(x-2)(x-3)}$ at the point where it cuts the x-axis

Solution

Equation of the curve is

$$y=\frac{(x-7)}{(x-2)(x-3)}$$

put y=0 in the above equation we get x=7

$$\frac{dy}{dx}=\frac{(x-2)\cdot(x-3)-(x-7)\cdot(2x-5)}{(x-2)^2\cdot(x-3)^2}$$

The slope of the tangent at point $(7,0)$ *is*

$$m_t=\left.\frac{dy}{dx}\right|_{(7,0)}=\frac{20}{400}=\frac{1}{20}$$

$$(y-0)=\frac{1}{20}(x-7)\Rightarrow x-20y-7=0$$

$$m_t\cdot m_n=-1$$

$$\Rightarrow m_n=\frac{-1}{1/20}=-20$$

Equation of the normal is

$$(y-0)=-20(x-7)\Rightarrow 20x+y-140=0$$

19. Find $\int\frac{\sin 2x}{(\sin^2 x+1)(\sin^2 x+3)}dx$

Solution

$$\int \frac{\sin 2x}{(\sin^2 x+1)(\sin^2 x+3)}dx$$

$$\Rightarrow I = \int \frac{2\sin x \cdot \cos x}{(\sin^2 x+1)(\sin^2 x+3)}dx$$

$let \sin^2 x + 3 = t \Rightarrow 2\sin x \cdot \cos x dx = dt$

Therefore,

$$I = \int \frac{dt}{(t-2)t}$$

$$\Rightarrow I = \frac{1}{2}\int\left(\frac{1}{t-2} - \frac{1}{t}\right)dt$$

$$\Rightarrow I = \frac{1}{2}\left[\ln(t-2) - \ln t\right] + c$$

$$\Rightarrow I = \frac{1}{2}\ln\left(\frac{t-2}{t}\right) + c$$

$$\Rightarrow I = \ln\sqrt{\frac{t-2}{t}} + c$$

$$\Rightarrow I = \ln\sqrt{\frac{\sin^2 x+1}{\sin^2 x+3}} + c$$

20. Prove that $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ and hence evaluate $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$

Solution

$let\; a+b-x=t$

$\Rightarrow dx=-dt$

$when\; x=a, t=b\; and\; x=b, t=a$

$$\int_a^b f(x)\,dx=-\int_b^a f(a+b-t)\,dt$$

$$=\int_a^b f(a+b-t)\,dt \qquad \left[\because \int_a^b f(x)\,dx=-\int_b^a f(x)\,dx\right]$$

$$=\int_a^b f(a+b-x)\,dx \qquad \left[\because \int_a^b f(x)\,dx=\int_a^b f(t)\,dt\right]$$

$$let\; I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1+\sqrt{\tan x}}=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos x}dx}{\sqrt{\cos x}+\sqrt{\sin x}}$$

$$I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\cos\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}dx}{\sqrt{\cos\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}+\sqrt{\sin\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}}$$

$$=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sqrt{\sin x}dx}{\sqrt{\sin x}+\sqrt{\cos x}} \qquad .....(iii)$$

$$2I=\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} dx=[x]_{\frac{\pi}{6}}^{\frac{\pi}{3}}=\frac{\pi}{3}-\frac{\pi}{6}=\frac{\pi}{6}$$

$$\Rightarrow I=\frac{\pi}{12}$$

21. Show that $\left(1+x^2\right)dy+2xydx=\cot x dx$

Solution

$\left(1+x^2\right)dy+2xydx=\cot xdx$

$\dfrac{dy}{dx}+\dfrac{2xy}{1+x^2}=\dfrac{\cot x}{1+x^2}$

The Linear differential equation is

IF=$e^{\int pdx}=e^{\int \frac{2x}{1+x^2}}=1+x^2$

the general solution is

y$\left(1+x^2\right)=\int\left[\dfrac{\cot x}{1+x^2}\left(1+x^2\right)\right]dx+C$

$=y\left(1+x^2\right)=\log\left[\sin x\right]+C$

22. let $\vec{a},\vec{b},\vec{c}$ be the three vectors such that $|\vec{a}|=1,|\vec{b}|=2,|\vec{c}|=3$. If the projection of $\vec{a}$ and $\vec{b}$ is equal to the projection of $\vec{c}$ along $\vec{a}$ and $\vec{b},\vec{c}$ are perpendicular to each other then find $|3\vec{a}-2\vec{b}+2\vec{c}|$

Solution

$|\vec{a}|=1,|\vec{b}|=2,|\vec{c}|=3$

the projection of $\vec{b}$ along $\vec{a}=\dfrac{\vec{b}\cdot\vec{a}}{|\vec{a}|}$

the projection of $\vec{c}$ along $\vec{a}=\dfrac{\vec{c}\cdot\vec{a}}{|\vec{a}|}$

$\Rightarrow\dfrac{\vec{b}\cdot\vec{a}}{|\vec{a}|}=\dfrac{\vec{c}\cdot\vec{a}}{|\vec{a}|}$

$\Rightarrow\vec{b}\cdot\vec{a}=\vec{c}\cdot\vec{a}$ .....(i)

$\left(3\vec{a}-2\vec{b}+2\vec{c}\right)\cdot\left(3\vec{a}-2\vec{b}+2\vec{c}\right)=9|\vec{a}|^2-6\vec{a}\cdot\vec{b}+6\vec{a}\cdot\vec{c}-6\vec{b}\cdot\vec{a}+4|\vec{b}|^2-4\vec{b}\cdot\vec{c}+6\vec{c}\cdot\vec{a}-4\vec{c}\cdot\vec{b}+4|\vec{c}|^2$

$|3\vec{a}-2\vec{b}+2\vec{c}|^2=9|\vec{a}|^2+4|\vec{b}|^2+4|\vec{c}|^2-12\vec{a}\cdot\vec{b}+12\vec{a}\cdot\vec{c}-8\vec{b}\cdot\vec{c}$

$|3\vec{a}-2\vec{b}+2\vec{c}|^2=9|\vec{a}|^2+4|\vec{b}|^2+4|\vec{c}|^2$

$\Rightarrow|3\vec{a}-2\vec{b}+2\vec{c}|^2=9\times1+4\times4+4\times9=61$

$\Rightarrow|3\vec{a}-2\vec{b}+2\vec{c}|=\sqrt{61}$

SECTION – D

23. Find the value of λ for which the following lines are perpendicular to each other

$$\frac{\mathbf{x-5}}{\mathbf{5\lambda+2}} = \frac{\mathbf{2-y}}{\mathbf{5}} = \frac{\mathbf{1-z}}{\mathbf{-1}}; \frac{\mathbf{x}}{\mathbf{1}} = \frac{\mathbf{y+\frac{1}{2}}}{\mathbf{2\lambda}} = \frac{\mathbf{z-1}}{\mathbf{3}}$$

hence, find whether the lines intersect or not

Solution

$$\frac{x-5}{5\lambda+2} = \frac{y-2}{-5} = \frac{z-1}{1} \qquad ...(1)$$

and

$$\frac{x}{1} = \frac{y+\frac{1}{2}}{2\lambda} = \frac{z-1}{3} \qquad ...(2)$$

$a_1 = 5\lambda + 2,\ b_1 = -5,\ c_1 = 1$ *and*

$a_2 = 1,\ b_2 = 2\lambda,\ c_2 = 3$

$a_1a_2 + b_1b_2 + c_1c_2 = 0$

$(5\lambda+2) - 5(2\lambda) + 1(3) = 0$

$-5\lambda + 5 = 0$

$\Rightarrow \lambda = -1$

24. If $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{3} \\ \mathbf{1} & \mathbf{-2} & \mathbf{1} \end{bmatrix}$**, find** $\mathbf{A^{-1}}$

hence, solve the following system of equations

$\mathbf{x + y + z = 6}$

$\mathbf{y + 3z = 11}$

$\mathbf{x - 2y + z = 0}$

Solution

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & -2 & 1 \end{bmatrix}$$

Cofactors

$A_{11} = 7, A_{12} = 3, A_{13} = -1$

$A_{21} = -3, A_{22} = 0, A_{23} = 3$

$A_{31} = 2, A_{32} = -3, A_{33} = 1$

$$A^{-1} = \frac{Adj(A)}{|A|}$$

$$Adj(A) = \begin{bmatrix} 7 & 3 & -1 \\ -3 & 0 & 3 \\ 2 & -3 & 1 \end{bmatrix}^T = \begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$$

$|A| = 9$

$$A^{-1} = \frac{1}{9}\begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}$$

For system of equations

$AX = B$

$X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 7 & -3 & 2 \\ 3 & 0 & -3 \\ -1 & 3 & 1 \end{bmatrix}\begin{bmatrix} 6 \\ 11 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 9 \\ 18 \\ 27 \end{bmatrix}$$

$x = 1,\ y = 2,\ z = 3$

25. Show that the height of a cylinder, which is open at the top, having a given surface area and greatest volume, is equal to the radius of its base.

Solution

$Let\ R\ be\ the\ radius$

$H\ be\ the\ height$

$V\ be\ the\ volume$

$S\ be\ the\ total\ surface\ area$

$$V = \pi R^2 H$$

$$S = \pi R^2 + 2\pi RH$$

$$\Rightarrow H = \frac{S - \pi R^2}{2\pi R}$$

$Substituting\ value\ of\ H\ in\ V$

$$V = \frac{1}{2}\left(SR - \pi R^3\right)$$

$$\frac{dV}{dR} = \frac{1}{2}\left(S - 3\pi R^2\right)$$

$$\frac{dV}{dR} = 0$$

$$\Rightarrow \frac{1}{2}\left(S - 3\pi R^2\right) = 0$$

$$R = \sqrt{\frac{S}{3\pi}}$$

$$\frac{d^2V}{dR^2} = \frac{1}{2}\left(0 - 6\pi R\right)$$

$$= -3\pi R$$

$$V\ is\ greatest\ when\ R = \sqrt{\frac{S}{3\pi}}$$

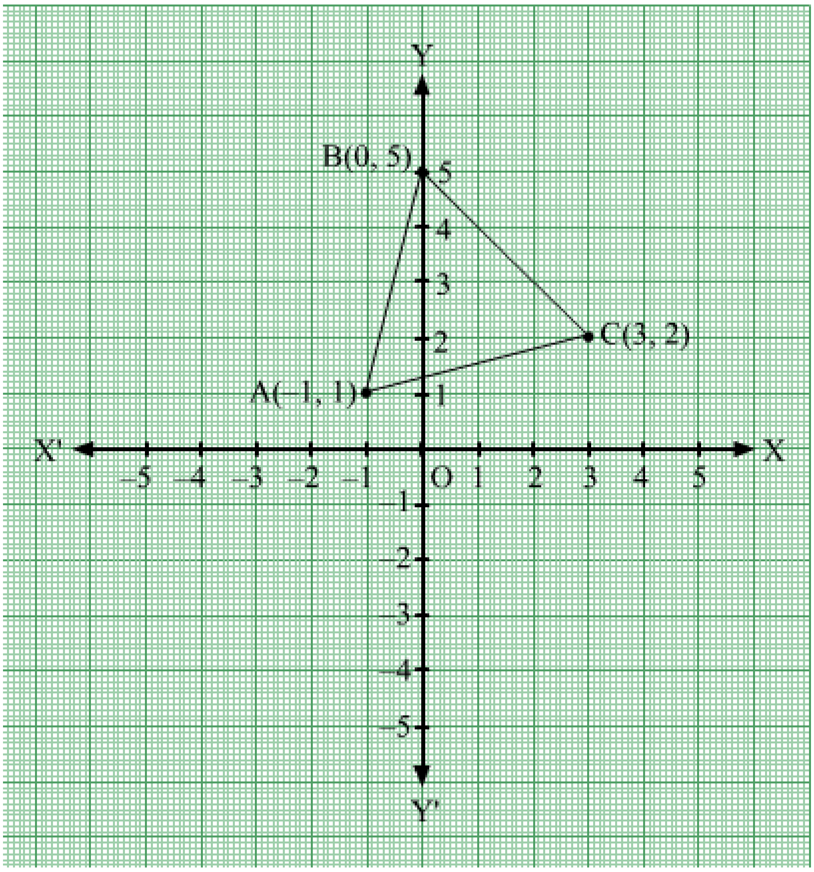
$$H = \frac{S - \pi \times \frac{S}{3\pi}}{2\pi\sqrt{\frac{S}{3\pi}}}$$

$$H = \frac{\frac{2S}{3}}{2\sqrt{\frac{\pi S}{3}}}$$

$$H = \sqrt{\frac{S}{3\pi}}$$

26. Find the area of the triangle whose vertices are (-1, 1), (0, 5) and (3, 2), using integration.

Solution



$Let\ A(-1,1),\ B(0,5)\ and\ C(3,2)$

$The\ equation\ of\ line\ AB\ is$

$$y-1=\frac{5-1}{0+1}(x+1)$$

$$y=4x+5$$

$The\ equation\ of\ line\ BC\ is$

$$y-5=\frac{2-5}{3-0}(x-0)$$

$$y=-x+5$$

$The\ equation\ of\ line\ CA\ is$

$$y-2=\frac{1-2}{-1-3}(x-3)$$

$$y=\frac{x}{4}+\frac{5}{4}$$

$Required\ area=Area\ of\ \Delta ABC$

$The\ equation\ of\ line\ CA\ is$

$$y-2=\frac{1-2}{-1-3}(x-3)$$

$$y=\frac{x}{4}+\frac{5}{4}$$

27. Find the equation of the plane passing through the intersection of the planes $\vec{r}\cdot\left(\hat{i}+\hat{j}+\hat{k}\right)=1$ and $\vec{r}\cdot\left(2\hat{i}+3\hat{j}-\hat{k}\right)+4=0$ and parallel to x-axis. Hence, find the distance of the plane from x-axis.

Solution

$$\vec{a}=2\hat{i}+5\hat{j}-3\hat{k},\vec{b}=-2\hat{i}-3\hat{j}+5\hat{k},\vec{c}=5\hat{i}+3\hat{j}-3\hat{k}$$

$$(\vec{r}-\vec{a})\cdot\left[\left(\vec{b}-\vec{a}\right)\times(\vec{c}-\vec{a})\right]=0$$

$$\Rightarrow\left[\vec{r}-\left(2\hat{i}+5\hat{j}-3\hat{k}\right)\right]\cdot\left[\left(-4\hat{i}-8\hat{j}+8\hat{k}\right)\times\left(3\hat{i}-2\hat{j}\right)\right]=0$$

$$\Rightarrow\left[\vec{r}-\left(2\hat{i}+5\hat{j}-3\hat{k}\right)\right]\cdot\left(2\hat{i}+3\hat{j}+4\hat{k}\right)=0$$

$$\begin{vmatrix} x-2 & y-5 & z+3 \\ -2-2 & -3-5 & 5+3 \\ 5-2 & 3-5 & -3+3 \end{vmatrix}=0$$

$$\Rightarrow\begin{vmatrix} x-2 & y-5 & z+3 \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix}=0$$

$$\Rightarrow(x-2)(16)-(y-5)(-24)+(z+3)(32)=0$$

$$\Rightarrow 2x+3y+4z=7$$

$$2(2\lambda+3)+3(2\lambda+1)+4(3\lambda+5)=7$$

$$\Rightarrow 22\lambda=-22$$

$$\Rightarrow \lambda=-1$$

Therefore, point of intersection is $(1,-1,2)$

28. There are two boxes I and II. Box I contains 3 red and 6 Black balls. Box II contains 5 red and black balls. One of the two boxes, box I and box II is selected at random and a ball is drawn at random. The ball drawn is found to be red. If the probability that this red ball comes out from box II is ' a find the value of n

Solution

$E_1 = selecting\ box\ I$

$E_2 = selecting\ box\ II$

$A = getting\ a\ red\ ball\ from\ selected\ box$

$P(E_1) = \frac{1}{2}, P(E_1) = \frac{1}{2}$

$P\left(\frac{A}{E_1}\right) = \frac{3}{9} = \frac{1}{3}$

$P\left(\frac{A}{E_2}\right) = \frac{5}{n+5}$

$$P\left(\frac{E_2}{A}\right) = \frac{P(E_2)P\left(\frac{A}{E_2}\right)}{P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right)}$$

$$\frac{3}{5} = \frac{\frac{1}{2} \times \frac{5}{n+5}}{\frac{1}{2} \times \frac{1}{3} + \frac{1}{2} \times \frac{5}{n+5}}$$

$$\frac{3}{5} = \frac{15}{n+20}$$

$(n+20)3 = 75$

$3n = 15$

$n = 5$

XII CBSE - BOARD - MARCH - 2018

CODE (65/2)

Date: 21.03.2018 **Mathematics - Solutions**

Section- A

1. If $a*b$ denotes the larger of 'a' and 'b' and if $a \circ b = (a*b)+3$, then write the value of $(5)\circ(10)$, where $*$ and $\circ$ are binary operations.

Sol: $(5)\circ(10) = (5*10)+3 = 10+3 = 13$

2. Find the magnitude of each of the two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude such that the angle between them is 60^0 and their scalar product is $\frac{9}{2}$

Sol: Given :

$|\bar{a}| = |\bar{b}|$ and $\theta = 60^o$ and $\bar{a}\cdot\bar{b} = \frac{9}{2}$

$\therefore \bar{a}\cdot\bar{b} = |\bar{a}||\bar{b}|\cos\theta$

$\frac{9}{2} = |\bar{a}|\ |\bar{a}|\cos 60^o$

$\frac{9}{2} = |\bar{a}|^2 \times \frac{1}{2}$

$|\bar{a}|^2 = 9$

$|\bar{a}| = 3 = |\bar{b}|$

3. If the matrix $A = \begin{bmatrix} 0 & a & -3 \\ 2 & 0 & -1 \\ b & 1 & 0 \end{bmatrix}$ is skew symmetric, find the values of 'a' and 'b'.

Sol: $\because$ A is skew symmetric matrix

$a_{12} = -a_{21} \Rightarrow a = -2$

and $a_{31} = -a_{13} \Rightarrow b = 3$

4. Find the value of $\tan^{-1}\sqrt{3}-\cot^{-1}\left(-\sqrt{3}\right)$

Sol: $\tan^{-1}\left(\sqrt{3}\right)-\cot^{-1}\left(-\sqrt{3}\right)=k\left(\text{say}\right)$

as $\cot^{-1}\left(-x\right)=\pi-\cot^{-1}x$

$\therefore k=\tan^{-1}\left(\sqrt{3}\right)-\left(\pi-\cot^{-1}\left(\sqrt{3}\right)\right)$

$=\frac{\pi}{3}-\left(\pi-\frac{\pi}{6}\right)$

$=\frac{\pi}{3}-\pi+\frac{\pi}{6}$

$=\frac{\pi}{2}-\pi$

$=-\frac{\pi}{2}$

Section- B

5. The total cost $C\left(x\right)$ associated with the production of x units of an item is given by

$C\left(x\right)=0.005x^3-0.02x^2+30x+5000$. Find the marginal cost when 3 units are produced, where by marginal cost we mean the instantaneous rate of total cost at any level of output.

Sol: $C\left(x\right)=0.005x^3-0.02x^2+30x+5000$

$\text{Marginal cost}\left(C_M\right)=\frac{d}{dx}\left(C\left(x\right)\right)=0.005\times 3x^2-0.02\times 2x+30$

$\because x=3$

$C_M=0.005\times 3\times 9-0.02\times 2\times 3+30$

$=0.135-0.12+30$

$=30.135-0.12$

$=30.015$

6. Differentiate $\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$ with respect to x.

Sol: Let $y=\tan^{-1}\left(\frac{1+\cos x}{\sin x}\right)$

$\therefore y = \tan^{-1}\left(\dfrac{2\cos^2 x/2}{2\sin x/2 \cos x/2}\right)$

$= \tan^{-1}(\cot x/2)$

$= \tan^{-1}\left(\tan\left(\dfrac{\pi}{2} - \dfrac{x}{2}\right)\right)$

$\therefore y = \dfrac{\pi}{2} - \dfrac{x}{2}$

$\dfrac{dy}{dx} = \dfrac{d}{dx}\left(\dfrac{\pi}{2} - \dfrac{x}{2}\right) = -\dfrac{1}{2}$

7. Given $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$ compute A^{-1} and show that $2A^{-} = 9I - A$.

Sol: $A = \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix}$

$|A| = 14 - 12 = 2$

$\therefore A_{11} = 7 \qquad A_{12} = 4 \qquad A_{31} = 3 \qquad A_{22} = 2$

$\text{adj}(A) = \begin{bmatrix} A_{11} & A_{22} \\ A_{21} & A_{22} \end{bmatrix}^T = \begin{bmatrix} 7 & 4 \\ 3 & 2 \end{bmatrix}^T = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

$\therefore A^{-1} = \dfrac{1}{|A|}\text{adj}(A) = \dfrac{1}{2}\begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

$\text{L.H.S.} = 2A^{-1} = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

$\text{R.H.S.} = 9I - A = \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix} = \begin{bmatrix} 7 & 3 \\ 4 & 2 \end{bmatrix}$

L.H.S. = R.H.S.

8. Prove that : $3\sin^{-1} x = \sin^{-1}(3x - 4x^3)$, $X \in \left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$

Sol: When $-\dfrac{1}{2} \le x \le \dfrac{1}{2}$

We have,

$-\dfrac{1}{2} \le x \le \dfrac{1}{2} \Rightarrow -\dfrac{\pi}{6} \le \theta \le \dfrac{\pi}{6} \Rightarrow -\dfrac{\pi}{2} \le 3\theta \le \dfrac{\pi}{2}$

Also, $-\frac{1}{2} \le x \le \frac{1}{2} \Rightarrow -1 \le 3x - 4x^3 \le 1$

$\therefore \sin 3\theta = 3x - 4x^3$

$\Rightarrow 3\theta = \sin^{-1}\left(3x - 4x^3\right)$

$\Rightarrow 3\sin^{-1} x = \sin^{-1}\left(3x - 4x^3\right)$

9. A black and a red die are rolled together. Find the conditional probability of obtaining the sum 8, given that the red die resulted in a number less than 4.

Sol: $S = \{(1,1),(1,2).....(6,6)\}$

$\therefore n(s) = 36$

$A =$ Red die resulted in a number less than 4.

$= \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3),(4,1),(4,2),(4,3),$

$(5,1),(5,2),(5,3),(6,1),(6,2),(6,3)\}$

$n(A) = {}^{18}C_1 = 18$

$B =$ sum of number is 8

$B = \{(4,4),(6,2),(2,6),(5,3),(3,5)\}$

$n(B) = {}^{5}C_1 = 5$

$A \cap B = \{(5,3),(6,2)\}$

$n(A \cap B) = {}^{2}C_1 = 2$

$\therefore P\left(\frac{B}{A}\right) =$ Probability of sum of number 8 when Red die resulted in a number less than

$4 = \frac{P(A \cap B)}{P(A)} = \frac{n(A \cap B)}{n(A)} = \frac{2}{18} = \frac{1}{9}$

10. If θ is the angle between two vectors $\hat{i} - 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$ find $\sin\theta$.

Sol: $\bar{a} = \hat{i} - 2\mathrm{j} + 3\hat{k},\ \bar{b} = 3\hat{i} - 2\hat{j} + \hat{k}$

$$\bar{a} \times \bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 3 & -2 & 1 \end{vmatrix} = (4)\hat{i} + 8\hat{j} + 4\hat{k}$$

$$|\bar{a} \times \bar{b}| = \sqrt{(4)^2 + (8)^2 + (4)^2} = \sqrt{16 + 64 + 16} = \sqrt{96}$$

$= 4\sqrt{6}$

$|\bar{a}| = \sqrt{(1)+(4)+9} = \sqrt{14}$

$|\bar{b}| = \sqrt{9+4+1} = \sqrt{14}$

$\sin\theta = \left|\frac{\sqrt{96}}{\sqrt{14}\times\sqrt{14}}\right| = \frac{4\sqrt{16}}{14} = \frac{2\sqrt{6}}{7}$

11. Find the differential equation representing the family of curves $y = ae^{bx+5}$, where *a* and *b* are arbitary constants.

Sol: $y = ae^{bx} \times e^{5}$

$y = ae^{bx} \times e^{5}$

$y = \alpha e^{bx}$ where $e^{5}a = \alpha$

Differentiate w.r.t. 'x'

$\frac{dy}{dx} = \alpha b e^{bx}$

$\therefore \frac{dy}{dx} = by$

$\frac{\frac{dy}{dx}}{y} = b$

Again differentiate w.r.t. 'x'

$\frac{y\frac{d^2y}{dx^2} - \frac{dy}{dx}\times\frac{dy}{dx}}{y^2} = 0$

$y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 = 0$

12. Evaluate : $\int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} dx$

Sol: $I = \int \frac{\cos 2x + 2\sin^2 x}{\cos^2 x} dx$

$I = \int \frac{1 - 2\sin^2 x + 2\sin^2 x}{\cos^2 x} dx$

$I = \int \sec^2 x \, dx$

$I = \tan x + C$

Section- C

13. If $y=\sin(\sin x)$, prove that $\frac{d^2y}{dx^2}+\tan x\frac{dy}{dx}+y\cos^2 x=0$

Sol: $y=\sin(\sin x)$ .....(1)

$$\frac{dy}{dx}=\cos(\sin x)\times\cos x \Rightarrow \frac{\frac{dy}{dx}}{\cos x}=\cos(\sin x).....(2)$$

$$\frac{d^2y}{dx^2}=-\cos(\sin x)\times\sin x-\cos x\ \sin(\sin x)\cos x \qquad ....(3)$$

Put (1) and (2) in (3)

$$\frac{d^2y}{dx^2}=-\left(\frac{\frac{dy}{dx}}{\cos x}\right)\times\sin x-y\cos^2 x$$

$$\frac{d^2y}{dx^2}=-\frac{dy}{dx}\tan x-y\cos^2 x$$

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}\tan x+y\cos^2 x=0$$

14. Find the particular solution of the differential equation $e^x\tan y dx+\left(2-e^x\right)\sec^2 y dy=0$, given that $y=\frac{\pi}{4}$ when $x=0$

Sol: $e^x\tan y\,dx+(2-e^x)\sec^2 y\,dy=0$

$$e^x\tan y\,dx=(e^x-2)\sec^2 y\,dy$$

$$\int\frac{e^x dx}{e^x-2}=\int\frac{\sec^2 y\,dy}{\tan y}$$

$$\ln\left|e^x-2\right|=\ln\left|\tan y\right|+\ln C$$

$$\ln\left|e^x-2\right|=\ln(C\tan y)$$

$$e^x-2=C\tan y$$

Given: $x=0,\ y=\frac{\pi}{4}$

$$e^o-2=C\tan\left(\frac{\pi}{4}\right)$$

$$e^{o}-2=C\tan\left(\frac{\pi}{4}\right)$$

$$1-2=C\times 1 \Rightarrow C=-1$$

$$\therefore e^{x}-2=-\tan y$$

$$e^{x}-2+\tan y=0$$

(OR)

Find the particular solution of the differential equation $\frac{dy}{dx}+2y\tan x=\sin x$, given that $y=0$ when $x=\frac{\pi}{3}$

Sol: $\frac{dy}{dx}+(2\tan x)y=\sin x$

$$\frac{dy}{dx}+py=Q$$

$P=2\tan x$ and $Q=\sin x$

$$I.F=e^{\int Pdx}=e^{2\int \tan x\,dx}=e^{2\ln\sec x}$$

$$=e^{\ln\sec^2 x}=\sec^2 x$$

Soln. $y(I.F)=\int Q(I.F)dx$

$$y\cdot\sec^2 x=\int \sin x\times\sec^2 x\,dx$$

$$y\sec^2 x=\int \tan x\sec x\,dx$$

$$y\sec^2 x=\sec x+C$$

Given $y=0$ $\quad x=\frac{\pi}{3}$

$$\sec\frac{\pi}{3}+C=0$$

$$C=-\sec\frac{\pi}{3}=-2$$

$$\therefore y\sec^2 x=\sec x-2$$

$$y\sec^2 x-\sec x+2=0$$

15. Find the shortest distance between the lines.

$$\vec{r}=\left(4\hat{i}-\hat{j}\right)+\lambda\left(\hat{i}+2\hat{j}-3\hat{k}\right) \text{ and } \vec{r}=\left(\hat{i}-\hat{j}+2\hat{k}\right)+\mu\left(2\hat{i}+4\hat{j}-5\hat{k}\right)$$

Sol: $\bar{r}=\left(4\hat{i}-\hat{j}\right)+\lambda\left(\hat{i}+2\hat{j}-3\hat{k}\right)=\bar{a}+\lambda\bar{b}$ (say)

$\bar{r}=\left(\hat{i}-\hat{j}+2\hat{k}\right)+\mu\left(2\hat{i}+4\hat{j}-5\hat{k}\right)=\bar{c}+\mu\bar{d}$ (say)

$\therefore \bar{c}-\bar{a}=\left(\hat{i}-\hat{j}+2\hat{k}\right)-\left(4\hat{i}-\hat{j}\right)=-3\hat{i}+0\hat{j}+2\hat{k}$

$$\therefore \bar{b}\times\bar{d}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & -3\\ 2 & 4 & -5\end{vmatrix}=2\hat{i}-\hat{j}+0\hat{k}$$

$|\bar{b}\times\bar{d}|=\sqrt{4+1}=\sqrt{5}$

$$\text{Shortest distance}=\left|\frac{(\bar{c}-\bar{a})\cdot(\bar{b}\times\bar{d})}{|\bar{b}\times\bar{d}|}\right|$$

$$=\left|\frac{-6}{\sqrt{5}}\right|=\frac{6}{\sqrt{5}}\text{ units}$$

16. Two numbers are selected at random (without replacement) from the first five positive integers. Let X denote the larger of the two numbers obtained. Find the mean and variance of X.

Sol: X can take values as 2,3,4,5 such that

$P(X=2)$ = probability that the larger of two number 2.

= prob. of getting 1 in first selection and 2 in second selection getting 2 in first selection and 1 in second selection.

$$\therefore P(X=2)=\frac{1}{5}\times\frac{1}{4}+\frac{1}{5}\times\frac{1}{4}=\frac{2}{20}$$

similarly,

$$\therefore P(X=3)=\frac{2}{5}\times\frac{1}{4}+\frac{1}{5}\times\frac{2}{4}=\frac{4}{20}$$

$$\therefore P(X=4)=\frac{3}{5}\times\frac{1}{4}+\frac{1}{5}\times\frac{3}{4}=\frac{6}{20}$$

$$\therefore P(X=5)=\frac{4}{5}\times\frac{1}{4}+\frac{1}{5}\times\frac{4}{4}=\frac{8}{20}$$

X	2	3	4	5
$P(X)$	$\frac{2}{20}$	$\frac{4}{20}$	$\frac{6}{20}$	$\frac{8}{20}$

$$E(X)=2\times\frac{2}{20}+3\times\frac{4}{20}+4\times\frac{6}{20}+5\times\frac{8}{20}$$

$$=\frac{80}{20}=4$$

$$E(X^2)=4\times\frac{2}{20}+9\times\frac{4}{20}+16\times\frac{6}{20}+25\times\frac{8}{20}$$

$$=\frac{340}{20}=17$$

$$V(X)=E(X^2)-(E(X))^2$$

$$=17-16$$
$$=1$$

17. Using propeties of determinants, prove that

$$\begin{vmatrix} 1 & 1 & 1+3x \\ 1+3y & 1 & 1 \\ 1 & 1+3z & 1 \end{vmatrix}=9(3xyz+xy+yz+zx)$$

Sol:

$$L.H.S.=\begin{vmatrix} 1 & 1 & 1+3x \\ 1+3y & 1 & 1 \\ 1 & 1+3z & 1 \end{vmatrix}$$

$C_1 \to C_1 - C_2$; $C_3 \to C_3 - C_2$

$$=\begin{vmatrix} 0 & 1 & 3x \\ 3y & 1 & 0 \\ -3z & 1+3z & -3z \end{vmatrix}$$

$$=(3\times 3)\begin{vmatrix} 0 & 1 & x \\ y & 1 & 0 \\ -z & 1+3z & -z \end{vmatrix}$$

$$=9[-1(-yz-0)+x(y+3zy+z)]$$

$$=9(yz+xy+3xyz+xz)$$

$= 9(3xyz + xy + yz + zx) = R.H.S.$

Hence proved.

18. Find the equations of the tangent and the normal, to the curve $16x^2 + 9y^2 = 145$ at the point (x_1, y_1) where $x_1 = 2$ and $y_1 > 0$

Sol: $\because P(x_1, y_1) \equiv (2, y_1)$ lies on $16x^2 + 9y^2 = 145$

$$16(2)^2 + 9y_1^2 = 145$$

$$9y_1^2 = 145 - 64$$

$$9y_1^2 = 81$$

$$y_1^2 = 9$$

$$y_1 = \pm 3$$

But $y_1 > 0 \quad \therefore \ y_1 = 3$

$\therefore \ P \equiv (2,3)$

$$16x^2 + 9y^2 = 145 \qquad \text{...(i)}$$

$$32x + 18y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-32x}{18y} = \frac{-16x}{9y}$$

Slope of tangent $= m_{(2,3)} = \dfrac{-16 \times 2}{9 \times 3} = \dfrac{-32}{27}$

Slope of normal $= m'_{(2,3)} = \dfrac{27}{32}$

Equation of tangent is,

$$(y - 3) = \frac{-32}{27}(x - 2)$$

$$27y - 81 = -32x + 64$$

$$32x + 27y - 145 = 0$$

Equation of normal is,

$$(y - 3) = \frac{27}{32}(x - 2)$$

$$32y - 96 = 27x - 54$$

$$27x - 32y - 54 + 96 = 0$$

$$27x - 32y + 42 = 0$$

(OR)

Find the intervals in which the function $f(x)=\frac{x^4}{4}-x^3-5x^2+24x+12$ is

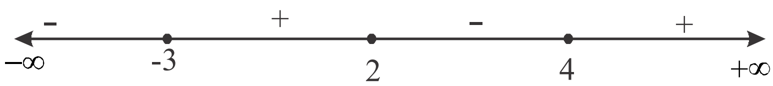
(a) strictly increasing, (b) strictly decreasing.

Sol: $f'(x)=x^3-3x^2-10x+24$

$f'(x)=(x+3)(x-2)(x-4)$

$f(x)$ is strictly increasing

if $f'(x)>0$



$\therefore x\in(-3,2)\cup(4,\infty)$

$f(x)$ is strictly decreasing if $f'(x)<0$

$\therefore x\in(-\infty,-3)\cup(2,4)$

19. Find : $\int\frac{2\cos x}{(1-\sin x)(1+\sin^2 x)}dx$

Sol: Let $I=\int\frac{2\cos x}{(1-\sin x)(1+\sin^2 x)}dx$

Let $\sin x=t$

$\cos dx=dt$

$\therefore I=\int\frac{2}{(1-t)(1+t^2)}dt$

Consider

$$\frac{2}{(1-t)(1+t^2)}=\frac{A}{1-t}+\frac{Bt+C}{t^2+1}$$

$$=\frac{A(t^2+1)+(Bt+C)(1-t)}{(1-t)(t^2+1)}$$

$\therefore 2=At^2+A+Bt+C-Bt^2-Ct$

$=(A-B)t^2+(B-C)t+(A+C)$

$\therefore A-B=0,\ B-C=0\quad A+C=2$

$A=1,\ B=1,\ C=1$

$$\therefore\ I=\int\left(\frac{1}{1-t}+\frac{2t}{2(t^2+1)}+\frac{1}{t^2+1}\right)dt$$

$$=-\log|1-t|+\frac{1}{2}\log\left|t^2+1\right|+\tan^{-1}(t)+C$$

$$=\frac{1}{2}\log\left|\frac{t^2+1}{(1-t)^2}\right|+\tan^{-1}(t)+C$$

$$=\frac{1}{2}\log\left|\frac{\sin^2 x+1}{(1-\sin x)^2}\right|+\tan^{-1}(\sin x)+C$$

20. Suppose a girl throws a die. If she gets 1 or 2 she tosses a coin three times and notes the number of tails. If she gets 3,4,5 or 6, she tosses a coin once and notes whether a 'head' or 'tail' is obtained. If she obtained exactly one 'tail', what is the probability that she threw 3,4,5 or 6 with the ride ?

Sol: Let A be the event that girl will get 1 or 2

$$P(A)=\frac{2}{6}=\frac{1}{3}$$

Let B be the event that girl will get 3, 4, 5 or 6

$$P(B)=\frac{4}{6}=\frac{2}{3}$$

$P\left(T/A\right)=$ Probability of exactly one till given she will get 1 or 2 $=\frac{3}{8}$

$P\left(T/B\right)=$ Probability of exactly one till given she will get 3, 4, 5 or 6 $=\frac{1}{2}$

$$P\left(B/T\right)=\frac{P(B)\times P\left(T/B\right)}{P(A)\times P\left(T/A\right)+P(B)\times P\left(T/B\right)}$$

$$=\frac{\frac{2}{3}\times\frac{1}{2}}{\frac{1}{3}\times\frac{3}{8}+\frac{2}{3}\times\frac{1}{2}}$$

$$=\frac{\frac{1}{3}}{\frac{1}{8}+\frac{1}{3}}$$

$$= \frac{\frac{1}{3}}{\frac{11}{8\times 3}}$$

$$= \frac{8}{11}$$

21. Let $\vec{a} = 4\hat{i} + 5\hat{j} - \hat{k},\ \vec{b} = \hat{i} - 4\hat{j} + 5\hat{k}$ and $\vec{c} = 3\hat{i} + \hat{j} - \hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{c}$ and $\vec{b}$ and $\vec{d}\cdot\vec{a} = 21$

Sol: Since $\overline{d}$ is perpendicular to both $\overline{c}$ and $\overline{b}$, therefore, if is parallel to $\overline{c}\times\overline{b}$

$$\therefore \overline{d} = \lambda(\overline{c}\times\overline{b})$$

$$= \lambda\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -1 \\ 1 & -4 & 5 \end{vmatrix}$$

$$= \lambda\left\{(5-4)\hat{i} - (15+1)\hat{j} + (-12-1)\hat{k}\right\}$$

$$= \lambda\left\{\hat{i} - 16\hat{j} - 13\hat{k}\right\}$$

Given that

$$\overline{d}\cdot\overline{a} = 21$$

$$\lambda\left\{\hat{i} - 16\hat{j} - 13\hat{k}\right\}\cdot\left(4\hat{i} + 5\hat{j} - \hat{k}\right) = 21$$

$$\lambda(4 - 80 + 13) = 21$$

$$\lambda = \frac{21}{-63} = -\frac{1}{3}$$

$$\therefore \overline{d} = -\frac{1}{3}\left(\hat{i} - 16\hat{j} - 13\hat{k}\right)$$

$$= \left(-\frac{1}{3}\right)\hat{i} + \left(\frac{16}{3}\right)\hat{j} + \left(\frac{13}{3}\right)\hat{k}$$

22. An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of material will be least when depth of the tank is half of its width . If the cost is to be least when depth of the tank is half of its width. If the cost is to be borne by nearby settled lower income families, for whom water will be provided, what kind of value is hidden in this question ?

Sol: Let the length, width and height of the open tank be x, x and y units respectively. Then, its volume is x^2y and the total suface area is $x^2 + 4xy$.

It is given that the tank can hold a given quantity of water. This means that its volume is constant. Let it be V. Then,

$V = x^2y$

The cost of the material will be least if the total surface area is least. Let S denote the total surface area. Then,

$S = x^2 + 4xy$

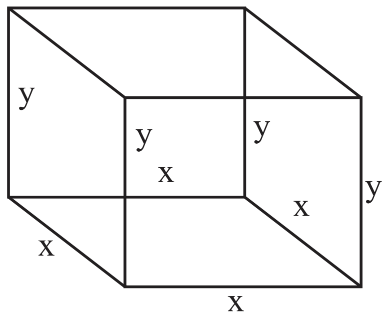
We have to minimize S object to the condition that the volume V is constant.

Now,

$$S = x^2 + 4xy$$

$$\Rightarrow \quad S = x^2 + \frac{4V}{x}$$

$$\Rightarrow \quad \frac{dS}{dx} = 2x - \frac{4V}{x^2} \text{ and } \frac{d^2S}{dx^2} = 2 + \frac{8V}{x^3}$$



The critical numbers of S are given by $\frac{dS}{dx} = 0.$

Now, $\frac{dS}{dx} = 0$

$$\Rightarrow \quad 2x - \frac{4V}{x^2} = 0$$

$$\Rightarrow \quad 2x^3 - 4V = 0$$

$$\Rightarrow \quad 2x^3 = 4x^2y$$

$$\Rightarrow \quad x = 2y$$

Clearly, $\frac{d^2S}{dx^2} = 2 + \frac{8V}{x^3} > 0$ for all x.

Hence, S is minimum when x = 2y i.e. the depth (height) of the tank is half of its width.

Comment : Base is directly proportional to height.

23. If $\left(x^2 + y^2\right)^2 = xy$, *find* $\frac{dy}{dx}$.

Sol: Given :

$$\left(x^2 + y^2\right)^2 = xy$$

$$x^4 + y^4 + 2x^2y^2 = xy$$

diff. w.r.t. x.

$$4x^3+4y^3\frac{dy}{dx}+2\left(2x^2y\frac{dy}{dx}+2xy^2\right)=\left(x\frac{dy}{dx}+y\right)$$

$$4y^3\frac{dy}{dx}+4x^2y\frac{dy}{dx}-x\frac{dy}{dx}=y-4x^3-4xy^2$$

$$\frac{dy}{dx}\left(4y^3+4x^2y-x\right)=y-4x^3-4xy^2$$

$$\frac{dy}{dx}=\frac{y-4x^3-4xy^2}{4y^3+4x^2y-x}$$

(OR)

If $x=a\left(2\theta-\sin 2\theta\right)$ and $y=a\left(1-\cos 2\theta\right)$, find $\frac{dy}{dx}$ when $\theta=\frac{\pi}{3}$.

Sol: $y=a[1-\cos 2\theta]$, $\quad \frac{dy}{d\theta}=a\left(0-2\sin 2\theta\right)$

$$\therefore \frac{dy}{d\theta}=-2a\sin 2\theta$$

$$x=a\left(2\theta-\sin 2\theta\right),\quad \frac{dx}{d\theta}=a\left(2-2\cos 2\theta\right)$$

$$\Rightarrow \frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{-2a\sin 2\theta}{2a[1-\cos 2\theta]}\quad\left(\because \frac{dx}{d\theta}\neq 0\right)$$

$$\Rightarrow \frac{-2\sin\theta\,\cos\theta}{2\sin^2\theta}=-\cot\theta$$

$$\therefore \frac{dy}{dx}=-\cot\left(\frac{\pi}{3}\right)=-\frac{1}{\sqrt{3}}$$

Section- D

24. Evaluate: $\int_0^{\pi/4}\frac{\sin x+\cos x}{16+9\sin 2x}dx$

Sol: Let $\mathrm{I}=\int_0^{\pi/4}\frac{\sin \mathrm{x}+\cos \mathrm{x}}{16+9\sin 2\mathrm{x}}\mathrm{dx}$

Here, we express the denominator in terms sin x - cos x which is integration of numerator.

Clearly, $(\sin x - \cos x)^2 = \sin^2 x + \cos^2 x - 2\sin x \cos x = 1 - \sin 2x$

$\Rightarrow \sin 2x = 1 - (\sin x - \cos x)^2$

$$\therefore\ I = \int_0^{\pi/4} \frac{\sin x + \cos x}{16 + 9\left\{1 - (\sin x - \cos x)^2\right\}} dx$$

$$\Rightarrow I = \int_0^{\pi/4} \frac{\sin x + \cos x}{25 - 9(\sin x - \cos x)^2} dx$$

Let $\sin x - \cos x = t.$ Then, $d(\sin x - \cos x) = dt \Rightarrow (\cos x + \sin x)dx = dt.$

Also, $x = 0 \Rightarrow t = \sin 0 - \cos 0 = -1$ and $x = \frac{\pi}{4} \Rightarrow t = \sin\frac{\pi}{4} - \cos\frac{\pi}{4} = 0$

$$\therefore\ I = \int_{-1}^{0} \frac{dt}{25 - 9t^2} = \frac{1}{9}\int_{-1}^{0} \frac{dt}{\frac{25}{9} - t^2} = \frac{1}{9}\int_{-1}^{0} \frac{dt}{\left(\frac{5}{3}\right)^2 - t^2}$$

$$\Rightarrow\ I = \frac{1}{9} \times \frac{1}{2(5/3)}\left[\log\left|\frac{5/3 + t}{5/3 - t}\right|\right]_{-1}^{0}$$

$$\Rightarrow I = \frac{1}{30}\left[\log 1 - \log\left(\frac{2/3}{8/3}\right)\right] = \frac{1}{30}\left[\log 1 - \log\left(\frac{1}{4}\right)\right] = \frac{1}{30}\left[\log 1 + \log 4\right] = \frac{1}{30}\log 4 = \frac{1}{15}\log 2$$

(OR)

Evaluate : $\int_1^3 \left(x^2 + 3x + e^x\right)dx$

Sol: $I = \int_1^3 (x^2 + 3x + e^x)dx = \int_a^b f(x)dx$ (say)

when $f(x) = x^2 + 3x + e^x$; $a = 1$, $b = 3$

$$h = \frac{b-a}{n} = \frac{3-1}{n} = \frac{2}{n}$$

$$f(a+rh) = f(1+rh) = (1+rh)^2 + 3(1+rh) + e^{1+rh}$$

$$= 4 + 5rh + r^2h^2 + e \times e^{rh}$$

$$= r^2h^2 + 5rh + 4 + e \times e^{rh}$$

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{r=1}^{n} h\, f(a+rh)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} h\,(r^2h^2 + 5rh + 4 + e \times e^{rh})$$

$$= \lim_{n\to\infty} \left(\sum_{r=1}^{n} r^2h^3 + 5\sum_{r=1}^{n} rh^2 + \sum_{r=1}^{n} 4h + e\sum_{r=1}^{n} e^{rh} \times h \right)$$

$$= \lim_{n\to\infty} \left(\frac{8}{n^3} \times \frac{n(n+1)(2n+1)}{6} + 5 \times \frac{4}{n^2} \times \frac{n(n+1)}{2} + 4 \times \frac{2}{n} \times n + e\left(e^h \left(\frac{e^{nh}-1}{e^h-1} \right) \right) \cdot h \right)$$

$$= \lim_{n\to\infty} \left[\left(\frac{8}{6} \times \frac{n}{n} \times \left(\frac{n}{n} + \frac{1}{n} \right) \times \left(\frac{2n}{n} + \frac{1}{n} \right) + \frac{20}{2} \times \frac{n}{n} \times \left(\frac{n}{n} + \frac{1}{n} \right) + 8 + \frac{e^{n+1}(e^2-1)}{\left(\frac{e^h-1}{h} \right)} \right) \right]$$

as $n \to \infty \ \therefore h \to 0$

$$\int_a^b f(x)dx = \frac{4}{3} \times 1 \times \lim_{n\to\infty} \left(1 + \frac{1}{n} \right) \times \lim_{n\to\infty} \left(2 + \frac{1}{n} \right) + \lim_{n\to\infty} 10 \times 1 \times \left(1 + \frac{1}{n} \right) + 8 + \lim_{h\to 0} \frac{e^{h+1}(e^2-1)}{\left(\frac{e^h-1}{h} \right)}$$

$$= \frac{4}{3} \times 1 \times 2 + 10 \times 1 \times 1 + 8 + \frac{e(e^2-1)}{1}$$

$$= \frac{8}{3} + 10 + 8 + e^3 - e$$

$$= \frac{8}{3} + 18 + e^3 - e$$

$$= \frac{62}{3} + e^3 - e$$

25. A factory manufactures two types of screws A and B, each type requiring the use of two machines, an automatic and a hand - operated. It takes 4 minutes on the automatic and 6 minutes on the hand operated machines to manufacture a packet of screws 'B'. Each machine is availble for at most 4 hours on any day. The manufacturer can sell a packet of screws 'A' at a profit of 70 paise and screws 'B' at a profit of Rs. 1. Assuming that he can sell all the screws he manufactrures, how many packets of each type should the factory owner produce in a day in order to maximize his profit ? Formulate the above LPP and solve it graphically and find the maximum profit.

Sol: Let the factory manufactures x screws of type A and y screws of type B on each day.

$\therefore x \geq 0, y \geq 0$

Given that

	Screw A	Screw B	Availibility
Automatic machine	4	6	$4\times 60 = 240$ minutes
Hand operate machine	6	3	$4\times 60 = 240$ minutes
Profit	70 paise	1 rupee	

The constraints are

$4x + 6y \le 240$

$6x + 3y \le 240$

Total profit

$z = 0.70x + 1y$

$\therefore$ $L.P.P.$ is

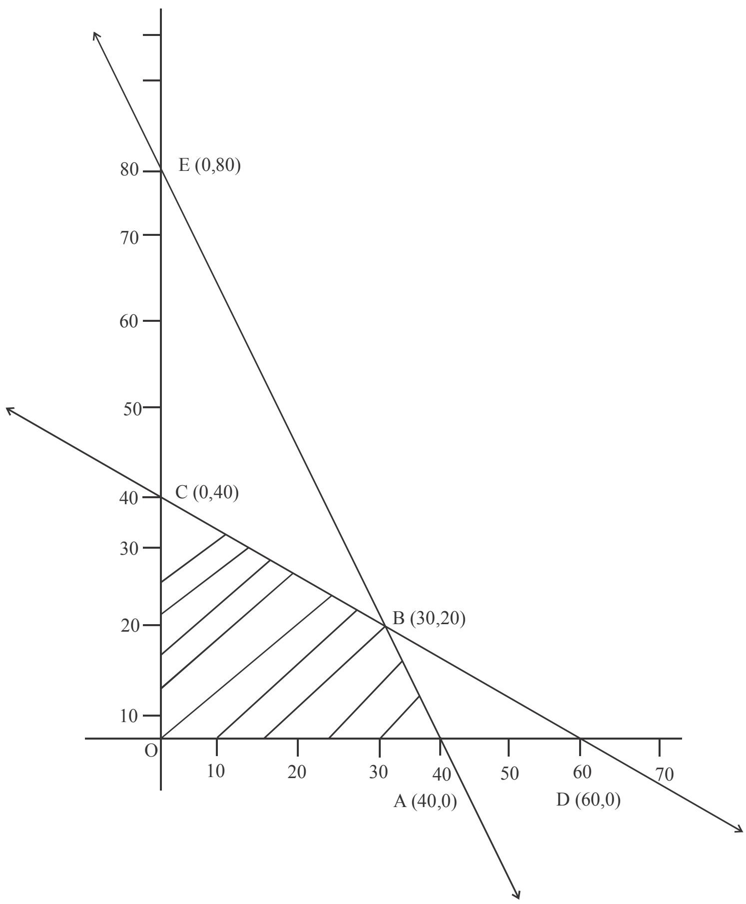
maximise $z = 0.7x + y$

subject to ,

$2x + 3y \le 120$

$2x + y \le 80$

$x \ge 0, y \ge 0$



$\therefore$ common feasible region is $OCBAO$

Correct point	$Z = 0.7x + y$
$A(40,0)$	$Z(A) = 28$
$B(30,20)$	$Z(B) = 41$ maximum
$C(0,40)$	$Z(C) = 40$
$O(0,0)$	$Z(O) = 0$

The maximum value of 'Z' is 41 at $(30,20)$. Thus the factory showed produce 30 packages at screw A and 20 packages of screw B to get the maximum profit of Rs.41

26. Let $A = \{x \in Z : 0 \le x \le 12\}$ show that $R = \{(a,b) : a,b \in A, |a-b|\}$ is divisible by 4} is an equivalence relation. Find the set of all elements related to 1. Also write the equivalence class [2].

Sol: We have,

$R = \{(a,b) : |a-b| \text{ is a multiple of } 4\}, \text{where } a,b \in A = \{x \in Z : 0 \le x \le 12\} = \{0,1,2,...,12\}.$

We observe the following properties of relation R.

Reflexivity : For any $a \in A$, we have

$|a-a| = 0$, which is a multiple of 4.

$\Rightarrow (a,a) \in R$

Thus, $(a,a) \in R$ for all $a \in A$.

So, R is reflexive.

Symmetry : Let $(a,b) \in R$. Then,

$(a,b) \in R$

$\Rightarrow \quad |a-b|$ is a multiple of 4

$\Rightarrow \quad |a-b| = 4\lambda$ for some $\lambda \in N$

$\Rightarrow \quad |b-a| = 4\lambda$ for some $\lambda \in N$ $\quad [\because |a-b| = |b-a|]$

$\Rightarrow \quad (b,a) \in R$

So, R is symmetric.

Transitivity : Let $(a,b) \in R$ and $(b,c) \in R$. Then,

$(a,b) \in R$ and $(b,c) \in R$

$\Rightarrow \quad |a-b|$ is a multiple of 4 and $|b-c|$ is a multiple of 4

$\Rightarrow \quad |a-b| = 4\lambda$ and $|b-c| = 4\mu$ for some $\lambda, \mu \in N$

$\Rightarrow \quad a-b = \pm 4\lambda$ and $b-c = \pm 4\mu$

$\Rightarrow \quad a-c = \pm 4\lambda \pm 4\mu$

$\Rightarrow \quad a - c$ is a multiple of 4

$\Rightarrow \quad |a - c|$ is a multiple of 4

$\Rightarrow \quad (a, c) \in R$

Thus, $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$

So, R is transitive.

Hence, R is an equivalence relation.

Let x be an element of A such that $(x, 1) \in R$. Then,

$|x - 1|$ is a multiple of 4

$\Rightarrow \quad |x - 1| = 0, 4, 8, 12$

$\Rightarrow \quad x - 1 = 0, 4, 8, 12$

$\Rightarrow \quad x = 1, 5, 9 \qquad [\because 13 \notin A]$

Hence, the set of all elements of A which are related to 1 is {1, 5, 9} i.e. [1] = [1, 5, 9].

& $[2] = [2, 6, 10]$

(OR)

Show that the function $f : R \to R$ defined by $f(x) = \frac{x}{x^2 + 1}, \forall x \in R$ is neither one – one nor onto.

Also, if $g : R \to R$ is defined as $g(x) = 2x - 1$ find $fog(x)$

Sol: $f : R \to R,\ f(x) = \frac{x}{x^2 + 1},\ \forall x \in R$

$$f(x_1) = \frac{x_1}{x_2^2 + 1}$$

$$f(x_1) = f(x_2)$$

$$\frac{x_1}{x_1^2 + 1} = \frac{x_2}{x_2^2 + 1}$$

$$x_1 x_2^2 + x_1 = x_2 x_1^2 + x_2$$

$$x_1 x_2^2 - x_2 x_1^2 + x_1 - x_2 = 0$$

$$x_1 x_2 (x_2 - x_1) - 1(x_2 - x_1) = 0$$

$$(x_1 x_2 - 1)(x_2 - x_1) = 0$$

$$x_1 x_2 = 1 \text{ or } x_1 = x_2$$

$\therefore f(x)$ is not one-one

also $y=\dfrac{x}{x^2+1}$

$x^2y-x+y=0$

$\Delta\geq 0$ if x is real

$\therefore B^2-4AC\geq 0$

$(-1)^2-4\times y\times y\geq 0$

$1-4y^2\geq 0$

$(1-2y)(1+2y)\geq 0$

$(2y-1)(2y+1)\leq 0$

$\therefore -\dfrac{1}{2}\leq y\leq\dfrac{1}{2}$

Codomain $\in R$

But range $\in\left[-\dfrac{1}{2},\dfrac{1}{2}\right]$

$\therefore$ Function is not onto

$f(x)=\dfrac{x}{x^2+1}$ as $f:R\to R$

$g(x)=2x-1$ as $g:R\to R$

$(fog)(x)=f(g(x))=\dfrac{g(x)}{(g(x)^2+1)}$

$=\dfrac{2x-1}{(2x-1)^2+1}$

$=\dfrac{2x-1}{4x^2-4x+1+1}$

$=\dfrac{2x-1}{4x^2-4x+2}$

27. Using integration, find the area of the region in the first quadrant enclosed by the x – axis, the line $y=x$ and the circle $x^2+y^2=32$

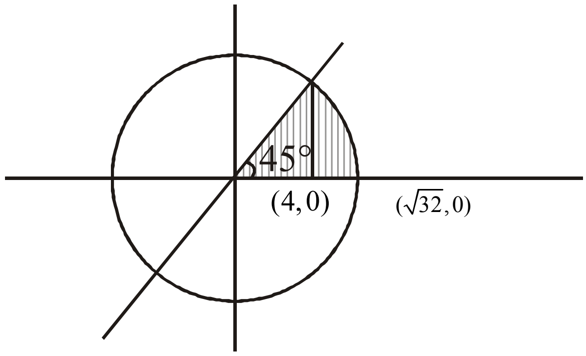
Sol: Put $y=x$ in $x^2+y^2=32$

$\therefore x^2+x^2=32$

$$2x^2 = 32$$

$$x^2 = 16$$

$$x = 4$$



$$A = \int_0^4 y_{\text{line}}dx + \int_4^{\sqrt{32}} y_{\text{circle}}dx$$

$$A = \int_0^4 xdx + \int_4^{\sqrt{32}} (\sqrt{32 - x^2})dx$$

$$= \left(\frac{x^2}{2}\right)_0^4 + \int_4^{\sqrt{32}} \sqrt{(\sqrt{32})^2 - x^2}\, dx$$

$$= (8) + \left(\frac{x}{2}\sqrt{32 - x^2} + \frac{32}{2}\sin^{-1}\left(\frac{x}{\sqrt{32}}\right)\right)^{\sqrt{32}}$$

$$= (8) + \left(0 + 16 \times \frac{\pi}{2} - \left(2\sqrt{16} + 16\sin^{-1}\left(\frac{4}{\sqrt{32}}\right)\right)\right)$$

$$= 8 + 8\pi - 8 - 16\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$$= 8\pi - 16 \times \frac{\pi}{4} = 8\pi - 4\pi = 4\pi \text{ sq units}$$

28. If $A = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix}$, find A^{-1}. Use it solve the system of equations.

$$2x - 3y + 5z = 11$$

$$3x + 2y - 4z = -5$$

$$x + y - 2z = -3$$

Sol: $\mathrm{A} = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$

$$\therefore |\mathrm{A}| = 2(-4+4) + 3(-6+4) + 5(3-2) = 0 - 6 + 5 = -1 \neq 0$$

$\text{Now, } A_{11} = 0,\ A_{12} = 2, A_{12} = 1$

$A_{21} = -1, A_{22} = -9, A_{23} = -5$

$A_{31} = 2, A_{32} = 23, A_{33} = 13$

$$\therefore\ A^{-1} = \frac{1}{|A|}(\text{adj}A) = -\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \quad ....(1)$$

Now, the given system of equations can be written in the form of $AX = B$, where

$$A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ x \end{bmatrix} \text{ and } B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$$

The solution of the systemof equations is given by $X = A^{-1}B$,

$X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix} \qquad [\text{Using}(1)]$$

$$= \begin{bmatrix} 0 - 5 + 6 \\ -22 - 45 + 39 \\ -11 - 25 + 39 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Hence, $x = 1, y = 2, \text{and } z = 3$

(OR)

Using elementary row transformations, find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$

Sol: $$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$$

$|A| = 1(-25 + 28) - 2(-10 + 14) + 3(-8 + 10)$

$= 3 - 2(4) + 3(2) = 9 - 8 = 1 \neq 0$

A^{-1} exists.

$A \cdot A^{-1} = I$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \Rightarrow R_2 - 2R_1$; $R_3 \Rightarrow R_3 + 2R_1$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$R_1 \Rightarrow R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$R_1 \Rightarrow R_1 - R_3; R_2 \Rightarrow R_2 - R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A^{-1} = \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$$

$$I \cdot A^{-1} = A^{-1} = \begin{bmatrix} 3 & -2 & -1 \\ -4 & 1 & -1 \\ 2 & 0 & 1 \end{bmatrix}$$

29. Find the distance of the point $(-1,-5,-10)$ from the point of intersection of the line $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda\left(3\hat{i} + 4\hat{j} + 2\hat{k}\right)$ and the plane $\vec{r} \cdot \left(\hat{i} - \hat{j} + \hat{k}\right) = 5$

Sol: Cartesian equation of line and plane,

$$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2} \qquad : (\text{Line})$$

$$x - y + z - 5 = 0 \qquad : (\text{Plane})$$

Let Q (α, β, γ) be point of intersection of line and plane which will satisfy both equation.

$$\frac{\alpha-2}{3} = \frac{\beta+1}{4} = \frac{\gamma-2}{2} = \lambda \text{ (say)}$$

$\alpha = 3\lambda + 2, \beta = 4\lambda - 1, \gamma = 2\lambda + 2$

also $\alpha - \beta + \gamma - 5 = 0$

$3\lambda + 2 - 4\lambda + 1 + 2\lambda + 2 - 5 = 0$

$\lambda = 0$

$\therefore\ \alpha = 2, \beta = -1, \gamma = 2 \Rightarrow Q \equiv (2, -1, 2)$

$$\ell(PQ) = \sqrt{(-1-2)^2 + (-5+1)^2 + (-10-2)^2}$$

$$= \sqrt{9+16+144}$$

$$= \sqrt{169}$$

$= 13$ units

SET-1 | # MATHEMATICS

Series GBM | Paper & Solution | **Code: 65/1**

Time: 3 Hrs. | Max. Marks: 100

General Instruction:

(i) All questions are compulsory

(ii) The question paper consists of **29** questions divided into four section A, B, C and D. Sections A comprises of questions of **one mark** each, Section B comprises of **8** questions of **two marks** each, Section C comprises of **11** questions of **four marks** each and Section D comprises of **6** questions of **six marks** each.

(iii) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.

(iv) There is no overall choice. However, internal choice has been provided in 3 questions of four marks each and 3 questions of six marks each. You have to attempt only one of the alternatives in all such questions.

(v) Use of calculators is **not** permitted. You may ask for logarithmic tables, if required.

SECTION – A

Question numbers 1 to 4 carry 1 mark each

1. If for any 2 × 2 square matrix A, A (adj A) $=\begin{bmatrix}8 & 0\\ 0 & 8\end{bmatrix}$, then write the value of |A|.

Solution:

$$A\text{ (adj A) }=\begin{bmatrix}8 & 0\\ 0 & 8\end{bmatrix},$$

by using property

A (adj A) = |A| I_n

$$\Rightarrow |A|I_n=\begin{bmatrix}8 & 0\\ 0 & 8\end{bmatrix}$$

$$\Rightarrow |A|I_n=8\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\Rightarrow |A|=8$$

2. Determine the value of 'k' for which the following function is continuous at x = 3:

$$f(x)=\begin{cases}\dfrac{(x+3)^2-36}{x-3} & ,x\neq 3\\ k & ,x=3\end{cases}$$

Solution:

$$\lim_{x\to 3}f(x)=\lim_{x\to 3}\frac{(x+3)^2-36}{x-3}$$

$$=\lim_{x\to 3}\frac{(x+3-6)(x+3+6)}{(x-3)}$$

= 12

given that f(x) is continuous at x = 3

$$\therefore \lim_{x\to 3}f(x)=f(3)$$

$$\Rightarrow k=12$$

3. Find: $\int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x} dx$

Solution:

$$\int \frac{\sin^2 x - \cos^2 x}{\sin x \cos x} dx$$

$$= 2\int \frac{-\cos 2x}{\sin x} dx$$

$$= -2\int \cot 2x \, dx$$

$$= \frac{-2\log|\sin 2x|}{2} + C$$

$$= -\log|\sin 2x| + C$$

4. Find the distance between the planes 2x – y + 2z = 5 and 5x – 2·5y + 5z = 20.

Solution:

2x – y + 2z = 5 …(1)

5x – 2·5 y + 5z = 20

or 2x – y + 2z = 8 …(2)

Distance between plane (1) & (2)

$$= \left|\frac{d_1 - d_2}{\sqrt{a^2 + b^2 + c^2}}\right| = \left|\frac{3}{\sqrt{9}}\right| = 1$$

SECTION - B

Question numbers 5 to 12 carry 2 marks each

5. If A is a skew-symmetric matrix of order 3, then prove that det A = 0.

Solution:

Let A = $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ be a skew symmetric matrix of order 3

$$\therefore \ |A| = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$$

$$|A| = -a(0 + bc) + b(ac - 0)$$

= – abc + abc = 0 Proved

6. Find the value of c in Rolle's theorem for the function $f(x) = x^3 - 3x$ in $[-\sqrt{3}, 0]$.

Solution:

$f(x) = x^3 - 3x$

(i) f(x) being a polynomial is continuous on $[-\sqrt{3}, 0]$

(ii) $f(-\sqrt{3}) = f(0) = 0$

(iii) $f'(x) = 3x^2 - 3$ and this exist uniquely on $[-\sqrt{3}, 0]$

$\therefore$ f(x) is derivable on $(-\sqrt{3}, 0)$

$\therefore$ f(x) satisfies all condition of Rolle's theorem

$\therefore$ There exist at least one $c \in (-\sqrt{3}, 0)$ where $f'(c) = 0$

$\Rightarrow 3c^2 - 3 = 0$

$\Rightarrow c = \pm 1 \Rightarrow c = -1$

7. The volume of a cub is increasing at the rate of $9\text{ cm}^3/s$. How fast is it surface area increasing when the length of an edge is 10 cm?

Solution:

Assumed volume of cube = V

Given that, $\frac{dV}{dt} = 9cm^3/\text{sec}$

$\frac{dA}{dt} = ?$

$l = 10\ cm$

$\frac{dV}{dt} = \frac{d}{dt}(l)^3 = 9 \Rightarrow 3l^2 \frac{dl}{dt} = 9$

$\frac{dl}{dt} = \frac{3}{l^2}$. ....(1)

Now $\frac{dA}{dt} = \frac{d}{dt}(6l^2) = 12l\frac{dl}{dt} = 12l \times \frac{3}{l^2}$ (form (1))

$= \frac{36}{l} = \frac{36}{10} = 3.6\ \text{cm}^2/\text{sec}$

8. Show that the function $f(x) = x^3 - 3x^2 + 6x - 100$ is increasing on R.

Solution: $f(x) = x^2 - 3x^2 + 6x - 100$

$f'(x) = 3x^2 - 6x + 6$

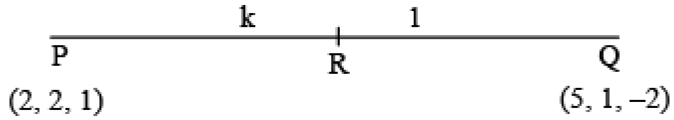
$f'(x) = 3(x^2 - 2x + 2)$

$f'(x) = 3[(x-1)^2 + 1]$

$f'(x) > 0$ for all $x \in R$

So, f(x) is increasing on R.

9. The x- coordinate of a point on the joining the points P(2, 2, 1) and Q(5, 1,-2) is 4. Find its z-coordinate**.**
Solution:



Let **R** divides **PQ** in the ratio **k**: 1

$$R\left(\frac{5k+2}{k+1}, \frac{k+2}{k+1}, \frac{-2k+1}{k+1}\right)$$

Given x co-ordinate of R = 4

$$\therefore \frac{5k+2}{k+1}=4$$

$$\Rightarrow k=2$$

$\therefore$ z co- ordinate = $\frac{-(2)+1}{2+1}=-1$

10. A die, whose faces are marked 1, 2, 3 in red and 4, 5, 6 in green, is tossed. Let A be the event "number obtained is even" and B be the event "number obtained is red". Find if A and B are independent events.

Solution:

A = {2, 4, 6} $P(A)=\frac{3}{6}=\frac{1}{2}$

B = {1, 2, 3}

A ∩ B = {2} $P(B)=\frac{3}{6}=\frac{1}{2}$

P(A ∩ B) = $\frac{1}{6}$

Here, P(A) P(B) $=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$

Since, P(A ∩ B) ≠ P(A) P(B), so events A and B are not independent events.

11. Tow tailors, A and B, earn ₹ 300 and ₹ 400 per day respectively A can stitch 6 shirts and 4 pairs of trousers while B can stitch 10 shirts and 4 pairs of trousers per day. To find how many days should each of them work and if it is desired to produce at least 60 shirts and 32 pairs of trousers at a minimum labor cost, formulate this as an LPP.
Solution:

	Tailor A	Tailor B	Minimum Total No.
No. of shirts	6	10	60
No. of trousers	4	4	32
Wage	Rs 300/day	Rs 400/day	

Let tailor A and B works for X days and Y days respectively

$\therefore\ x\geq 0, \quad y\geq 0$

Minimum number of shirts = 60

$\therefore\ 6x+10y \ge 60$

$3x+5y \ge 8$

Minimum no of trouser = 32

$\therefore\ 4x+4y \ge 32$

$\Rightarrow\ x+\ y \ge 8$

Let z be the total labor cost

$\therefore$ z = 300 x + 400 y

$\therefore$ The given L. P. problem reducers to: z = 300x + 400y

x $\ge$ 0, y $\ge$ 0, 3x + 5y $\ge$ 30 and x + y $\ge$ 8

12. Find: $\int \frac{dx}{5-8x-x^2}$

Solution:

$$= -\int \frac{dx}{\{(x+4)^2-21\}}$$

$$= \int \frac{dx}{(\sqrt{21})^2-(x+4)^2}$$

$$= \frac{1}{2\sqrt{21}} \log\left|\frac{\sqrt{21}+(x+4)}{\sqrt{21}-(x+4)}\right| + C$$

SECTION – C

Question numbers 13 to 23 carry 4 marks each

13. If $\tan^{-1}\frac{x-3}{x-4}+\tan^{-1}\frac{x+3}{x+4}=\frac{\pi}{4}$, then find the value of x.

Solution:

$$\tan^{-1}\left[\frac{\frac{x-3}{x-4}+\frac{x+3}{x+4}}{1-\left(\frac{x^2-9}{x^2-16}\right)}\right]=\frac{\pi}{4}$$

$$\frac{(x+4)(x-3)+(x+3)(x-4)}{(x^2-16)-(x^2-9)}=1$$

$$2x^2-24=-7$$

$$2x^2=-7+24$$

$$x^2=\frac{17}{2}$$

$$x=\pm\sqrt{\frac{17}{2}}$$

14. Using properties of determinants, prove that

$$\begin{vmatrix} a^2+2a & 2a+1 & 1 \\ 2a+1 & a+2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a-1)^3$$

OR

Find matrix A such that

$$\begin{pmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{pmatrix} A = \begin{pmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{pmatrix}$$

Solution:

Use $R_1 = R_1 - R_2$; $R_2 = R_2 - R3$; $R3 = R3$

L. H. S.

$$= \begin{vmatrix} a^2-1 & a-1 & 0 \\ 2a-2 & a-1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} (a-1)(a+1) & (a-1) & 0 \\ 2(a-1) & (a-1) & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

Taking common $(a-1)^2$

$$= (a-1)^2 \begin{vmatrix} (a+1) & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

$$= (a-1)^2[(a+1)(1-0)-1(2-0)]$$

$$= (a-1)^2[(a+1)-2]$$

$$= (a-1)^3$$

= R. H. S.

OR

Let matrix A is

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}$$

$$\begin{bmatrix} 2a-c & 2b-d \\ a & b \\ -3a+4c & -3b+4d \end{bmatrix} = \begin{bmatrix} -1 & -8 \\ 1 & -2 \\ 9 & 22 \end{bmatrix}$$

Comparing both the sides

$2a - c = -1$,

$2b - d = -8$

After solving we get

C = 3, d = – 4

So, $A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$

15. If $x^y + y^x = a^b$, then find $\frac{dy}{dx}$.

OR

If e^y(x + 1) = 1, then show that $\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2$.

Solution:

We have $x^y + y^x = a^b$

Differentiating W. r. t . x, we get $\frac{d}{dx}(x^y) + \frac{d}{dx}(y^x) = 0.$ ……(1)

Let u = x^y ∴ log u = y log x

$\Rightarrow \frac{1}{u}\frac{du}{dx} = y.\frac{1}{x} + \log x.\frac{dy}{dx}; \Rightarrow \frac{du}{dx} = u\left(\frac{y}{x} + \log x\frac{dy}{dx}\right)$

or $\frac{d}{dx}(x^y) = x^y\left(\frac{y}{x} + \log x\frac{dy}{dx}\right)$ ……(2)

Let v = y^x ∴ $\log v = x \log y$

$\Rightarrow \frac{1}{v}\frac{dv}{dx} = x.\frac{1}{y}\frac{dy}{dx} + \log y.1; \Rightarrow \frac{dy}{dx} = v\left(\frac{x}{y}\frac{dy}{dx} + \log y\right)$

or $\frac{d}{dx}(y^x) = y^x\left(\frac{x}{y}\frac{dy}{dx} + \log y\right)$ ……(3)

Using (2) and (3) in (1),

We get. $x^y\left(\frac{y}{x} + \log x\frac{dy}{dx}\right) + y^x\left(\frac{x}{y}\frac{dy}{dx} + \log y\right) = 0.$ …….(4)

$\Rightarrow (x^y \log x + xy^{x-1})\frac{dy}{dx} = -(y^x \log y + yx^{y-1})$ or $\frac{dy}{dx} = -\frac{y^x \log y + yx^{y-1}}{x^y \log x + xy^{x-1}}$

OR

Let e^y (x + 1) = 1

$e^y(1) + (x + 1)e^y \dfrac{dy}{dx} = 0$

$$\Rightarrow (x+1)\frac{dy}{dx}+1=0 \quad ......(1)$$

Again differentiating W. r. t . x,

$$\therefore\ (x+1)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right).1=0$$

$$\frac{d^2y}{dx^2}=-\frac{\frac{dy}{dx}}{(x+1)}$$

$$\frac{d^2y}{dx^2}=-\frac{dy}{dx}.\frac{dy}{dx} \text{ [equation (1)]}$$

$$\frac{d^2y}{dx^2}=\left(\frac{dy}{dx}\right)^2$$

16. Find: $\displaystyle\int\frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)}d\theta$

Solution:

$$\int\frac{\cos\theta}{(4+\sin^2\theta)(5-4\cos^2\theta)}d\theta$$

$$=\int\frac{\cos\theta}{(4+\sin^2\theta)(5-4(1-\sin^2\theta))}$$

$$\int\frac{\cos\theta\ d\theta}{(\sin^2\theta+4)(4\sin^2\theta+1)}$$

Put sin θ = t

Cosθ dθ = dt

$$\therefore\ I=\int\frac{1}{(4+t^2)(1+4t^2)}dt$$

Consider

$$\frac{1}{(4+t^2)(1+4t^2)}=\frac{At+B}{4+t^2}+\frac{Ct+D}{1+4t^2}$$

$$1=(At+B)(1+4t^2)+(Ct+D)(4+t^2)$$

$$=At+B+4At^3+4Bt^2+4Ct+Ct^3+4D+Dt^2$$

$$=(4A+C)t^3+(4B+D)t^2+(A+4C)t+(B+4D)$$

$$4A+C=0\Rightarrow C=-4A$$

$$4B+D=0\Rightarrow D=-4B$$

$$A+4C=0\Rightarrow A=-4C$$

$$B+4D=1$$

By solving we get A = 0, B = $-\frac{1}{15},\ C=\frac{4}{15}$

$$\therefore\frac{1}{(4+t^2)(1+4t^2)}=\frac{-1/15}{4+t^2}+\frac{4/15}{1+4t^2}$$

$$\therefore I=-\frac{1}{15}\int\frac{1}{4+t^2}dt+\frac{4}{15}\times\frac{1}{4}\int\frac{1}{\frac{1}{4}+t^2}dt$$

$$=-\frac{1}{15}\times\frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right)+\frac{1}{15}\times\frac{1}{\frac{1}{2}}\tan^{-1}\left(\frac{t}{1/2}\right)+C$$

$$=-\frac{1}{30}\tan^{-1}\left(\frac{t}{2}\right)+\frac{2}{15}\tan^{-1}(2t)+C$$

$$=\frac{2}{15}\tan^{-1}(2\sin\theta)-\frac{1}{30}\tan^{-1}\left(\frac{\sin\theta}{2}\right)+C$$

17. Evaluate: $\int_0^{\pi}\frac{x\tan x}{\sec x+\tan x}dx$

OR

Evaluate: $\int_1^4\{|x-1|+|x-2|+|x-4|\}dx$

Solution:

$$I=\int_0^{\pi}\frac{x\tan x}{\sec x+\tan x}dx \qquad ...(1)$$

$$I=\int_0^{\pi}\frac{x(\pi-x)(-\tan x)}{-\sec x-\tan x}dx$$

$$I=\int_0^{\pi}\frac{(\pi-x)\tan x}{\sec x+\tan x}dx \qquad ...(2)$$

Adding (1) & (2)

$$2I = \int_0^{\pi} \frac{\pi \tan x}{\sec x + \tan x} dx$$

$$\Rightarrow 2I = 2\pi \int_0^{\pi/2} \frac{\tan x}{\sec x + \tan x} dx$$

$$\left\{ \because \int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx \text{ whenever } f(2a - x) = f(x) \right\}$$

$$I = \pi \int_0^{\pi/2} \frac{\tan x}{\sec x + \tan x} dx$$

$$I = \pi \int_0^{\pi/2} \frac{\tan x(\sec x - \tan x)}{\sec^2 x - \tan^2 x} dx$$

$$I = \pi \int_0^{\pi/2} (\sec x \tan x - \tan^2 x) dx$$

$$= \pi \int_0^{\pi/2} (\sec x \tan x - \sec^2 x + 1) dx$$

$$I = \pi \left[\sec x - \tan x + x \right]_0^{\pi/2}$$

$$= \pi \left[\lim_{x \to \frac{\pi^-}{2}} \left(\sec x - \tan x \right) + \frac{\pi}{2} - \sec 0 \right]$$

$$= \pi \lim_{x \to \frac{\pi^-}{2}} \frac{1 - \sin x}{\cos x} + \frac{\pi^2}{2} - \pi$$

$$= \pi \lim_{x \to \frac{\pi^-}{2}} \frac{1 - \sin^2 x}{\cos x(1 + \sin x)} + \frac{\pi^2}{2} - \pi$$

$$= \frac{\pi^2}{2} - \pi$$

OR

Let f (x) = | x – 1| +| x – 2 | + |x – 4|

We have three critical points x = 1, 2, 4

(i) when x < 1

(ii) when 1 ≤ x < 2

(iii) when 2 ≤ x < 4

(iv) when x ≥ 4

F (x) = – (x – 1) – (x – 2) – (x – 4)	if	x < 1
= (x – 1) – (x – 2) – (x – 4)	if	1 ≤ x < 2
= (x – 1) + (x – 2) – (x – 4)	if	2 ≤ x < 4
= (x – 1) + (x – 2) + (x – 4)	if	x ≤ 4

∴ f(x) = –3x + 7	if	x < 1
= –x + 5	if	1 ≤ x < 2
= x + 1	if	2 ≤ x < 4
= 3x –7	if	x ≥ 4

$$\therefore I=\int_1^4 f(x)dx$$

$$\therefore I=\int_1^2 f(x)dx+\int_2^4 f(x+1)dx$$

$$\therefore I=\int_1^2(-x+5)dx+\int_2^4(x+1)dx$$

$$=\left[-\frac{x^2}{2}+5x\right]_1^2+\left[\frac{x^2}{2}+x\right]_2^4$$

$$=\left(-\frac{4}{2}+10\right)-\left(-\frac{1}{2}+5\right)+\left(\frac{16}{2}+4\right)-\left(\frac{4}{2}+2\right)$$

$$=8-\frac{9}{2}+12-4=\frac{23}{2}$$

18. Solve the differential equation $(\tan^{-1} x - y)dx = (1 + x^2)\, dy$.

Solution:

We have

$$\frac{dy}{dx}=\frac{\tan^{-1}x-y}{1+x^2}$$

$$\frac{dy}{dx}+\frac{y}{1+x^2}=\frac{\tan^{-1}x}{1+x^2}$$

$$I.F.=e^{\int\frac{1}{1+x^2}dx}=e^{\tan^{-1}x}$$

$$y.e^{\tan^{-1}}=\int\frac{\tan^{-1}x}{1+x^2}\times e^{\tan^{-1}}dx$$

Put t = $\tan^{-1}$

$$dt=\frac{1.dx}{1+x^2}$$

$$=t.e^t-\int 1.e^t dt$$

$$y.e^{\tan^{-1}}x=te^t-e^t+c$$

$$y.e^{\tan^{-1}}x=(\tan^{-1}x-1)e^{\tan^{-1}x}+c$$

$$y=\tan^{-1}x-1+ce^{\tan^{-1}x}$$

19. Show that the points A, B, C with position vectors $2\hat{i}-\hat{j}+\hat{k}, \hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ respectively, are the vertices of a right-angled triangle, Hence find the area of the triangle.

Solution:

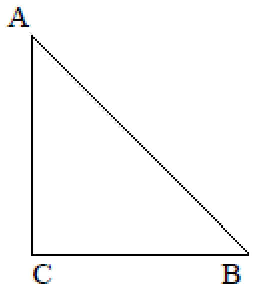
$\overrightarrow{AB} = -\hat{i} - 2\hat{j} - 6\hat{k}$

$\overrightarrow{BC} = 2\hat{i} - \hat{j} + \hat{k}$

$\overrightarrow{CA} = \hat{i} - 3\hat{j} - 5\hat{k}$

$\overrightarrow{BC}.\overrightarrow{CA} = 0$

$\overrightarrow{BC} \perp \overrightarrow{CA}$



$\therefore \Delta ABC$ is a right angled triangle

$$\Delta = \frac{1}{2}|\overrightarrow{BC}||\overrightarrow{AC}|$$

$$\Delta = \frac{1}{2}\sqrt{4+1+1}\sqrt{1+9+25}$$

$$= \frac{1}{2}\sqrt{6}\sqrt{35}$$

$$= \frac{1}{2}\sqrt{210}$$

20. Find the value of λ, if four points with position vectors $3\hat{i} + 6\hat{j} + 9\hat{k}, \hat{i} + 2\hat{j} + 3\hat{k}, 2\hat{i} + 3\hat{j} + \hat{k}$ and $4\hat{i} + 6\hat{j} + \lambda\hat{k},$ are coplanar.

Solution:

We have

P.V. of A $= 3\hat{i} + 6\hat{j} + 9\hat{k}$

P.V. of B $= \hat{i} + 2\hat{j} + 3\hat{k}$

$\overrightarrow{AB} = -2\hat{i} - 4\hat{j} - 6\hat{k}$

$\overrightarrow{AD} = -\hat{i} - 3\hat{j} - 8\hat{k}$

$\overrightarrow{AD} = \hat{i} + (\lambda - 9)\hat{k}$

Now,

$$\overrightarrow{AB}.(\overrightarrow{AC}\times\overrightarrow{AD})\Rightarrow\begin{vmatrix}-2 & -4 & -6\\ -1 & -3 & -8\\ 1 & 0 & (\lambda-9)\end{vmatrix}=0$$

$\Rightarrow -2(-3\lambda + 27) + 4(-\lambda + 9 + 8) - 6(0 + 3) = 0$

$\Rightarrow 6\lambda - 54 - 4\lambda + 68 - 18 = 0$

$2\lambda - 4 = 0$

$\lambda = 2$

$\because$ $\overrightarrow{AB}, \overrightarrow{AC}, \overleftrightarrow{AD}$ are coplanar and so the points A, B, C and D are coplanar.

21. There are 4 cards numbered 1, 3, 5 and 7, one number on one card. Two cards are drawn at random without replacement. Let X denote the sum of the numbers on the two drawn cards. Find the mean and variance of X.

Solution:

X denote sum of the numbers so, X can be 4, 6, 8, 10, 12

X	Number on card	P(x)	X P(x)	X^2 P(x)
4	(1, 3)	$\frac{1}{4}\times\frac{1}{3}\times 2=\frac{1}{6}$	2/3	8/3
6	(1, 6)	$\frac{1}{4}\times\frac{1}{3}\times 2=\frac{1}{6}$	1	6
8	(3, 5) or (1, 7)	$\frac{1}{4}\times\frac{1}{3}\times 2+\frac{1}{4}\times\frac{1}{3}\times 2=\frac{1}{3}$	8/3	64/3
10	(3, 7)	$\frac{1}{4}\times\frac{1}{3}\times 2=\frac{1}{6}$	5/3	50/3
12	(5, 7)	$\frac{1}{4}\times\frac{1}{3}\times 2=\frac{1}{6}$	2	24

Mean = $\Sigma XP(x) = 8$

Variance $= \Sigma X^2 P(x) - (\Sigma X P(x))^2 = \frac{212}{3} - 64 = \frac{20}{3}$

22. Of the students in a school, it is known that 30% have 100% attendance and 70% students are irregular. Previous year results report that 70% of all students who have 100% attendance attain A grade and 10% irregular students attain A grade in their annual examination. At the end of the year, one student is chosen at random from the school and he was found to have an A grade. What is the probability that the student has 100% attendance? Is regularity required only in school? Justify your answer.

Solution:

Let E_1 be students having 100% attendance

E_2 be students having irregular attendance

E be students having A grade

$$P(E_1) = \frac{30}{100} \qquad P(E_2) = \frac{70}{100}$$

$$P\left(\frac{E}{E_1}\right) = \frac{70}{100} \times \frac{30}{100} = 21\%$$

$$P\left(\frac{E}{E_2}\right) = \frac{10}{100} \times \frac{70}{100} = 7\%$$

By Baye's theorem,

$$\text{So, } P\left(\frac{E_1}{E}\right) = \frac{P(E_1)P\left(\frac{E}{E_1}\right)}{P(E_1)P\left(\frac{E}{E_2}\right) + P(E_2)P\left(\frac{E}{E_2}\right)} = \frac{\frac{30}{100} \times \frac{21}{100}}{\frac{30}{100} \times \frac{21}{100} + \frac{70}{100} \times \frac{7}{100}} = \frac{63}{63+49} = \frac{63}{112}$$

23. Maximize Z = x + 2y

Subject to the constraints

$x + 2y \geq 100$

$2x - y \leq 0$

$2x + y \leq 200$

$x, y \geq 0$

Solve the above LPP graphically.

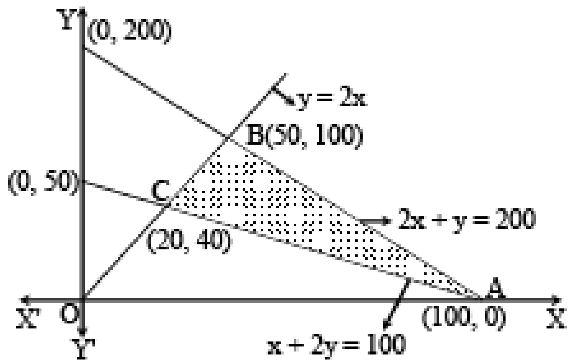
Solution:

$x + 2y = 100$

$2x - y = 0$ ……(1)

$2x + y = 200$ ……(2)

$x = 0, y = 0$ ……(3)



Corner points are A (100, 0), B(50, 100), C(20, 40)

Corner points	Z = x + 2y	
A(100, 0)	100	← minimum
B(50, 100)	250	← maximum
C(20, 40)	100	← minimum

Maximum at point B and maximum value 250

SECTION – D

Question numbers 24 to 29 carry 6 marks each

24. Determine the product $\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$ and use it to solve the system of equations

x – y + z = 4, x – 2y – 2z = 9, 2x + y +3z = 1.

Solution:

Product of the matrices

$$\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -4+4+8 & 4-8+4 & -4-8+12 \\ -7+1+6 & 7-2+3 & -7-2+9 \\ 5-3-2 & -5+6-1 & 5+6-3 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix} = 8I_3$$

Hence $\begin{bmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{bmatrix}^{-1} = \frac{1}{8}\begin{bmatrix} -4 & 4 & 4 \\ -7 & 1 & 3 \\ 5 & -3 & -1 \end{bmatrix}$

$$\begin{bmatrix}1 & -1 & 1\\ 1 & -2 & -2\\ 2 & 1 & 3\end{bmatrix}^{-1}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}4\\ 9\\ 1\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}1 & -1 & 1\\ 1 & -2 & -2\\ 2 & 1 & 3\end{bmatrix}^{-1}=\begin{bmatrix}4\\ 9\\ 1\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{8}\begin{bmatrix}-4 & 4 & 4\\ -7 & 1 & 3\\ 5 & -3 & -1\end{bmatrix}=\begin{bmatrix}4\\ 9\\ 1\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{8}\begin{bmatrix}-16 & +36 & +4\\ -28 & +9 & +3\\ 20 & -27 & -1\end{bmatrix}$$

$$\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{8}\begin{bmatrix}24\\ -16\\ -8\end{bmatrix}$$

$$x=\frac{24}{8},\ y=\frac{-16}{8},\ z=\frac{-8}{8}$$

x = 3, y = –2, z = –1

25. Consider $f:R-\left\{-\frac{4}{3}\right\}\rightarrow R-\left\{\frac{4}{3}\right\}$ given by f (x) $=\frac{4x+3}{3x+4}$. Show that f is bijective. Find the inverse of *f* and hence find $f^{-1}(0)$ and x such that $f^{-1}(x) = 2$.

OR

Let A = Q × Q and let * be a binary operation on A defined by (a, b) * (c, d) = (ac, b + ad) for (a, b), (c, d) ∈ A. Determine, whether * is commutative and associative. Then, with respect to * on A

(i) Find the identity element in A.

(ii) Find the invertible elements of A.

Solution:

$$f(x)=\frac{4x+3}{3x+4},\ x\in R-\left\{-\frac{4}{3}\right\}$$

F is one – one $\rightarrow$

Let $x_1, x_2 \in R-\left\{-\frac{4}{3}\right\}$ and $f(x_1) = f(x_2)$

$$\Rightarrow \frac{4x_1+3}{3x_1+4}=\frac{4x_2+3}{3x_2+4}$$

$$\Rightarrow 12x_1x_2+16x_1+9x_2+12=12x_1x_2+9x_1+16x_2+12$$

$$\Rightarrow 7x_1=7x_2 \Rightarrow x_1=x_2$$

$\therefore$ f is one – one

F is onto $\rightarrow$

Let k $\in R-\left\{\frac{4}{3}\right\}$ be any number

$f(x)=k \Rightarrow \frac{4x+3}{3x+4}$

$\Rightarrow 4x+3=3kx+4k$

$\Rightarrow x=\frac{4k-3}{4-3k}$

Also $\frac{4k-3}{4-3k}=-\frac{4}{3}$

implies – 9 = – 16 (which is impossible)

$\therefore f\left(\frac{4k-3}{4-3k}\right)$ = k i.e. f is onto

$\therefore$ The function f is invertible i.e. f^{-1} exist inverse of f

Let f^{-1} (x) = k

f (k) = x

$\Rightarrow \frac{4k+3}{3k+4}=x$

$\Rightarrow k=\frac{4x-3}{4-3x}$

$\therefore f^{-1}(x)=\frac{4x+3}{4-3x},\ \ x\in R-\left\{-\frac{4}{3}\right\}$

$f^{-1}(0)=-\frac{3}{4}$

and when

f^{-1} (x) = 2

$\Rightarrow \frac{4x-3}{4-3x}=2$

$\Rightarrow 4x-3=8-6x$

$\Rightarrow 10x=11$

$\Rightarrow x=\frac{11}{10}$

OR

(i) Let (e, f) be the identify element for *

$\therefore$ for (a, b) $\in$ Q × Q, we have

(a, b) * (e, f) = (a, b) = (e, f) * (a, b)

$\Rightarrow$ (ae, af + b) = (a, b) = (ea, eb + f)

$\Rightarrow$ ae = a, af + b = b, a = ea, b = eb + f

$\Rightarrow$ e = 1, af = 0, e = 1, b = (1) b + f

($\because$ a need not be '0')

$\Rightarrow$ e = 1, f = 0, e = 1, f = 0

$\therefore$ (e, f) = (1, 0) $\in$ Q × Q

$\therefore$ (1, 0) is the identify element of A

(ii) Let (a, b) $\in$ Q × Q
Let (c, d) $\in$ Q × Q
such that
(a, b) * (c, d) = (1, 0) = (c, d) * (a, b)
$\Rightarrow$ (ac, ad + b) = (1, 0) = (ca, cb + d)
$\Rightarrow$ ac = 1, ad + b = 0, ca = 1, cb + d = 0

$$\Rightarrow c = \frac{1}{a}, d = -\frac{b}{a}, \left(\frac{1}{a}\right)b + d = 0 (a \neq 0)$$

$$\therefore (c, d) = \left(\frac{1}{a}, \frac{-b}{a}\right)(a \neq 0)$$

$$\therefore \text{for } a \neq 0, (a, b)^{-1} = \left(\frac{1}{a}, \frac{-b}{a}\right)$$

26. Show that the surface area of a closed cuboid with square base and given volume is minimum, when it is a cube.

Solution: If each side of square base is x and height is h then volume

$V = x^2h \Rightarrow h = \frac{V}{x^2}$

S is surface area then

$$S = 4hx + 2x^2 = 4\left(\frac{V}{x^2}\right)x + 2x^2$$

$$\Rightarrow S = \frac{4V}{x} + 2x^2$$

Diff. w. r. to x

$$\frac{dS}{dx} = -\frac{4V}{x^2} + 4x \text{ and } \frac{d^2S}{dx^2} = +\frac{8V}{x^3} + 4$$

Now $\frac{dS}{dx} = 0 \Rightarrow 4x = \frac{4V}{x^2}$

$$\Rightarrow x^3 = V \Rightarrow x = V^{1/3}$$

at x = $V^{1/3}$, $\frac{d2S}{dx^2} > 0$

$\Rightarrow$ S is minimum when x = $V^{1/3}$

and $h = \frac{V}{x^2} = \frac{V}{V^{2/3}} = V^{1/3} \Rightarrow x = h$

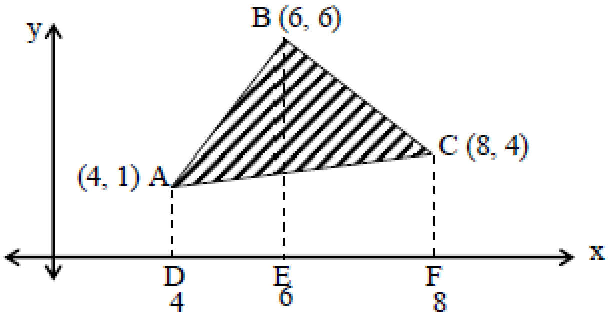
$\Rightarrow x = h$ means it is a cube

27. Using the method of integration, find the area of the triangle ABC, coordinates of whose vertices are A (4, 1), B (6, 6) and C (8, 4).

OR

Find the area enclosed between the parabola $4y = 3x^2$ and the straight line $3x - 2y + 12 = 0$.

Solution:



Equation of AB is y – 1 = $\frac{6-1}{6-4}(x-4)$

$\Rightarrow$ 2y – 2 = 5x – 20

$\Rightarrow y = \frac{5x}{2} - 9$

Equation of BC is

$\Rightarrow y - 6 = \frac{4-6}{8-6}(x-6)$

$\Rightarrow$ y = – x + 12

Equation of AC is

$\Rightarrow y - 1 = \frac{4-1}{8-4}(x-4)$

$\Rightarrow$ 4y – 4 = 3x – 12

$\Rightarrow y = \frac{3x}{4} - 2$

Area of ΔABC = area ABED + area BEFC – area ADFC

$$= \int_4^6 \left(\frac{5x}{2} - 9\right) dx + \int_6^8 (-x + 12)\, dx - \int_4^6 \left(\frac{3x}{4} - 2\right) dx$$

$$= \left| \left(\frac{5x^2}{4} - 9x\right)_4^6 \right| + \left| \left(\frac{-x^2}{2} + 12x\right)_6^8 \right| - \left| \left(\frac{3x^2}{8} - 2x\right)_4^6 \right| = 7 \text{ sq units}$$

OR

Parabola 4y = 3x^2 …(1)

line 3x – 2y + 12 = 0 …(2)

from (2) $y = \frac{3x+12}{2}$

putting this value of y is (1) we get

6x + 24 = 3x^2

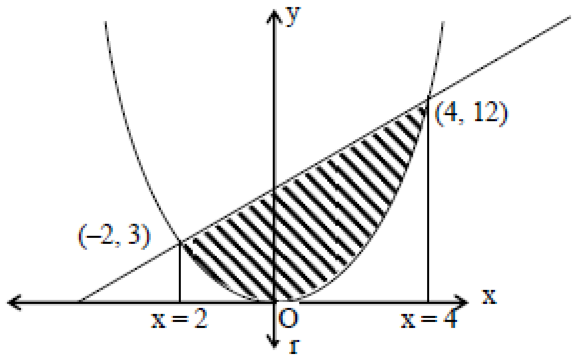
$\Rightarrow$ x = 4, – 2

when x = 4 then y = 12

x = – 2 then y = 3

Required area

$$= \int_{-2}^{4} (y \text{ of line})\, dx - \int_{-2}^{4} (y \text{ of parabola})\, dx$$



$$= \int_{-2}^{4} \left(\frac{3x+12}{2} - \frac{3x^2}{4} \right) dx$$

$$= \frac{3}{4} \int_{-2}^{4} (8 + 2x - x^2)\, dx$$

$$= \frac{3}{4} \left[8x + x^2 - \frac{x^3}{3} \right]_{-2}^{4} = 27 \text{ sq. units}$$

28. Find the particular solution of the differential equation $(x - y)\dfrac{dy}{dx} = (x + 2y)$, given that y = 0 when x = 1.

Solution:

$$(x - y)\frac{dy}{dx} = (x + 2y)$$

$$\frac{dy}{dx} = \frac{x + 2y}{x - y}$$

Let y = Vx

$$\frac{dy}{dx} = V + x\frac{dV}{dx}$$

$$\Rightarrow V + x\frac{dV}{dx} = \frac{x + 2(Vx)}{x - Vx}$$

$$\Rightarrow V + x\frac{dV}{dx} = \frac{1 + 2V}{1 - V}$$

$$\Rightarrow x\frac{dV}{dx} = \frac{1 + 2V - V + V^2}{1 - V}$$

$$\Rightarrow \int \frac{1 - V}{1 + V + V^2} dV = \int \frac{dx}{x}$$

$$\Rightarrow -\frac{1}{2}\int \left\{ \frac{(2V + 1) - 3}{1 + V + V^2} \right\} dV = \int \frac{dx}{x}$$

$$\Rightarrow -\frac{1}{2}\left[\int \frac{2V + 1}{1 + V + V^2} dV - 3\int \frac{dV}{1 + V + V^2} \right] = \int \frac{dx}{x}$$

$$\Rightarrow -\frac{1}{2}\log|1+V+V^2|+\frac{3}{2}\int\frac{dV}{\left(V+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\log|x|+C$$

$$\Rightarrow -\frac{1}{2}\log|1+V+V^2|+\frac{3}{2}\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{V+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)=\log|x|+C$$

$$\Rightarrow -\frac{1}{2}\log\left|1+\frac{y}{x}+\frac{y^2}{x^2}\right|+\sqrt{3}\tan^{-1}\left(\frac{2\frac{y}{x}+1}{\sqrt{3}}\right)=\log|x|+C$$

we have y = 0 when x = 1

$$\Rightarrow 0+\sqrt{3}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)=0+C$$

$$\Rightarrow C=\sqrt{3}\tan^{-1}\frac{1}{\sqrt{3}}$$

∴ Solution

$$\Rightarrow -\frac{1}{2}\log\left|1+\frac{y}{x}+\frac{y^2}{x^2}\right|+\sqrt{3}\tan^{-1}\left(\frac{2\frac{y}{x}+1}{\sqrt{3}}\right)=\log|x|+\sqrt{3}\tan^{-1}\frac{1}{\sqrt{3}}$$

29. Find the coordinates of the point where the line through the points (3, –4, –5) and (2, –3, 1), crosses the plane determined by the points (1, 2, 3), (4, 2, –3) and (0, 4, 3).

OR

A variable plane which remains at a constant distance 3p from the origin cuts the coordinate axes at A, B, C. Show that the locus of the centroid of triangle ABC is $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2}$.

Solution:

Equation of line passing through

(3, – 4, – 5) and (2, – 3, 1)

$$\frac{x-3}{-1}=\frac{y+4}{1}=\frac{z+5}{6} \qquad ...(1)$$

Equation of plane passing through

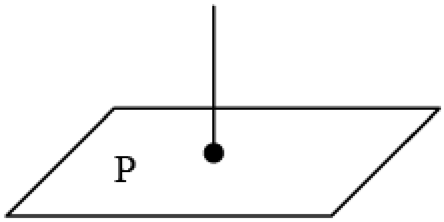
(1, 2, 3) (4, 2, – 3) and (0, 4, 3)

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 3 & 0 & -6 \\ -1 & 2 & 0 \end{vmatrix}=0$$

⇒ (x – 1) (12) – (y – 2) (– 6) + (z – 3) (6) = 0

⇒ 2x + y + z – 7 = 0 …(2)

Let any point on line (1)



is P (– k + 3, k – 4, 6k – 5)

it lies on plane

$\therefore$ 2(– k + 3) + k – 4 + 6k – 5 – 7 = 0

5k = 10

$\Rightarrow$ k = 2

$\therefore$ P (1, –2, 7)

OR

Let the equation of plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \quad ...(1)$$

It cut the co-ordinate axes at A, B and C

$\therefore$ A (a, 0, 0), B (0, b, 0), C (0, 0, c)

Let the centroid of ΔABC be (x, y, z)

$$\therefore \left(x=\frac{a}{3}, y=\frac{b}{3}, z=\frac{c}{3}\right) \quad ...(2)$$

given that distance of plane (1) from origin is 3p

$$\therefore \left|\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}\right|=3p$$

$$\Rightarrow \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{9p^2}$$

from (2)

$$\Rightarrow \frac{1}{9x^2}+\frac{1}{9y^2}+\frac{1}{9z^2}=\frac{1}{9p^2}$$

$$\Rightarrow \frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{p^2} \text{ Proved}$$

CBSE-XII-2015 EXAMINATION

SET-1

MATHEMATICS

Series SSO/1 Paper & Solution **Code: 65/1/1/D**

Time: 3 Hrs. Max. Marks: 100

General Instructions:

*(i) **All** questions are compulsory.*
*(ii) Please check that this Question Paper contains **26** Questions.*
(iii) Marks for each question are indicated against it.
*(iv) Questions **1** to **6** in Section-**A** are Very Short Answer Type Questions carrying **one** mark each.*
*(v) Questions **7** to **19** in Section-**B** are Long Answer **I** Type Questions carrying **4** marks each.*
*(vi) Questions **20** to **26** in Section-**C** are Long Answer **II** Type Questions carrying **6** marks each*
(vii) Please write down the serial number of the Question before attempting it.

SECTION – A

Question numbers **1** to **6** carry **1** mark each.

1. If $\vec{a} = 7\hat{i} + \hat{j} - 4\hat{k}$ and $\vec{b} = 2\hat{i} + 6\hat{j} + 3\hat{k}$, then find the projection of $\vec{a}$ on $\vec{b}$.

Solution:

$$p = \frac{\vec{a}.\vec{b}}{|\vec{b}|} = \frac{8}{7}$$

2. Find λ, if the vectors $\vec{a} = \hat{i} + 3\hat{j} + \hat{k}, \vec{b} = 2\hat{i} - \hat{j} - \hat{k}$ and $\vec{c} = \lambda\hat{j} + 3\hat{k}$ are coplanar.

Solution:

$$\begin{vmatrix} 1 & 3 & 1 \\ 2 & -1 & -1 \\ 0 & \lambda & 3 \end{vmatrix} = 0 \Rightarrow \lambda = 7$$

3. If a line makes angles $90^0, 60^0$ and θ with x, y and z- axis respectively, where θ is Acute, then find θ.

Solution:

$$\cos^2 \pi/2 + \cos^2 \pi/3 + \cos^2\theta = 1 \Rightarrow \theta = \frac{\pi}{6}$$

4. Write the element a_{23} of a 3 × 3 matrix A = (a_{ij}) whose elements a_{ij} are given $a_{ij} = \frac{|i-j|}{2}$.

Solution:

$$a_{23} = \frac{|2-3|}{2} = \frac{1}{2}$$

5. Find the differential equation representing the family of curves $v = \frac{A}{r} + B$, where A and B are arbitrary constants.

Solution:

$$\frac{dv}{dr} = -\frac{A}{r^2}, \Rightarrow r^2\frac{d^2v}{dr^2} + 2r\frac{dv}{dr} = 0$$

6. Find the integrating factor of the differential equation

$$\left(\frac{e^{-2}\sqrt{x}}{\sqrt{x}}-\frac{y}{\sqrt{x}}\right)\frac{dx}{dy}=1.$$

Solution:

$$I.F=\int_e \frac{1}{\sqrt{x}}dx=e^{2\sqrt{x}}$$

SECTION – B

Question numbers **7** to **19** carry **4** marks each.

7. If $A=\begin{pmatrix}2 & 0 & 1\\ 2 & 1 & 3\\ 1 & -1 & 0\end{pmatrix}$ find $A^2-5A+4I$ and hence find a matrix X such that $A^2-5A+4I+X=O$

OR

If $A=\begin{bmatrix}1 & -2 & 3\\ 0 & -1 & 4\\ -2 & 2 & 1\end{bmatrix}$, find $(A')^{-1}$.

Solution:

Getting $A^2=\begin{pmatrix}5 & -1 & 2\\ 9 & -2 & 5\\ 0 & -1 & -2\end{pmatrix}$

$$A^2-5A+4I=\begin{pmatrix}5 & -1 & 2\\ 9 & -2 & 5\\ 0 & -1 & -2\end{pmatrix}+\begin{pmatrix}-10 & 0 & -5\\ -10 & -5 & -15\\ -5 & 5 & 0\end{pmatrix}+\begin{pmatrix}4 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 4\end{pmatrix}$$

$$=\begin{pmatrix}-1 & -1 & -3\\ -1 & -3 & -10\\ -5 & 4 & 2\end{pmatrix}$$

$$\therefore X=\begin{pmatrix}1 & 1 & 3\\ 1 & 3 & 10\\ 5 & -4 & -2\end{pmatrix}$$

OR

$$A'=\begin{pmatrix}1 & 0 & -2\\ -2 & -1 & 2\\ 3 & 4 & 1\end{pmatrix}$$

$$|A'|=1(-9)-2(-5)=-9+10=1\neq 0$$

$$\text{Adj } A'=\begin{pmatrix}-9 & -8 & -2\\ 8 & 7 & 2\\ -5 & -4 & -1\end{pmatrix}$$

$$\therefore (A')^{-1} = \begin{pmatrix} -9 & -8 & -2 \\ 8 & 7 & 2 \\ -5 & -4 & -1 \end{pmatrix}$$

8. if f(x) $= \begin{bmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{bmatrix}$, using properties of determinants find the value of F(2x) - f(x).

Solution:

$$f(x) = \begin{vmatrix} a & -1 & 0 \\ ax & a & -1 \\ ax^2 & ax & a \end{vmatrix}$$

$R_2 \to R_2 - xR_1$ *and* $R_3 \to R_3 - x^2R^1$

$$f(x) = \begin{vmatrix} a & -1 & 0 \\ 0 & a+x & -1 \\ 0 & ax+x^2 & a \end{vmatrix}$$ (For bringing 2 zeroes in any row/column

$$\therefore f(x) = a(a^2 + 2ax + x^2) = a(x+a)^2$$

$$\therefore f(2x) - f(x) = a[2x+a]^2 - a(x+a)^2$$

$$= a\,x\,(3x+2a)$$

9. Find : $\int \frac{dx}{\sin x + \sin 2x}$

OR

Integrate the following w. r. t. x

$$\frac{x^2 - 3x + 1}{\sqrt{1-x^2}}$$

Solution:

$$\int \frac{dx}{\sin x + \sin 2x} = \int \frac{dx}{\sin x\,(1 + 2\cos x)} = \int \frac{\sin x \cdot dx}{(1-\cos x)(1+\cos x)(1+2\cos x)}$$

$$= -\int \frac{dt}{(1-t)(1+t)(1+2t)}$$ where cos x = t

$$= \int \left(\frac{-1/6}{1-t} + \frac{1/2}{1+t} - \frac{4/3}{1+2t} \right) dt$$

$$= +\frac{1}{6}\log|1-t| + \frac{1}{2}\log|1+t| - \frac{2}{3}\log|1+2\cos x| + c$$

OR

$$\int \frac{x^2 - 3x + 1}{\sqrt{1-x^2}}\,dx = \int \frac{2 - 3x - (1-x^2)}{\sqrt{1-x^2}}\,dx$$

$$= 2\int \frac{1}{\sqrt{1-x^2}}\,dx - 3\int \frac{x}{\sqrt{1-x^2}}\,dx - \int \sqrt{1-x^2}\,dx$$

$$= 2\sin^{-1} x + 3\sqrt{1-x^2} - \frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\sin^{-1} x + c$$

or $= \frac{3}{2}\sin^{-1} x + \frac{1}{2}(6-x)\sqrt{1-x^2} + c$

10. Evaluate : $\int_{-\pi}^{\pi} (\cos ax - \sin bx)^2\,dx$

Solution:

$$I = \int_{-\pi}^{\pi} \left(\cos ax - \sin bx\right)^2\,dx = \int_{-\pi}^{\pi} \left(\cos^2 ax + \sin^2 bx\right)dx - \int_{-\pi}^{\pi} 2\cos ax \sin bx\,dx$$

$$= I_1 - I_2$$

$I_1 = 2\int_0^{\pi} \left(\cos^2 ax + \sin^2 bx\right)dx$ (being an even fun.)

$I_2 = 0$ (being an odd fun.)

$$\therefore I = I_1 = \int_0^{\pi} \left(1 + \cos 2ax + 1 - \cos 2bx\right)dx$$

$$= \left[2x + \frac{\sin 2ax}{2a} - \frac{\sin 2bx}{2b}\right]_0^{\pi}$$

$$= \left[2\pi + \frac{1}{2a}\cdot \sin 2a\pi - \frac{\sin 2b\pi}{2b}\right] \text{ or } 2\pi$$

11. A bag A contains 4 black and 6 red balls and bag B contains 7 black and 3 red balls. A die is thrown. If 1 or 2 appears on it, then bag A is chosen, otherwise bag b. If two balls are drawn at random (without replacement) from the select bag, find the probability of one of them being red and another black.

OR

An unbiased coin is tossed 4 times. Find the mean and variance of the number of heads obtained.

Solution:

. Let E_1 : selecting bag A, and E_2 : selecting bag B.

$$\therefore P(E_1) = 1/3,\ P(E_2) = 2/3$$

Let A : Getting one Red and one balck ball

$$\therefore P\left(A/E_1\right) = \frac{{}^4C_1 \cdot {}^6C_1}{{}^{10}C_2} = \frac{8}{15},\ P\left(A/E_2\right) = \frac{{}^7C_1 \cdot {}^3C_1}{{}^{10}C_2} = \frac{7}{15}$$

$$P(A) = P(E_1)\cdot P(A/E_1) + P(E_2)\cdot P\left(A/E_2\right)$$

$$= \frac{1}{3}\cdot\frac{8}{15} + \frac{2}{3}\cdot\frac{7}{15} = \frac{22}{45}$$

OR

x	:	0	1	2	3	4
P (x)	:	${}^4C_0\left(\frac{1}{2}\right)^4$	${}^4C_1\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)$	${}^4C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2$	${}^4C_3\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3$	${}^4C_4\left(\frac{1}{2}\right)^4$
	:	$=\frac{1}{16}$	$=\frac{4}{16}$	$=\frac{6}{16}$	$=\frac{4}{16}$	$=\frac{1}{16}$
x P (x)	:	0	$\frac{4}{16}$	$\frac{12}{16}$	$\frac{12}{16}$	$\frac{4}{16}$
x^2P (x)	:	0	$\frac{4}{16}$	$\frac{24}{16}$	$\frac{36}{16}$	$\frac{16}{16}$

$$Mean = \sum x\,P(x) = \frac{32}{16} = 2$$

$$\text{Variance } = \sum x^2\,P(x) - \left(\sum xP(x)\right)^2 = \frac{80}{16} - (2)^2 = 1$$

12. If $\vec{r} = x\hat{i} + y\hat{j} + z k$, find $(\vec{r}\times\hat{i})\ .\ (\vec{r}\times j) + xy$

Solution:

$$\vec{r}\times\vec{i} = \left(x\hat{i} + y\hat{j} + z\hat{k}\right)x\hat{i} = -y\hat{k} + z\hat{j}$$

$$\vec{r}\times\vec{j} = \left(x\hat{i} + y\hat{j} + z\hat{k}\right)\hat{j} = x\hat{k} - z\hat{i}$$

$$\left(\vec{r}\times\hat{i}\right), \left(\vec{r}\times\vec{j}\right) = \left(o\hat{i} + z\hat{j} - y\hat{k}\right)\cdot\left(-z\hat{i} + o\hat{j} + x\hat{k}\right) = -xy$$

$$\left(\vec{r}\times\hat{i}\right)\cdot\left(\vec{r}\times\vec{j}\right) + xy = -xy + xy + 0$$

13. Find the distance between the point (-1 , -5, -10) and the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane x – y + z = 5.

Solution:

. Any point on the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ *is* $(3\lambda + 2, 4\lambda - 1, 12\lambda + 2)$

If this is the point of intersection with plane x – y + z = 5

Then $3\lambda + 2 - 4\lambda + 1 + 12\lambda + 2 - 5 = 0 \Rightarrow \lambda = 0$

$\therefore$ Point of intersection is (2, –1, 2)

Required distance $= \sqrt{(2+1)^2 + (-1+5)^2 + (2+10)}^2 = 13$

14. If $\sin\left[\cot^{-1}(x+1)\right] = \cos\ (\tan^{-1}x)$, then find x.

OR

If $(\tan^{-1}x)^2 + (\cot^{-1}x)^2 = \frac{5\pi^2}{8}$, then find x.

Solution:

Writing $\cot^{-1}(x+1)=\sin^{-1}\frac{1}{\sqrt{1+(x+1)^2}}$

and $\tan^{-1}x=\cos^{-1}\frac{1}{\sqrt{1+x^2}}$

$$\therefore \sin\left(\sin^{-1}\frac{1}{\sqrt{1+(x+1)^2}}\right)=\cos\left(\cos^{-1}\frac{1}{\sqrt{1+x^2}}\right)$$

$$1+x^2+2x+1=1+x^2 \Rightarrow x=-\frac{1}{2}$$

OR

$$\left(\tan^{-1}x\right)^2+\left(\cot^{-1}x\right)^2=\frac{5\pi^2}{8}\Rightarrow\left(\tan^{-1}x\right)^2+\left(\frac{\pi}{2}-\tan^{-1}x\right)^2=\frac{5\pi^2}{8}$$

$$\therefore 2\left(\tan^{-1}x\right)^2-\pi\tan^{-1}x-\frac{3\pi^2}{8}=0$$

$$\tan^{-1}x=\frac{\pi\pm\sqrt{\pi^2+3\pi^2}}{4}=3\pi/4,\ -\pi/4$$

$$\Rightarrow x=-1$$

15. If $y=\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right), x^2\le 1$, then find $\frac{dy}{dx}$.

Solution:

Putting $x^2=\cos\theta$, *we get*

$$y=\tan^{-1}\left(\frac{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}\right)$$

$$=\tan^{-1}\left(\frac{\cos\theta/2+\sin\theta/2}{\cos\theta/2-\sin\theta/2}\right)=\tan^{-1}\left(\frac{1+\tan\theta/2}{1-\tan\theta/2}\right)$$

$$y=\frac{\pi}{4}+\theta/2=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$$

$$\frac{dy}{dx}=-\frac{1}{2}\frac{1}{\sqrt{1-x^4}}\cdot 2x=-\frac{x}{\sqrt{1-x^4}}$$

16. If $x = a\cos\theta + b\sin\theta$, $y = a\sin\theta - b\cos\theta$, show that $y^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=0$.

Solution:

$$\frac{dx}{d\theta}=-a\sin\theta+b\cos\theta$$

$$\frac{dy}{d\theta}=a\cos\theta+b\sin\theta$$

$$\therefore \frac{dy}{dx}=\frac{a\cos\theta+b\sin\theta}{a\sin\theta+b\cos\theta}=-\frac{x}{y}$$

Or $y\frac{dy}{dx}+x=0$

$$\therefore y\frac{d^2y}{dx^2}+\frac{dy}{dx}\cdot\frac{dy}{dx}+1=0$$

17. The side of an equilateral triangle is increasing at the rate of 2 cm/s. At what rate is its area increasing when the side of the triangle is cm ?

Solution:

Let x be the side of an equilateral triangle

$$\therefore \frac{dx}{dt}=2cm/s.$$

Area (A) $=\frac{\sqrt{3x^2}}{4}$

$$\Rightarrow \frac{dA}{dt}=\frac{\sqrt{3}}{2}x\frac{dx}{dt}$$

$$\Rightarrow \frac{dA}{dt}=\frac{\sqrt{3}}{2}\cdot(20)\cdot(2)=20\sqrt{3}\,cm^2/_s$$

18. Find : $\int(x+3)\sqrt{3-4x-x^2}dx.$

Solution:

Writing $x+3=-\frac{1}{2}(-4-2x)+1$

$$\therefore \int(x+3)\sqrt{3-4x-x^2}\,dx=-\frac{1}{2}\int(-4-2x)\sqrt{3-4x-x^2}\,dx+\int\sqrt{7-(x+2)^2}\,dx$$

$$=-\frac{1}{3}(3-4x-x^2)^{3/2}+\frac{x+2}{2}\sqrt{3-4x-x^2+}\frac{7}{2}\sin^{-1}\left(\frac{x+2}{\sqrt{7}}\right)+c$$

19. Three schools A, B and C organized a mela for collecting found for helping the rehabilitation of flood victims. They sold hand made fans, mats and plates from recycled material at a cost of Rs. 25 , Rs. 100, Rs. 50, each. The number of articles sold are given below :

School / Article	A	B	C
Hand-fans	40	25	35
Mats	50	40	50
Plates	20	30	40

Find the found collected by each school separately by selling the above articles, Also Find the total founds collected for the purpose. Write one value generated by the above situation.

Solution:

$$\begin{matrix} & HF. & M & P \\ A & 40 & 50 & 20 \\ B & 25 & 40 & 30 \\ C & 35 & 50 & 40 \end{matrix}\begin{pmatrix}25\\100\\50\end{pmatrix}=\begin{pmatrix}7000\\6125\\7875\end{pmatrix}$$

Funds collected by school A : Rs. 7000,
School B : Rs. 6125, School C : Rs. 7875
Total collected : Rs. 21000
For writing one value

SECTION – C

Question numbers **20** to **26** carry **6** marks each.

20. Let N denote the set of all natural numbers and R be the relation on N × N defined by (a, b) R (c, d) if ad (b + c) = bc (a + d). show that R is an equivalence relation.

Solution:

$\forall\, a, b \in N, (a,b)R(a,b)\ as\ ab\,(b+a) = ba(a+b)$

$\therefore$ R is reflexive .... (i)

Let (a, b) R (c, d) for (a, b), (c, d) $\in N \times N$

$\therefore$ ad (b + c) = bc (a + d) .... (ii)

Also (c, d) R (a, b) $\because$ cb (d + a) = da (c + b) (using ii)

$\therefore$ R is symmetric .... (iii)

Let (a, b) R (c, d) and (c, d) R (e, f), for a, b, c, d, e, f $\in N$

$\therefore$ ad (b + c) = bc (a + d) and cf (d + e) = de (c + f)

$\therefore \frac{b+c}{bc} = \frac{a+d}{ad}\ and\ \frac{d+e}{de} = \frac{c+f}{cf}$

i.e $\frac{1}{c}+\frac{1}{b}=\frac{1}{d}+\frac{1}{a}\ and\ \frac{1}{e}+\frac{1}{d}=\frac{1}{f}+\frac{1}{c}$

adding we get $\frac{1}{c}+\frac{1}{b}+\frac{1}{e}+\frac{1}{d}=\frac{1}{d}+\frac{1}{a}+\frac{1}{f}+\frac{1}{c}$

$\Rightarrow$ af (b + e) = be (a + f)

Hence (a, b) R (e, f) $\therefore$ R is transitive .... (iv)

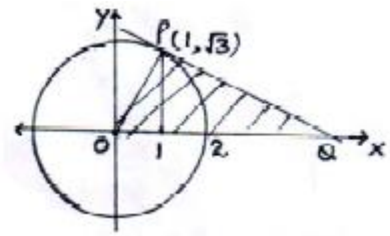
Form (i), (iii) and (iv) R is an equivalence relation

21. Using integration find the area of the triangle formed by positive x- zxis and tangent and normal to the circle $x^2 + y^2 = 4$ at $(1, \sqrt{3})$.

OR

Evaluate $\int_1^3 (e^{2-3x} + x^2 + 1)$ as a limit of a sum.

Solution:



Eqn. of normal (OP) : y = $\sqrt{3}x$

Eqn. of tangent (PQ) is

$$y-\sqrt{3}=-\frac{1}{\sqrt{3}}(x-1)\ i.e\ y=\frac{1}{\sqrt{3}}(4-x)$$

Coordinates of Q (4, 0)

$$\therefore \text{ Req. area } =\int_0^1 \sqrt{3}x\,dx+\int_1^4 \frac{1}{\sqrt{3}}(4-x)\,dx$$

$$=\sqrt{3}\left.\frac{x^2}{2}\right]_0^1+\frac{1}{\sqrt{3}}\left[4x-\frac{x^2}{2}\right]_1^4$$

$$=\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{3}}\left[16-8-4+\frac{1}{2}\right]=2\sqrt{3}\text{ sq. units}$$

OR

$$\int_1^3\left(e^{2-3x}+x^2+1\right)dx\ here\ h=\frac{2}{n}$$

$$=\lim_{h\to 0}h\left[f(1)+f(1+h)+f(1+2h)+\dots\dots+f(1+(n-1)h)\right]$$

$$=\lim_{h\to 0}h\left[\left(e^{-1}+2\right)+\left(e^{-1-3h}+2+2h+h^2\right)+\left(e^{-1-6h}+2+4h+4h^2\right)+\dots\dots\right.$$
$$\left.+\left(e^{-1-3(n-1)h}+2+2(n-1)h+(n-1)^2h^2\right)\right]$$

$$=\lim_{h\to 0}h\left[e^{-1}\left(1+e^{-3h}+e^{-6h}+\dots..+e^{-3(n-1)h}\right)+2n+2h\left(1+2+\dots.+(n-1)^2\right)\right]$$

$$=\lim_{h\to 0}h\left(e^{-1}\cdot\frac{e^{-3nh}-1}{e^{-3n}-1}\cdot h+2nh+2\frac{nh(nh-h)}{2}+\frac{nh(nh-h)(2nh-h)}{6}\right)$$

$$=e^{-1}\frac{\left(e^{-6}-1\right)}{-3}+4+4+\frac{8}{3}=-e^{-1}\frac{e^{-6}-1}{3}+\frac{32}{3}$$

22. Solve the differential equation : $(\tan^{-1}y-x)dy=(1+y^2)dx.$

OR

Find the particular solution of the differential equation $\frac{dy}{dx}=\frac{xy}{x^2+y^2}$ given that y = 1, When x = 0 .

Solution:

Given differential equation can be written as

$$\frac{dx}{dy}+\frac{1}{1+y^2}\cdot x=\frac{\tan^{-1}y}{1+y^2}$$

$\therefore$ Integrating factor is $e^{\tan^{-1}y}$

$\therefore$ Solution is : $x \cdot e^{\tan^{-1}y} = \int \frac{\tan^{-1} y \cdot e^{\tan^{-1}y}}{1+y^2} dy$

$\Rightarrow x \cdot e^{\tan^{-1}y} = \int t\, e^t dt \text{ where } \tan^{-1} y = t$

$= t\, e^t - e^t + c = e^{\tan^{-1}y}\left(\tan^{-1} y - 1\right) + c$

$or\ x = \tan^{-1} y - 1 + c\, e^{-\tan^{-1} y}$

OR

Given differential equation is $\frac{dy}{dx} = \frac{y/x}{1+\left(y/x\right)^2}$

Putting $\frac{y}{x} = v$ to get $v + x\frac{dv}{dx} = \frac{v}{1+v^2}$

$\therefore x\frac{dv}{dx} = \frac{v}{1+v^2} - v = \frac{|-v^3}{1+v^2}$

$\Rightarrow \int \frac{v^2+1}{v^3} dv = -\int \frac{dx}{x}$

$\Rightarrow \log|v| - \frac{1}{2v^2} = -\log|x| + c$

$\therefore \log y - \frac{x^2}{2y^2} = c$

$x = 0,\ y = 1 \Rightarrow c = 0 \therefore \log y - \frac{x^2}{2y^2} = 0$

23. If lines $\frac{\mathrm{x}-1}{2} = \frac{\mathrm{y}+1}{3} = \frac{\mathrm{z}-1}{4}$ and $\frac{\mathrm{x}-3}{1} = \frac{\mathrm{y}+\mathrm{k}}{2} = \frac{\mathrm{z}}{1}$ intersect , then find the value of K and hence find the equation of the plane containing these lines.

Solution:

. Any point on line $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ $is\ (2\lambda+1, 3\lambda-1, 4\lambda+1)$

$\therefore \frac{2\lambda+1-3}{1} = \frac{3\lambda-1-k}{2} = \frac{4\lambda+1}{1} \Rightarrow \lambda = -\frac{3}{2}$, hence k $= \frac{9}{2}$

Eqn. of plane containing three lines is

$$\begin{vmatrix} x-1 & y+1 & z-1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

$\Rightarrow -5(x-1) + 2(y+1) + 1(z-1) = 0$

i.e. $5x - 2y - z - 6 = 0$

24. If A and B are two independent events such that $P(\bar{A}\cap B)=\frac{2}{15}$ and $P(A\cap\bar{B})=\frac{1}{6}$, then find P (A) and P (B).

Solution:

$$P\left(\bar{A}\cap B\right)=\frac{2}{15}\Rightarrow P\left(\bar{A}\right)\cdot P\left(B\right)=\frac{2}{15}$$

$$P\left(A\cap\bar{B}\right)=\frac{1}{6}\Rightarrow P\left(\bar{A}\right)\cdot P\left(\bar{B}\right)=\frac{1}{6}$$

$$\therefore\left(1-P\left(A\right)\right)P\left(B\right)=\frac{2}{15}\ or\ P\left(B\right)-P\left(A\right)\cdot P\left(B\right)=\frac{2}{15}\ \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

$$P\left(A\right)\left(1-P\left(B\right)\right)=\frac{1}{6}\ or\ P\left(A\right)-P\left(A\right)\cdot P\left(B\right)=\frac{1}{6}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

From (i) and (ii) $P\left(A\right)-P\left(B\right)=\frac{1}{6}-\frac{2}{15}=\frac{1}{30}$

Let P (A) = x, P (B) = y $\therefore x=\left(\frac{1}{30}+y\right)$

(i) $\Rightarrow y\left|-\left(\frac{1}{30}+y\right)y=\frac{2}{15}\right.\therefore\ 30y^2-29y+4=0$

Solving to get $y={}^{1}\!/_{6}\ or\ y={}^{4}\!/_{5}$

$\therefore x={}^{1}\!/_{5}\ or\ x={}^{5}\!/_{6}$

Hence $P\left(A\right)={}^{1}\!/_{5},\ \ P\left(B={}^{1}\!/_{6}\right)$ OR $P\left(A\right)={}^{5}\!/_{6},\ \ P\left(B={}^{4}\!/_{5}\right)$

25. Find the local maxima and local minima, of the function $f(x)=\sin x,\ 0<x<2\pi.$ Also find the local maximum and local minimum values.

Solution:

$$f\left(x\right)=\sin x-\cos x,\ 0<x<2\pi$$

$$f'\left(x\right)=0\ \Rightarrow\cos x+\sin x=0\ or\ \tan x=-1,$$

$$\therefore x={}^{3\pi}\!/_{4},\ \frac{7\pi}{4}$$

$f''\left({}^{3\pi}\!/_{4}\right)=-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}$ i.e ve so, $x={}^{3\pi}\!/_{4}$ is LocalMaxima

and $f''\left({}^{7\pi}\!/_{4}\right)=-\frac{1}{\sqrt{2}}+\frac{1}{2}$ i.e ve so, $x={}^{7\pi}\!/_{4}$ is LocalMinima

Local Maximum value $=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\sqrt{2}$

Local Minimum value $=-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}=\sqrt{2}$

26. Find graphically, the maximum value of z = 2x + 5y, subject to constraints given below :

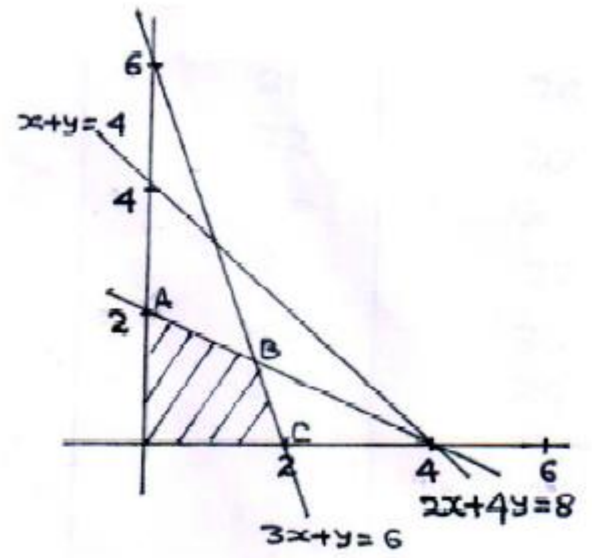
$2x + 4y \leq 8.$

$3x + y \leq 6$

$x + y \leq 4$

$x \geq 0,\ y \geq 0$

Solution:



Correct graphs of three lines
Correctly shading
feasible region
Vertices are
A (0, 2), B (1.6, 1.2), C (2, .0)
Z = 2x + 5y is maximum
at A (0, 2) and maximum value = 10

CBSE-XII-2014 EXAMINATION

MATHEMATICS

Paper & Solution

Code: 65/1

Time: 3 Hrs. Max. Marks: 100

General Instructions:

(i) **All** question are compulsory.

(ii) The question paper consists of **29** questions divided into three sections A, B and C. Section A comprises of **10** questions of **one mark** each, Section B comprises of **12** questions of **four marks** each and Section C comprises of **7** questions of **six marks** each.

(iii) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the question.

(iv) There is no overall choice. However, internal choice has been provided in 4 questions of four marks each and 2 questions of six marks each. You have to attempt only one of the alternatives in all such questions.

(v) Use of calculators is **not** permitted. You may ask for logarithmic tables, if required.

SECTION A

Question numbers 1 to 10 carry 1 mark each.

1. If $R=\{(x,y): x+2y=8\}$ is a relation on N, write the range of R.

Solution:

$R=\{(x,y): x+2y=8\}$ is a relation on N

Then we can say 2y = 8 – x

$$y=4-\frac{x}{2}$$

so we can put the value of x, x = 2, 4, 6 only

we get y = 3 at x = 2
we get y = 2 at x = 4
we get y = 1 at x = 6
so range = {1, 2, 3} Ans.

2. *If* $\tan^{-1}x+\tan^{-1}y=\frac{\pi}{4}, xy<1,$ then write the value of x + y + xy.

Solution:

$$\tan^{-1}x+\tan^{-1}y=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\frac{x+y}{1-xy}=\frac{\pi}{4}$$

$$\Rightarrow \frac{x+y}{1-xy}=\tan\frac{\pi}{4}$$

$$\Rightarrow \frac{x+y}{1-xy}=1\, or,\, x+y=1-xy$$

$$or,\, x+y+xy=1\, Ans.$$

3. If A is a square matrix such that $A^2 = A$, then write the value of $7A - (I + A)^3$, where I is an identity matrix.

Solution:

$A^2 = A$

$7A - (I + A)^3$

$7A - [(I + A)^2(I + A)] = 7A - [(II + AA + 2AI)(I + A)]$

$= 7A - [I + A^2 + 2AI][I + A]$

$= 7A - [I + A + 2A][I + A]$

$= 7A - [I + 3A][I + A]$

$= 7A - [II + IA + 3AI + 3A^2]$

$= 7A - [I + A + 3A + 3A]$

$= 7A - [I + 7A]$

$= - I$ Ans.

4. If $\begin{bmatrix} x-y & z \\ 2x-y & w \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$, find the value of x + y.

Solution:

If $\begin{bmatrix} x-y & z \\ 2x-y & w \end{bmatrix} = \begin{bmatrix} -1 & 4 \\ 0 & 5 \end{bmatrix}$ then x + y = ?

we can compare the element of 2 matrices. so

$x - y = -1$ … (1)

$2x - y = 0$ … (2)

On solving both eq^n we get → x = 1, y = 2

so x + y = 3 Ans.

5. If $\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$, find the value of x.

Solution:

$$\begin{vmatrix} 3x & 7 \\ -2 & 4 \end{vmatrix} = \begin{vmatrix} 8 & 7 \\ 6 & 4 \end{vmatrix}$$

on expanding both determinants we get

$12x + 14 = 32 - 42$

$12x + 14 = -10$

$12x = -24$

$x = -2$ Ans.

6. If $f(x) = \int_0^x t \sin t \, dt$, then write the value of $f'(x)$.

Solution:

$$f(x) = \int_0^x t \sin t \, dt$$

$\Rightarrow f'(x) = 1 \cdot x \sin x - 0$

= x sin x Ans.

7. Evaluate :

$$\int_2^4 \frac{x}{x^2+1}dx$$

Solution:

$$I = \int_2^4 \frac{x}{x^2+1}dx$$

Put $x^2+1=t$

$$\Rightarrow 2xdx = dt \qquad xdx = \frac{1}{2}dt$$

$$\left|\begin{array}{l} at\ x=2 \\ t=5 \\ at\ x=4 \\ t=17 \end{array}\right.$$

$$\therefore I = \int_4^{17} \frac{1/2}{t}dt$$

$$= \frac{1}{2}\left[\log|t|\right]_4^{17}$$

$$= \frac{1}{2}[\log 17 - \log 4]$$

$$= \frac{1}{2}\log(17/4)\ Ans.$$

8. Find the value of 'p' for which the vectors $3\hat{i}+2\hat{j}+9\hat{k}$ *and* $\hat{i}-2p\hat{j}+3\hat{k}$ are parallel.

Solution:

Let $\vec{a} = 3\hat{i}+2\hat{j}+9\hat{k}, \vec{b} = \hat{i}-2p\hat{j}+3\hat{k}$

If $\vec{a}, \vec{b}$ are parallel vector then their exist a, λ such that

$$\vec{a} = \lambda\vec{b}$$

So $\left(3\hat{i}+2\hat{j}+9\hat{k}\right) = \lambda\left(\hat{i}-2p\hat{j}+3\hat{k}\right)$

compare $3 = \lambda$ | $2 = -2p\lambda$ | $9 = 3\lambda$, $\lambda = 3$

put λ = 3 in 2= –2pλ

2 = –2p.3

$$p = -\frac{1}{3}\ Ans.$$

9.Find $\vec{a}.(\vec{b}\times\vec{c})$, *if* $\vec{a} = 2\hat{i}+\hat{j}+3\hat{k}, \vec{b} = -\hat{i}+2\hat{j}+\hat{k}$ *and* $\vec{c} = 3\hat{i}+\hat{j}+2\hat{k}.$

Solution:

If $\vec{a} = 2\hat{i} + \hat{j} + 3\hat{k}, \vec{b} = -\hat{i} + 2\hat{j} + \hat{k}, \vec{c} = 3\hat{i} + \hat{j} + 2\hat{k}$

Then $\vec{a}.\left(\vec{b} \times \vec{c}\right) = \begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix}$

expand along R_1 = 2[4 – 1] – 1 [–2 – 3] + 3 [– 1 – 6]
= 6 + 5 – 21 = – 10

10. If the Cartesian equations of a line are $\frac{3-x}{5} = \frac{y+4}{7} = \frac{2z-6}{4}$, write the vector equation for the line.

Solution:

Cartesian eq^n of line is $\frac{3-x}{5} = \frac{y+4}{7} = \frac{2z-6}{4}$,

we can write it as $\frac{x-3}{-5} = \frac{y+4}{7} = \frac{z-3}{2}$

so vector eq^n is $\vec{r} = \left(3i - 4j + 3k\right) + \lambda\,(-5\hat{i} + 7\hat{j} + 2\hat{k})$

where λ is a constant

SECTION B

Question numbers 11 to 22 carry 4 marks each.

11. If the function f : R → R be given by f(x) = x^2 + 2 and g : R → R be given by g(x) = $\frac{x}{x-1}$, x ≠ 1, find fog and gof and hence find fog (2) and gof (–3).

Solution:

$f : R \to R;\ f(x) = x^2 + 2$

$g : R \to R;\ g(x) = \frac{x}{x-1},\ x \neq 1$

$fog = f(g(x))$

$= f\left(\frac{x}{x-1}\right) = \left(\frac{x}{x-1}\right)^2 + 2$

$= \frac{x^2}{(x-1)^2} + 2$

$= \frac{x^2 + 2(x-1)^2}{(x-1)^2}$

$= \frac{x^2 + 2x^2 - 4x + 2}{(x-1)^2}$

$= \frac{3x^2 - 4x + 2}{(x-1)^2}$

$$gof = g(f(x))$$
$$= g(x^2+2)$$
$$\frac{(x^2+2)}{(x^2+2)-1} = \frac{x^2+2}{x^2+1} = 1+\frac{1}{x^2}$$

$$\therefore\ fog(2) = \frac{3(2)^2-4(2)+2}{(2-1)^2} = 6$$
$$gof(-3) = 1+\frac{1}{(-3)^2+1} = \frac{11}{10} = 1\frac{1}{10}$$

12. Prove that $\tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right] = \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x, \frac{-1}{\sqrt{2}} \le x \le 1$

OR

If $\tan^{-1}\left(\frac{x-2}{x-4}\right)+\tan^{-1}\left(\frac{x+2}{x+4}\right) = \frac{\pi}{4}$, find the value of x.

Solution:

$$\tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right] = \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x, \frac{1}{\sqrt{2}} \le x \le 1$$

In LHS

put x = cos 2θ

$$\tan^{-1}\left[\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right]$$
$$= \tan^{-1}\left[\frac{\sqrt{1+2\cos^2\theta-1}-\sqrt{1-1+2\sin^2\theta}}{\sqrt{1+2\cos^2\theta-1}+\sqrt{1-1+2\sin^2\theta}}\right]$$
$$= \tan^{-1}\left[\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right]$$
$$= \tan^{-1}\left[\frac{1-\tan\theta}{1+\tan\theta}\right]$$
$$= \tan^{-1}\left[\frac{\tan(\pi/4)-\tan\theta}{1+\tan(\pi/4).\tan\theta}\right]$$
$$= \tan^{-1}\left[\tan(\pi/4)-\theta\right]$$
$$= \frac{\pi}{4}-\theta \quad as \begin{Bmatrix} x = \cos 2\theta \\ so, \theta \frac{\cos^{-1}x}{2} \end{Bmatrix}$$
$$= \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x = RHS \qquad proved$$

OR

$$\tan^{-1}\left(\frac{x-2}{x-4}\right)+\tan^{-1}\left(\frac{x+2}{x+4}\right)=\frac{\pi}{4} \quad (1)$$

Use formula, $\tan^{-1}\left[\dfrac{\frac{x-2}{x-4}+\frac{x+2}{x+4}}{1-\left(\frac{x-2}{x-4}\right).\left(\frac{x+2}{x+4}\right)}\right]=\dfrac{\pi}{4}$

$$\Rightarrow \tan^{-1}\left[\frac{(x-2)(x+4)+(x+2).(x-4)}{(x-4).(x+4)-(x-2).(x+2)}\right]=\frac{\pi}{4}$$

$$\Rightarrow \frac{(x-2)(x+4)+(x+2).(x-4)}{(x-4).(x+4)-(x-2).(x+2)}=1$$

$$\Rightarrow \frac{x^2-8+2x+x^2-8-2x}{x^2-16-x^2+4}=1$$

$$\Rightarrow \frac{2x^2-16}{-12}=1$$

$$\Rightarrow 2x^2=-12+16=4$$

$$\Rightarrow x^2=2 \qquad \Rightarrow x=\pm\sqrt{2}$$

13. Using properties of determinants, prove that

$$\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix}=x^3$$

Solution:

To prove, $\begin{vmatrix} x+y & x & x \\ 5x+4y & 4x & 2x \\ 10x+8y & 8x & 3x \end{vmatrix}=x^3$

$$\text{LHS }=\begin{vmatrix} x & x & x \\ 5x & 4x & 2x \\ 10x & 8x & 3x \end{vmatrix}+\begin{vmatrix} y & x & x \\ 4y & 4x & 2x \\ 8y & 8x & 3x \end{vmatrix}$$

$$=x^3\begin{vmatrix} 1 & 1 & 1 \\ 5 & 4 & 2 \\ 10 & 8 & 3 \end{vmatrix}+yx^2\begin{vmatrix} 1 & 1 & 1 \\ 4 & 4 & 2 \\ 8 & 8 & 3 \end{vmatrix}$$

Applying $C_1 \rightarrow C_1 - C_2$, $C_2 \rightarrow C_2 - C_3$ in the first determinant

$$= x^3\begin{vmatrix} 0 & 0 & 1 \\ 1 & 2 & 2 \\ 2 & 5 & 3 \end{vmatrix} + yx^2 \times 0$$

As the first two columns of the 2^{nd} determinant are same.
Expanding the first determinant through R_1

$$= x^3.1.\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = x^3(5-4)$$

$= x^3 = RHS$ thus proved

14. Find the value of $\frac{dy}{dx}$ at $\theta = \frac{\pi}{4}$, if $x = ae^{\theta}(\sin\theta - \cos\theta)$ and $y = ae^{\theta}(\sin\theta + \cos\theta)$.

Solution:

$$y = ae^{\theta}(\sin\theta + \cos\theta)$$
$$x = ae^{\theta}(\sin\theta - \cos\theta)$$

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$ (Applying parametric differentiation) … (1)

Now, $\frac{dy}{d\theta} = ae^{\theta}(\cos\theta - \sin\theta) + ae^{\theta}(\sin\theta + \cos\theta)$

$= 2ae^{\theta}(\cos\theta)$ (Applying product Rule)

$$\frac{dx}{d\theta} = ae^{\theta}(\cos\theta + \sin\theta) + ae^{\theta}(\sin\theta - \cos\theta)$$
$$= 2ae^{\theta}(\sin\theta)$$

Substituting the values of $\frac{dy}{d\theta}$ and $\frac{dx}{d\theta}$ in (1)

$$\frac{dy}{dx} = \frac{2ae^{\theta}\cos\theta}{2ae^{\theta}\sin\theta} = \cot\theta$$

Now $\frac{dy}{dx}$ at $\theta = \frac{\pi}{4}$

$$[\cot\theta]_{\theta=\pi/4} = \cot\frac{\pi}{4} = 1.$$

15. If $y = Pe^{ax} + Qe^{bx}$, show that

$$\frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + aby = 0.$$

Solution:

If y = $Pe^{ax} + Qe^{bx}$ …(1)

$$\frac{dy}{dx} = aPe^{ax} + bQe^{bx} \quad ...(2)$$

$$\frac{d2y}{dx^2} = a^2Pe^{ax} + b^2Qe^{bx} \quad ...(3)$$

multiplying … (1) by ab
we get, aby = $abPe^{ax} + abQe^{bx}$ … (4)
multiplying (2) by (a + b)

we get,, $(a+b)\frac{dy}{dx} = (a+b)(aPe^{ax} + bQe^{bx}) = (a^2Pe^{ax} + b^2Pe^{bx}) + (abPe^{ax} + abQe^{bx})$

or, $(a^2bPe^{ax} + b^2Qe^{bx}) - (a+b)\frac{dy}{dx} + (abPe^{ax} + abQe^{bx})$

or, $\frac{d^2y}{dx^2} - (a+b)\frac{dy}{dx} + aby = 0$

16. Find the value(s) of x for which y = $[x(x-2)]^2$ is an increasing function.

OR

Find the equations of the tangent and normal to the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $\left(\sqrt{2}a, b\right)$.

Solution:

y = $[x(x-2)]^2$

we know, for increasing function we have $f'(x) \geq 0$

$$\therefore\ f'(x) = 2[x(x-2)]\left[\frac{d}{dx}x(x-2)\right]$$

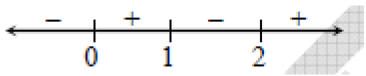
Or, $f'(x) = 2[x(x-2)]\frac{d}{dx}(x^2 - 2x)$

= 2x(x – 2) (2x – 2)
= 4x(x – 2) (x – 1)

For $f'(x) \geq 0$

i.e., 4x(x – 1) (x – 2) ≥ 0

the values of x are :



$x \in [0,1] \cup [2, \infty]$

OR

The slope of the tangent at $\left(\sqrt{2}\,a, b\right)$ to the curve $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\frac{2x}{a^2} - \frac{2yy'}{b^2} = 0$$

$$\Rightarrow y' = \frac{b^2 x}{a^2 y}\Bigg]_{(\sqrt{2}a, b)} = \frac{b^2\sqrt{2}a}{a^2 b} = \frac{b\sqrt{2}}{a}$$

The equation of the tangent :

$y - b = \frac{b\sqrt{2}}{a}(x - \sqrt{2}a)$ {using point-slope form : $y - y_1 = m(x - x_1)$}

$$ay - ab = b\sqrt{2}x - 2ab$$

or $b\sqrt{2}x - ay - ab = 0$

Normal :

The slope of the normal = $\frac{-1}{dy/dx}$

$$= \frac{-1}{\frac{b\sqrt{2}}{a}} = -\frac{a}{b\sqrt{2}}$$

Equation of Normal :

$$y - b = \frac{-a}{b\sqrt{2}}(x - \sqrt{2}a)$$

$$yb\sqrt{2} - b^2\sqrt{2} = -ax + \sqrt{2}a^2$$

or $ax + b\sqrt{2}y - \sqrt{2}(a^2 + b^2) = 0$

17. Evaluate :

$$\int_0^{\pi} \frac{4x\sin x}{1 + \cos^2 x}dx$$

OR

Evaluate :

$$\int \frac{x+2}{\sqrt{x^2+5x+6}}\,dx$$

Solution:

$$I=\int_0^{\pi}\frac{4x\sin x}{1+\cos^2 x}\,dx$$

$$I=\int_0^{\pi}\frac{4(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}\,dx \quad \left\{Applying \int f(a-x)=\int f(x)\right.$$

$$I=\int_0^{\pi}\frac{4\pi\sin x}{1+\cos^2 x}\,dx-\int_0^{\pi}\frac{4x\sin x}{1+\cos^2 x}\,dx$$

Or,

$$I=\int_0^{\pi}\frac{4\pi\sin x}{1+\cos^2 x}\,dx-I$$

$$2I=4\pi\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}\,dx$$

$$2I=4\pi.2\times\int_0^{\pi/2}\frac{\sin x}{1+\cos^2 x}\,dx \quad \left\{\text{Applying} \int_0^{2a} f(x)dx=2\int_0^{2} f(x)dx \quad \text{if f (2a – x)= f (x)}\right.$$

$$I=4\pi\int_0^{\pi/2}\frac{\sin x}{1+\cos^2 x}\,dx$$

put cos x = t $\Rightarrow -\sin x dx = dt$

as well for x = 0, $x=\pi/2$

t = 1 t = 0

$$\therefore I=4\pi\int_1^{0}\frac{-dt}{1+t^2}$$

$$I=4\pi\int_0^{1}\frac{dt}{1+t^2} \quad \left\{\int_a^b f(x)dx=-\int_b^a f(x)dx\right.$$

$$I=4\pi\left[\tan^{-1}1-\tan^{-1}0\right]$$

$$=4\pi\times\frac{\pi}{4}=\pi^2.$$

OR

$$\int \frac{x+2}{\sqrt{x^2+5x+6}}dx$$

put , x + 2 $=\lambda\left(\frac{d}{dx}(x^2+5x+6)\right)+\mu$

$x+2=2\lambda x+5\lambda+\mu$

comparing coefficients of x both sides

$1=2\lambda \Rightarrow \lambda=1/2$

comparing constant terms both sides,

$2=5\lambda+\mu$

or, $2=5\left(\frac{1}{5}\right)+\mu$

or, $\mu=2-\frac{5}{2}=\frac{-1}{2}$

$$\therefore \int \frac{x+2}{\sqrt{x^2+5x+6}}dx=\int \frac{\frac{1}{2}(2x+5)-\frac{1}{2}}{\sqrt{x^2+5x+6}}dx \quad \{as\ x+2=\lambda(2x+5)+\mu\}$$

$$\therefore I=\underset{(I_1)}{\int \frac{\frac{1}{2}(2x+5)}{\sqrt{x^2+5x+6}}dx}-\underset{(I_2)}{\frac{1}{2}\int \frac{dx}{\sqrt{x^2+5x+6}}}$$

$\therefore I=I_1-I_2$

$$I_1=\frac{1}{2}\int \frac{(2x+5)}{\sqrt{x^2+5x+6}}dx, \quad \text{put } x^2+5x+6=t$$

$\therefore\ (2x+5)dx=dt$

$$=\frac{1}{2}\int \frac{dt}{\sqrt{t}}=\frac{1}{2}\left(\frac{t^{-1/2+1}}{-\frac{1}{2}+1}\right)+C=t^{1/2}+C=\sqrt{x^2+5x+6}+C$$

$$I_2=\frac{1}{2}\frac{dx}{\sqrt{x^2+5x+6}}$$

$$=\frac{1}{2}\int \frac{dx}{\sqrt{x^2+5x+\frac{25}{4}-\frac{25}{4}+6}}=\frac{1}{2}\int \frac{dx}{\sqrt{\left(x+\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2}}$$

$$\frac{1}{2}.\log\left[\left(x+\frac{5}{2}\right)+\sqrt{\left(x+\frac{5}{2}\right)^2-\left(\frac{1}{2}\right)^2}\right]+C$$

$$\frac{1}{2}.\log\left[\left(x+\frac{5}{2}\right)+\sqrt{x^2+5x+6}\right]+C$$

Substituting the values of I_1 and I_2 in (1)
we get,

$$I=\sqrt{x^2+5x+6}+\frac{1}{2}\log\left[\left(x+\frac{5}{2}\right)+\sqrt{x^2+5x+6}\right]+c$$

18. Find the particular solution of the differential equation $\frac{dy}{dx}=1+x+y+xy,$ given that y = 0 when x = 1.

Solution:

$$\frac{dy}{dx}=(1+x)+y(1+x)$$

Or, $\frac{dy}{dx}=(1+y)(1+x)$

Or, $\frac{dy}{1+y}=(1+x)dx$

$$\int\frac{dy}{1+y}=\int(1+x)dx$$

$$\log|1+y|=x+\frac{x^2}{2}+C$$

given y = 0 when x = 1

i.e., $\log|1+0|=1+\frac{1}{2}+C$

$$\Rightarrow C=-\frac{3}{2}$$

$\therefore$ The particular solution is

$$\log|1+y|=\frac{x^2}{2}+x-\frac{3}{2}.$$

or the answer can expressed as

$$\log|1+y| = \frac{x^2+2x-3}{2}$$

or $1+y = e^{(x^2+2x-3)/2}$

or, $y = e^{(x^2+2x-3)} - 1.$

19. Solve the differential equation $(1+x^2)\frac{dy}{dx} + y = e^{\tan -1x}$.

Solution:

$$(1+x^2)\frac{dy}{dx} + y = e^{\tan -1x}$$

$$\frac{dy}{dx} + \frac{y}{1+x^2} = \frac{e\tan^{-1}}{1+x^2}$$

It is a linear differential equation of 1[st] order.
comparing with standard LDE

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$P(x) = \frac{1}{1+x^2}; Q(x) = \frac{e\tan^{-1x}}{1+x^2}$$

Integrating factor $IF = e^{\int Pdx} = e^{\int \frac{1}{1+x^2}dx} = e^{\tan -1x}$

Solution of LDE

$$y.IF = \int IF\ Q(x)dx + C$$

$$\therefore y.e^{\tan -1x} = \int e^{\tan -1x}.\frac{e^{\tan -1x}}{1+x^2}dx + C$$

$$y.e^{\tan^{-1}x} = \int \frac{(e^{\tan^{-1}x})^2}{1+x^2}dx + C \quad ....(1) \text{ y}$$

To solving $\int \frac{(e^{\tan^{-1}x})^2}{1+x^2}dx$

Put $e^{\tan^{-1}x} = t$

or $e^{\tan^{-1}x}.\frac{1}{1+x^2} = dt$

$$\therefore \int \frac{e^{\tan^{-1}x}.e^{\tan^{-1}x}}{1+x^2}dx = \int t\,dt$$

$$= \frac{t^2}{2} + C = \frac{(e^{\tan^{-1}x})^2}{2} + C$$

Substituting in (1)

$$y.e^{\tan^{-1}x} = \frac{(e^{\tan^{-1}x})^2}{2} + C$$

20. Show that the four points A, B, C and D with position vectors $4\hat{i}+5\hat{j}+\hat{k}, -\hat{j}-\hat{k}, 3\hat{i}+9\hat{j}+4\hat{k}$ and $4(-\hat{i}+\hat{j}+\hat{k})$ respectively are coplanar.

OR

The scalar product of the vector $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of vectors $\vec{b}=2\hat{i}+4\hat{j}-5\hat{k}$ and $\vec{c}=\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to one. Find the value of λ and hence find the unit vector along $\vec{b}+\vec{c}$.

Solution:

If P.V of $\vec{A}=4\hat{i}+5\hat{j}+\hat{k}$

$\vec{B}=-\hat{j}-\hat{k}$

$\vec{C}=3\hat{i}+9\hat{j}+4\hat{k}$

$\vec{D}=4(-\hat{i}+\hat{j}+\hat{k})$

Points $\vec{A}, \vec{B}, \vec{C}, \vec{D}$ all Coplanar if $\left[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}\right]=0 \quad \Rightarrow (1)$

So, $\overrightarrow{AB} = P.V.\text{ of } \vec{B} - P.V.\text{ of } \vec{A} = \ -4\hat{i}-6\hat{j}-2\hat{k}$

$\overrightarrow{AC} = P.V.\text{ of } \vec{C} - P.V.\text{ of } \vec{A} = \ -\hat{i}-4\hat{j}+3\hat{k}$

$\overrightarrow{AD} = P.V.\text{ of } \vec{D} - P.V.\text{ of } \vec{A} = \ -8\hat{i}-\hat{j}+3\hat{k}$

So, so for $\left[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}\right]$

$$= \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix}$$

expand along $R_1 \rightarrow$

– 4[12 + 3] + 6[–3 + 24] – 2[1 +32]
= – 60 + 126 – 66

$= 0$

So, we can say that point A, B, C, D are Coplanar proved

OR

Given $\rightarrow \quad \vec{a} = \hat{i} + \hat{j} + \hat{k}$

$\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$

$\vec{c} = \lambda\hat{i} + 2\hat{j} - 3\hat{k}$

So, $\vec{b} + \vec{c} = (2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}$

Unit vector along $(\vec{b} + \vec{c}) = \dfrac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{(2+\lambda)^2 + 36 + 4}}$

$= \dfrac{(2+\lambda)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{(2+\lambda)^2 + 40}}$

given that dot product of $\vec{a}$ with the unit vector of $\vec{b} + \vec{c}$ is equal to 1

So, apply given condition

$\dfrac{(2+\lambda) + 6 - 2}{\sqrt{(2+\lambda)^2 + 40}} = 1$

$\Rightarrow 2 + \lambda + 4 = \sqrt{(2+\lambda)^2 + 40}$

Squaring $36 + \lambda^2 + 12\lambda = 4 + \lambda^2 + 4\lambda + 40$

$\Rightarrow 8\lambda = 8$

$\Rightarrow \lambda = 1.$

21. A line passes through (2, –1, 3) and is perpendicular to the lines

$\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$ and

$\vec{r} = (2\hat{i} - \hat{j} - 3\hat{k}) + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$. Obtain its equation in vector and Cartesian form.

Solution:

Line L is passing through point $= (2\hat{i} - \hat{j} + 3\hat{k})$

If $L_1 \Rightarrow \vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$

$L_2 \Rightarrow \vec{r} = (2\hat{i} - \hat{j} - 3\hat{k}) + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$

Let dr of line L = a_1, a_2, a_3

The eq^n of L in vector form $\Rightarrow$

$\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + k(a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$

k is any constant.
so by condition that L1is perpendicular to L $a_1a_2 + b_1b_2 + c_1c_2 = 0$

$2a_1 - 2a_2 + a_3 = 0$ … (1)

and also

$L \perp L_2$

so, $a_1 + 2a_2 + 2a_3 = 0$ …(2)
Solve (1), (2)
$3a_1 + 3a_3 = 0$

$\Rightarrow a_3 = -a_1$

put it in (2)
$a_1 + 2a_2 - 2a_1 = 0$

$a_2 = \frac{a_1}{2}$ let

so dr of L = $\left(a_1, \frac{a_1}{2}, -a_1\right)$

so we can say dr of L $= \left(1, \frac{1}{2}, -1\right)$

so eq^n of L in vector form

$\vec{r} = (2\hat{i} - \hat{j} + 3\hat{k}) + k\left(\hat{i} + \frac{\hat{j}}{2} - \hat{k}\right)$

3-D form $\rightarrow \frac{x-2}{1} = \frac{y+1}{1/2} = \frac{z-3}{-1}$

22. An experiment succeeds thrice as often as it fails. Find the probability that in the next five trials, there will be at least 3 successes.
Solution:

In Binomial distribution

$(p+q)^n = {}^nC_0.p^n + {}^nC_1.p^{n-1}.q^1 + {}^nC_2.p^{n-2}.q^2 + ... + {}^nC_n.q^n$

if p = probability of success
q = prob. of fail
given that p = 3q …(1)
we know that p + q = 1

so, 3q + q = 1

$q = \frac{1}{4}$

So, $p = \frac{3}{4}$

Now given $\Rightarrow$ n = 5 we required minimum 3 success
$(p + q)^5 = {}^5C_0.p^5 + {}^5C_1.p^4.q^1 + {}^5C_2.p^3.q^2$

$$= {}^5C_0.\left(\frac{3}{4}\right)^5 + {}^5C_1.\left(\frac{3}{4}\right)^4.\left(\frac{1}{4}\right) + {}^5C_2.\left(\frac{3}{4}\right)^3.\left(\frac{1}{4}\right)^2$$

$$= \frac{3^5}{4^5} + \frac{5.3^4}{4^5} + \frac{10.3^3}{4^5}$$

$$= \frac{3^5 + 5.3^4 + 10.3^3}{4^5} = \frac{3^3[9 + 15 + 10]}{4^5} = \frac{34 \times 27}{16 \times 64} = \frac{459}{512}.$$

SECTION C

Question numbers 23 to 29 carry 6 marks each.

23. Two schools A and B want to award their selected students on the values of sincerity, truthfulness and helpfulness. The school A wants to award ₹ x each, ₹ y each and ₹ z each for the three respective values to 3, 2 and 1 students respectively with a total award money of ₹ 1,600. School B wants to spend ₹ 2,300 to award its 4, 1 and 3 students on the respective values (by giving the same award money to the three values as before). If the total amount of award for one prize on each value is ₹ 900, using matrices, find the award money for each value. Apart from these three values, suggest one more value which should be considered for award.

Solution:

Let Matrix D represents number of students receiving prize for the three categories :

D =

Number of students of school	SINCERITY	TRUTHFULNESS	HELPFULNESS
A	3	2	1
B	4	1	3
One student for each value	1	1	1

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ where x, y and z are rupees mentioned as it is the question, for sincerity, truthfulness and helpfulness respectively.

$E = \begin{bmatrix} 1600 \\ 2300 \\ 900 \end{bmatrix}$ is a matrix representing total award money for school A, B and for one prize for each value.

We can represent the given question in matrix multiplication as :
DX = E

or $\begin{bmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1600 \\ 2300 \\ 900 \end{bmatrix}$

Solution of the matrix equation exist if | D | ≠ 0

i.e., $\begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 3[1-3] - 2[4-3] + 1[4-1]$

= – 6 – 2 + 3
= – 5
therefore, the solution of the matrix equation is
$X = D^{-1} E$

To find D^{-1}; $D^{-1} = \frac{1}{|D|} adj(D)$

Cofactor Matrix of D

$= \begin{bmatrix} -2 & -1 & 3 \\ -1 & 2 & -1 \\ 5 & -5 & -5 \end{bmatrix}$

Adjoint of D = adj (D)

$= \begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$

{transpose of Cofactor Matrix}

$\therefore D^{-1} = \frac{1}{-5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$

Now, $X = D^{-1}E$

$= \frac{1}{-5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}\begin{bmatrix} 1600 \\ 2300 \\ 900 \end{bmatrix}$

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 200 \\ 300 \\ 400 \end{bmatrix}$

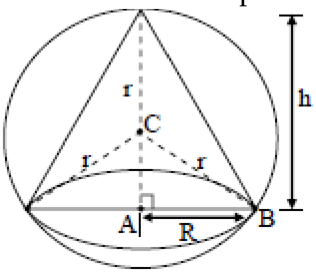
∴ x = 200, y = 300, z = 400. Ans.
Award can also be given for Punctuality.

24. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$. Also show that the maximum volume of the cone is $\frac{8}{27}$ of the volume of the sphere.

Solution:

Let R and h be the radius and height of the cone.

r be the radius of sphere.



To show $h = \frac{4r}{3}$

and Maximum Volume of Sphere

$= \frac{8}{27}$ Volume of Sphere

In ΔABX, $AC = h - r$

$\therefore (h-r)^2 + R^2 = r^2$ {Pythagorus Theorem}

$\Rightarrow R^2 = r^2 - (h-r)^2$

Volume of cone : $V = \frac{1}{3}\pi R^2 h$

or, $V = \frac{1}{3}\pi(r^2 - (h-r)^2)h$

$V = \frac{1}{3}\pi[r^2 - h^2 - r^2 + 2hr]h$

$V = \frac{1}{3}\pi[2h^2 r - h^3]$

For maxima or minima, $\frac{dV}{dh} = 0$

Now, $\frac{dV}{dh} = \frac{1}{3}\pi[4hr - 3h^2]$

Putting, $\frac{dV}{dh} = 0$

We get $4hr = 3h^2$

$\Rightarrow h = \frac{4r}{3}$

$\frac{d^2V}{dh^2} = \frac{1}{3}\pi[4r - 6h]$

Putting $h = \frac{4r}{3}$

$\frac{d^2V}{dh^2} = \frac{1}{3}\pi\left(4r - \frac{6.4r}{3}\right)$

$= -\frac{1}{3}\pi[4r]$

Which is less than zero, therefore

$h=\frac{4r}{3}$ is a Maxima

and the Volume of the cone at $h=\frac{4r}{3}$

will be maximum,

$V=\frac{1}{3}\pi R^2 h$

$=\frac{1}{3}\pi[r^2-(h-r)^2]h$

$=\frac{1}{3}\pi\left[r^2-\left(\frac{4r}{3}-r\right)^2\right]\left[\frac{4r}{3}\right]$

$=\frac{1}{3}\pi\left[\frac{8r^2}{9}\right]\left[\frac{4r}{3}\right]$

$=\frac{8}{27}\left(\frac{4\pi r^3}{3}\right)$

$=\frac{8}{27}$ (Volume of the sphere)

25. Evaluate :

$\int\frac{1}{\cos^4 x+\sin^4 x}dx$

Solution:

$\int\frac{1}{\cos^4 x+\sin^4 x}dx$

$=\int\frac{\frac{1}{\cos^4 x}dx}{1+\tan^4 x}$

$=\int\frac{\sec^2 x\sec^2 x dx}{1+\tan^4 x}$

$=\int\frac{(1+\tan^2 x)\sec^2 x dx}{1+\tan^4 x}$

put tan x = t ⇒ $\sec^2$x dx = dt

$=\int\frac{(1+t^2)dt}{1+t^4}$

$=\int\frac{(1/t^2+1)dt}{\frac{1}{t^2}+t^2}$ {dividing each by t^2}

$=\int\frac{\left(1+1/t^2\right)dt}{\left(t-1/t\right)^2+2}$

Put $t-\frac{1}{t}=z \quad \Rightarrow\left(1+\frac{1}{t^2}\right)dt=dz$

$=\int\frac{dz}{z^2+2}=\frac{1}{\sqrt{2}}\tan^{-1}z+C$

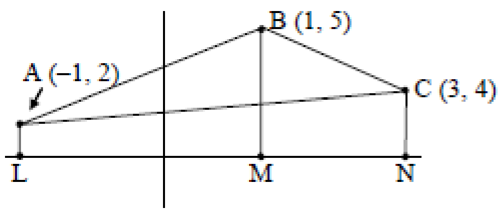
$=\frac{1}{\sqrt{2}}\tan^{-1}\left(\tan x-\frac{1}{\tan x}\right)+C$

$=\frac{1}{\sqrt{2}}\tan^{-1}(\tan x-\cot x)+C$

26. Using integration, find the area of the region bounded by the triangle whose vertices are (– 1, 2), (1, 5) and (3, 4).

Solution:

Let A = (–1, 2)
B = (1, 5)
C = (3, 4)



We have to find the area of ΔABC

Find eq^n of Line AB $\rightarrow y-5=\left(\frac{2-5}{-1-1}\right)\cdot(x-1)$

$y-5=\frac{3}{2}(x-1)$

2y – 10 = 3x – 3
3x – 2y + 7 = 0 …(1)

$y=\frac{3x+7}{2}$

Eq^n of BC $\rightarrow\ y-4=\left(\frac{5-4}{1-3}\right)\cdot(x-3)$

$y-4=\frac{1}{-2}(x-3)$

$2y-8=-x+3$

$x+2y-11=0 \qquad$ ....(2)

$y=\frac{11-x}{2}$

Eq^n of AC $\rightarrow\ y-4=\left(\frac{2-4}{-1-3}\right)\cdot(x-3)$

$y-4=\frac{1}{2}(x-3)\Rightarrow 2y-8=x-3$

x – 2y + 5 = 0 …(3)

$\Rightarrow y = \frac{x+5}{2}$

So, required area = $\int_{-1}^{1}\left(\frac{3x+7}{2}\right)dx + \int_{1}^{3}\left(\frac{11-x}{2}\right)dx - \int_{-1}^{3}\left(\frac{x+5}{2}\right)dx$

$= \frac{1}{2}\left[\frac{3x^2}{2}+7x\right]_{-1}^{1} + \frac{1}{2}\left[11x-\frac{x^2}{2}\right]_{1}^{3} - \frac{1}{2}\left[\frac{x^2}{2}+5x\right]_{-1}^{3}$

$= \frac{1}{2}\left[\left(\frac{3}{2}+7\right)-\left(\frac{3}{2}-7\right)\right] + \frac{1}{2}\left[\left(33-\frac{9}{2}\right)-\left(11-\frac{1}{2}\right)\right] - \frac{1}{2}\left[\left(\frac{9}{2}+15\right)-\left(\frac{1}{2}-5\right)\right]$

$= \frac{1}{2}[14+22-4-24] = \frac{1}{2}[36-28] = 4$ square unit

27. Find the equation of the plane through the line of intersection of the planes x + y + z = 1 and 2x + 3y + 4z = 5 which is perpendicular to the plane x – y + z = 0. Also find the distance of the plane obtained above, from the
origin.

OR

Find the distance of the point (2, 12, 5) from the point of intersection of the line
$\vec{r} = 2\hat{i} - 4\hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r}.(\hat{i} - 2\hat{j} + \hat{k}) = 0$.

Solution:

Eq^n of given planes are

$P_1 \Rightarrow x + y + z - 1 = 0$

$P_2 \Rightarrow 2x + 3y + 4z - 5 = 0$

Eq^n of plane through the line of intersection of planes P_1, P_2 is

$P_1 + \lambda P_2 = 0$

$(x + y + z - 1) + \lambda(2x + 3y + 4z - 5) = 0$

$(1 + 2\lambda) x + (1 + 3\lambda) y + (1 + 4\lambda) z + (-1 - 5\lambda) = 0$ … (1)

given that plane represented by eq^n (1) is perpendicular to plane

x – y + z = 0

so we use formula $a_1a_2 + b_1b_2 + c_1c_2 = 0$

so $(1 + 2\lambda).1 + (1 + 3\lambda).(-1) + (1 + 4\lambda).1 = 0$

$1 + 2\lambda - 1 - 3\lambda + 1 + 4\lambda = 0$

$3\lambda + 1 = 0$

$\lambda = \frac{-1}{3}$

Put $\lambda = -\frac{1}{3}$ in eq^n (1) so we get

$\left(1-\frac{2}{3}\right)x + (1-1)y + \left(1-\frac{4}{3}\right)z + \frac{2}{3} = 0$

$\frac{x}{3} - \frac{z}{3} + \frac{2}{3} = 0$

x – z + 2 = 0 Ans.

OR

General points on the line:

x = 2 + 3λ, y = – 4 + 4λ, z = 2 + 2λ

The equation of the plane :

$\vec{r}.(\hat{i} - 2\hat{j} + \hat{k}) = 0$

The point of intersection of the line and the plane :

Substituting general point of the line in the equation of plane and finding the particular value of λ.

$[(2+3\lambda)\hat{i} + (-4+4\lambda)\hat{j} + (2+2\lambda)\hat{k}].(\hat{i} - 2\hat{j} + \hat{k}) = 0$

$(2+3\lambda).1 + (-4+4\lambda)(-2) + (2+2\lambda).1 = 0$

$12 - 3\lambda = 0$ or, $\lambda = 4$

$\therefore$ the point of intersection is :

(2 + 3 (4), –4 +4(4), 2 + 2(4)) = (14, 12, 10)

Distance of this point from (2, 12, 5) is

$= \sqrt{(14-2)^2 + (12-12)^2 + (10-5)^2}$ {Applying distance formula}

$= \sqrt{12^2 + 5^2}$

= 13 Ans.

28. A manufacturing company makes two types of teaching aids A and B of Mathematics for class XII. Each type of A requires 9 labour hours of fabricating and 1 labour hour for finishing. Each type of B requires 12 labour hours for fabricating and 3 labour hours for finishing. For fabricating and finishing, the maximum labour hours available per week are 180 and 30 respectively. The company makes a profit of ₹ 80 on each piece of type A and ₹120 on each piece of type B. How many pieces of type A and type B should be manufactured per week to get a maximum profit? Make it as an LPP and solve graphically. What is the maximum profit per week?

Solution:

Let pieces of type A manufactured per week = x

Let pieces of type B manufactured per week = y

Companies profit function which is to be maximized : Z = 80x + 120y

	Fabricating hours	Finishing hours
A	9	1
B	12	3

Constraints : Maximum number of fabricating hours = 180

$\therefore 9x + 12y \le 180 \Rightarrow 3x + 4y \le 60$ K

Where 9x is the fabricating hours spent by type A teaching aids, and 12y hours spent on type B.

and Maximum number of finishing hours = 30

$\therefore x + 3y \le 30$

where x is the number of hours spent on finishing aid A while 3y on aid B.

So, the LPP becomes :

Z (MAXIMISE) = 80x + 120 y

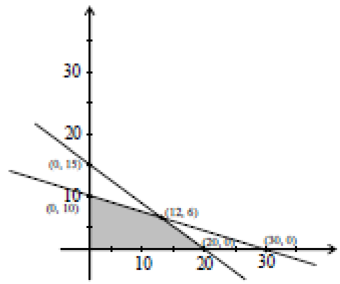
Subject to $3x + 4y \le 60$

$x + 3y \le 30$

$x \ge 0$

$y \ge 0$

Solving it Graphically :



Z = 80x + 120y at (0, 15)
= 1800
Z = 1200 at (0, 10)
Z = 1600 at (20, 0)
Z = 960 + 720 at (12, 6)
= 1680
Maximum profit is at (0, 15)
∴ Teaching aid A = 0
Teaching aid B = 15
Should be made

29. There are three coins. One is a two-headed coin (having head on both faces), another is a biased coin that comes up heads 75% of the times and third is also a biased coin that comes up tails 40% of the times. One of the three coins is chosen at random and tossed, and it shows heads. What is the probability that it was the two-headed coin?

OR

Two numbers are selected at random (without replacement) from the first six positive integers. Let X denote the larger of the two numbers obtained. Find the probability distribution of the random variable X, and hence find the mean of the distribution.

Solution:
If there are 3 coins.
Let these are A, B, C respectively
For coin A $\rightarrow$ Prob. of getting Head P(H) = 1
For coin B $\rightarrow$ Prob. of getting Head P(H) = $\frac{3}{4}$
For coin C $\rightarrow$ Prob. of getting Head P(H) = 0.6
we have to find $P\left(A/H\right)$ = Prob. of getting H by coin A
So, we can use formula

$$P\left(A/H\right)=\frac{P\left(H/A\right).P(A)}{P\left(H/A\right).P(A)+P\left(H/B\right).P(B)+P\left(H/C\right).P(C)}$$

Here P(A) = P(B) = P(C) = $\frac{1}{3}$ (Prob. of choosing any one coin)

$$P\left(H/A\right)=1, P\left(H/B\right)=\frac{3}{4}, P\left(H/C\right)=0.6$$

Put value in formula so

$$P\left(A/H\right)=\frac{1.\frac{1}{3}}{1.\frac{1}{3}+\frac{3}{4}.\frac{1}{3}+\frac{1}{3}(0.6)}=\frac{1}{1+0.75+0.6}$$

$$=\frac{100}{235}$$

$$=\frac{20}{47}\,Ans.$$

OR

First six numbers are 1, 2, 3, 4, 5, 6.

X is bigger number among 2 number so

Variable (X)	2	3	4	5	6
Probability P(X)					

if X = 2

for P(X) = Prob. of event that bigger of the 2 chosen number is 2

So, Cases = (1, 2)

So, $P(X)=\frac{1}{{}^6C_2}=\frac{1}{15}$....(1)

if X = 3

So, favourable cases are = (1, 3), (2, 3)

$P(x)=\frac{2}{{}^6C_2}=\frac{2}{15}$....(2)

if X = 4 $\Rightarrow$ favourable casec = (1, 4), (2, 4), (3, 4)

$P(X)=\frac{3}{15}$....(3)

if X = 5 $\Rightarrow$ favourable casec = (1, 5), (2, 5), (3, 5), (4, 5)

$P(X)=\frac{4}{15}$....(4)

if X = 6 $\Rightarrow$ favourable casec = (1, 6), (2, 6), (3, 6), (4, 6), (5, 6)

$P(X)=\frac{5}{15}$....(5)

We can put all value of P(X) in chart, So

Variable (X)	2	3	4	5	6
Probability P(X)	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{4}{15}$	$\frac{5}{15}$

and required mean $=2.\left(\frac{1}{15}\right)+3.\left(\frac{2}{15}\right)+4\left(\frac{3}{15}\right)+5.\left(\frac{4}{15}\right)+6.\left(\frac{5}{15}\right)$

$$=\frac{70}{15}=\frac{14}{3}\text{ Ans.}$$

CBSE-XII-2013 EXAMINATION

MATHEMATICS

Paper & Solution

Code: 65/1

Time: 3 Hrs. Max. Marks: 70

General Instructions:

(i) All questions are compulsory.

(ii) The question paper consists of 29 questions divided into three sections A, B and C. Section A comprises of10 questions of one mark each, Section B comprises of 12 questions of four marks each and Section Comprises of 7 questions of six marks each.

(iii) All questions in Section A are to be answered in one word, one sentence or as per the exact requirement of the questions.

(iv) There is no overall choice, However, internal choice has been provided in 4 questions of four marks each and 2 questions of six marks each. You have to attempt only one of the alternatives in all such questions.

(v) Use of calculators is not permitted. You may ask for logarithmic tables, if required.

1. Write the principal value of $\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3}$

Solution:

$$\tan^{-1}(\sqrt{3}) = \pi/3$$

$$\cot^{-1}(-\sqrt{3}) = \pi - \pi/6$$

Hence

$$\pi/3 - (\pi - \pi/6) = -\pi/2$$

2. Write the value of $\tan^{-1}\left[2\sin\left(2\cos^{-1}\frac{\sqrt{3}}{2}\right)\right]$.

Solution:

$$\because \quad \cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}$$

$$= \tan^{-1}(2\sin(2.\pi/6))$$

$$= \tan^{-1}\left(2.\sin\frac{\pi}{3}\right)$$

$$= \tan^{-1}\left(2.\frac{\sqrt{3}}{2}\right) = \tan^{-1}\sqrt{3} = \pi/3$$

3. For what value of x, is the matrix $A = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ x & -3 & 0 \end{bmatrix}$ a skew-symmetric matrix ?

Sol. The value of determinant of skew symmetric matrix of odd order is always equal to zero

$$\begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ x & -3 & 0 \end{bmatrix} = 0$$

$$-1(0-3x)-2(3-0)=0$$

$$\Rightarrow \quad 3x-6=0 \quad \Rightarrow \quad \boxed{x=2}$$

4. If matrix $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and A^2 = kA, then write the value of k.

Sol. Given A2 = kA

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = k\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \Rightarrow \boxed{k=2}$$

5. Write the differential equation representing the family of curves y = mx, where m is an arbitrary constant.

Sol. y = mx .....(1)

differentiating with respect to x, we get

dy/dx = m

∴ differential equation of curve

$$y = \frac{xdy}{dx}$$

6. If A_{ij} is the cofactor of the element a_{ij} of the determinant $\begin{bmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{bmatrix}$, then write the value of $a_{32} \cdot A_{32}$.

Sol. $$\begin{bmatrix} 2 & -3 & 5 \\ 6 & 0 & 4 \\ 1 & 5 & -7 \end{bmatrix}$$

$A_{32} = (-1)^{3+2} M_{32}$ where M_{32} is the minor of a_{32}.

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix}$$

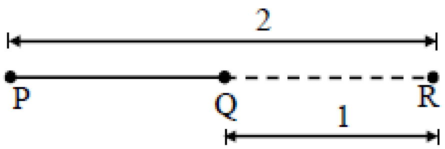
$$A_{32} = -\begin{vmatrix} 2 & 5 \\ 6 & 4 \end{vmatrix} \Rightarrow A_{32} = -(8-30)$$

$$\boxed{A_{32} = 22}$$

$$\therefore \quad a_{32}A_{32} = 5(22) = 110$$

7. P and Q are two points with position vectors $3\vec{a} - 2\vec{b}$ and $\vec{a} + \vec{b}$ respectively. Write the position vector of point R which divides the line segment PQ in the ratio 2 : 1externally. 1

Sol. P.V. of P is $3\vec{a}-2\vec{b}$



P.V. of Q is $\vec{a}+\vec{b}$

Point R divides segment PQ in ratio 2 : 1 externally.

$$\text{P.V.of R} = \frac{(\text{P.V.of p})1-(\text{P.V.of Q})(2)}{1-2}$$

$$\text{P.V.of R} = \frac{(3\vec{a}-2\vec{b})(1)-(\vec{a}+\vec{b})(2)}{1-2} = \frac{\vec{a}-4\vec{b}}{-1}$$

$$\text{P.V.of R} = 4\vec{b}-\vec{a}$$

8. Find $|\vec{x}|$, if for a unit vector $\vec{a}$, $(\vec{x}-\vec{a}).(\vec{x}+\vec{a})=15$.

Sol. Given $|\vec{a}|=1$

$$(\vec{x}-\vec{a}).(\vec{x}+\vec{a})=15$$

$$|\vec{x}|^2-|\vec{a}|^2=15$$

$$|\vec{x}|^2-1=15$$

$$|\vec{x}|^2=15+1$$

$$|\vec{x}|^2=16$$

$$|\vec{x}|=4$$

9. Find the length of the perpendicular drawn from the origin to the plane 2x – 3y + 6z + 21 = 0.

Sol. $p=\left|\frac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}\right|$

$$p=\left|\frac{0+0+0+21}{\sqrt{2^2+3^2+6^2}}\right| \Rightarrow p=\frac{21}{\sqrt{49}} \Rightarrow p=\frac{21}{7} \Rightarrow p=3$$

10. The money to be spent for the welfare of the employees of a firm is proportional to the rate of change of its total revenue (marginal revenue). If the total revenue (in rupees) received from the sale of x units of a product is given by $R(x) = 3x^2 + 36x + 5$, find the marginal revenue, when x = 5, and write which value does the equations indicate.

Sol. $R(x) = 3x^2 + 36x + 5$

$$MR = \frac{dR}{dx} = 6x^2 + 36$$

when $x = 5$

$$MR = 30 + 36 + 66$$

11. Consider $f : R_+ \rightarrow [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse f^{-1} of f given by

$f^{-1}(y) = \sqrt{y-4}$, where R_+ is the set of all non-negative real numbers.

Sol. $f : R^+ \rightarrow [4.\infty)$

$$f(x) = x^2 + 4$$

$$f(x) = x^2 > 4 \qquad \therefore \text{ (one} - \text{one)}$$

$$\text{As } f(x) = x^2 + 4 \geq 4$$

$$\Rightarrow \text{ Rage} = [4.\infty) = \text{co} - \text{domain}$$

$$\therefore \text{ onto}$$

$$\text{Further}: y = x^2 + 4$$

so f is invertible.

$$\Rightarrow y - 4 = x^2 \quad \Rightarrow x = \pm\sqrt{y-4}$$

$$\text{As } x > 0 \text{ so } x = \sqrt{y-4}$$

$$\therefore y = \sqrt{x-4} = f^{-1}(x)$$

$$\text{Or } f^{-1}(y) - \sqrt{y-4}$$

12. Show that :

$$\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3}$$

OR

Solve the folowing equation :

$$\cos(\tan^{-1} x) = \sin\left(\cot^{-1}\frac{3}{4}\right)$$

Solution:

$$\text{Let } \frac{1}{2}\sin^{-1}\frac{3}{4} = \theta \text{ then } \frac{3}{4} = \sin 2\theta$$

$$\text{Now } \tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \tan\theta$$

$$\text{if } \sin 2\theta = \frac{3}{4} \text{ then } \frac{2\tan\theta}{1+\tan^2\theta} = \frac{3}{4}$$

$$8\ \tan\theta = 3 + 3\tan^2\theta$$

$$3\tan^2\theta - 8\ \tan\theta + 3 = 0$$

$$\tan\theta = \frac{8 \pm \sqrt{64 - 4\times3\times3|}}{6}$$

$$\tan\theta = \frac{8\pm\sqrt{28}}{6} = \frac{4\pm\sqrt{7}}{3}$$

$$\tan\theta = \frac{4+\sqrt{7}}{3} \text{ or } \frac{4-\sqrt{7}}{3}$$

$$\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right) = \frac{4-\sqrt{7}}{3} \text{ Hence proved.}$$

OR

$$\cos(\tan^{-1}x)$$

$$\text{LHS. lrt } \tan^{-1}x = \theta \Rightarrow x = \tan\theta$$

$$\cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}} = \frac{1}{\sqrt{1+x^2}}$$

$$\text{Hence } \cos(\tan^{-1}x) = \frac{1}{\sqrt{1+x^2}}$$

$$\text{R.H.S Let } \cot^{-1}\frac{3}{4} = \theta \Rightarrow \frac{3}{4} = \cot\theta$$

$$\text{then } \sin\theta = \frac{1}{\sqrt{1+\cot^2\theta}} = \frac{1}{\sqrt{1+\frac{9}{16}}} = \frac{4}{5}$$

$$\text{Now LHS} = \text{RHS}$$

$$\frac{1}{\sqrt{1+x^2}} = \frac{4}{5}$$

$$25 = 16 + 16x^2$$

$$x^2 = \frac{9}{16} \qquad \Rightarrow x = \frac{3}{4}$$

13. Using properties of determinants, prove the following :

$$\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} = 9y^2(x+y)$$

Solution:

$$\begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix} = 9y^2(x+y)$$

$$\text{LHS} \begin{vmatrix} x & x+y & x+2y \\ x+2y & x & x+y \\ x+y & x+2y & x \end{vmatrix}$$

Now, apply $C_1 \to C_1 + C_2 + C_3$

$$\begin{vmatrix} 3x+3y & x+y & x+2y \\ 3x+3y & x & x+y \\ 3x+3y & x+2y & x \end{vmatrix}$$

$$3(x+y)\begin{vmatrix} 3x+3y & x+y & x+2y \\ 3x+3y & x & x+y \\ 3x+3y & x+2y & x \end{vmatrix}$$

$$3(x+y)\begin{vmatrix} -y & 2y \\ -2y & y \end{vmatrix}$$

$$3y^2(x+y)\begin{vmatrix} -1 & 2 \\ -2 & 1 \end{vmatrix}$$

$3y^2(x+y)\,(-1+4) = 9y^2(x+y)$. Hence proved.

14. If $y^x = e^{y-x}$, prove that $\frac{dy}{dx} = \frac{(1+\log y)^2}{\log y}$

Sol. $y^x = e^{y-x}$

$\Rightarrow x \log_e y = y - x$ ....(1)

Differentiating w.r.t.x

$$\Rightarrow \log_e y + x.\frac{1}{y}\frac{dy}{dx} = \frac{dy}{dx} - 1$$

$$\Rightarrow \log_e y + 1 = \frac{dy}{dx}\left(1 - \frac{x}{y}\right)\{\text{form}(1)\ \frac{x}{y} = \frac{1}{1+\log_e y}\}$$

$$\Rightarrow \log_e y + 1 = \frac{dy}{dx}\left(1 - \frac{1}{1+\log_e y}\right)$$

$$\Rightarrow (\log_e y + 1) = \frac{dy}{dx}\left(1 - \frac{\log_e y}{1+\log_e y}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1+\log_e y)^2}{\log_e y}$$

15. Differentiate the following with respect to x :

$$\sin^{-1}\left(\frac{2^{x+1}.3^x}{1+(36)^x}\right)$$

Solution:

$$y = \sin^{-1}\left[\frac{2^{x+1}.3^x}{1+(36)^x}\right]$$

$$y = \sin^{-1}\left[\frac{2^x.2.3^x}{1+(36)^x}\right]$$

$$y = \sin^{-1}\left[\frac{2.(6)^x}{1+(6)^{2x}}\right]$$

$$y = 2\tan^{-1}(6)^x$$

$$\frac{dy}{dx} = \frac{2}{1+(6)^{2x}}.6^x \log 6$$

$$\frac{dy}{dx} = \frac{2.6^x \log 6}{1+(36)^x}$$

16. Find the value of k, for which $f(x) = \begin{cases} \dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{if } -1 \le x < 0 \\ \dfrac{2x+1}{x-1}, & \text{if } 0 \le x < 1 \end{cases}$ is continuous at $x = 0$.

OR

If $x = a\cos^3\theta$ and $y = a\sin^3\theta$, then find the value of $\dfrac{d^2y}{dx^2}$ at $\theta = \dfrac{\pi}{6}$.

Solution:

$$f(x) = \begin{cases} \dfrac{\sqrt{1+kx}-\sqrt{1-kx}}{x}, & \text{if } -1 \le x < 0 \\ \dfrac{2x+1}{x-1}, & \text{if } 0 \le x < 1 \end{cases}$$

function f(x) is continuous at x = 0

$$\therefore \quad f(0)=\lim_{x\to 0} f(x)$$

$$\Rightarrow \frac{0+1}{0-1}=\lim_{x\to 0}\left(\frac{\sqrt{1+kx}-\sqrt{1-kx}}{x}\right)$$

$$\Rightarrow -1=\lim_{x\to 0}\left(\frac{\sqrt{1+kx}-\sqrt{1-kx}}{x}\right)\left(\frac{\sqrt{1+kx}+\sqrt{1-kx}}{\sqrt{1+kx}+\sqrt{1-kx}}\right)$$

$$\Rightarrow -1=\lim_{x\to 0}\frac{(1+2k)-(1-kx)}{x[\sqrt{1+kx}+\sqrt{1-kx}]}$$

$$\Rightarrow -1=\lim_{x\to 0}\frac{2k}{\sqrt{1+kx}+\sqrt{1-kx}}$$

$$\Rightarrow -1=\frac{2k}{2} \quad \Rightarrow \ k=-1$$

OR

$x = a\cos^3\theta$ and $y = a\sin^3\theta$

$\frac{dx}{d\theta} = -3a\cos^{2\theta}\sin\theta$ and $\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$

$x = a\cos^3\theta$ and $y = a\sin^3\theta$

$\frac{dx}{d\theta} = -3a\cos^{2\theta}\sin\theta$ and $\frac{dy}{d\theta} = 3a\sin^2\theta\,\frac{\cos\theta}{}$

$$\Rightarrow \frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$

$$\Rightarrow \frac{dy}{dx}=-\tan\theta$$

$$\Rightarrow \frac{d^2y}{dx^2}=-\sec^2\theta\frac{1}{(-3a\cos^2\theta.\sin\theta)}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{1}{3a}\sec^4\theta\,\mathrm{cosec}\,\theta$$

$$\left(\frac{d^2y}{dx^2}\right)_{\theta\frac{\pi}{6}}=\frac{1}{3a}\left(\frac{2}{\sqrt{3}}\right)^4.2=\frac{32}{27a}$$

17. Evaluate :

$$\int\frac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}$$

OR

Evaluate :

$$\int\frac{x+2}{\sqrt{x^2+2x+3}}dx$$

Solution:

$$\int \frac{\cos 2x - \cos 2\alpha}{\cos x - \cos \alpha} dx$$

$$= \int \frac{(2\cos^2 x - 1) - (2\cos^2 \alpha - 1)}{\cos x - \cos \alpha} dx$$

$$= \int \frac{2(\cos^2 x - 1)(2\cos^2 \alpha - 1)}{\cos x - \cos \alpha} dx$$

$$= 2\int (\cos x + \cos \alpha) dx$$

$$= 2(\sin x + x \cos \alpha) + c$$

OR

$$I = \int \frac{x+2}{\sqrt{x^2 + 2x + 3}} dx$$

$$I = \int \left\{ \frac{(x+1)+1}{\sqrt{x^2 + 2x + 3}} \right\} dx$$

$$I = \int \left\{ \frac{(x+1)}{\sqrt{x^2 + 2x + 3}} \right\} dx + \int \left\{ \frac{1}{\sqrt{x^2 + 2x + 3}} \right\} dx$$

$$I = I_1 + I_2$$

In I_1 let $x^2 + 3 = t^2$

$\therefore \quad (2x + 2)dx = 2t\ 2t$

$\Rightarrow (x+1)dx = tdt$

$$\therefore \quad I_1 = \int \frac{t.dt}{t} = t$$

$$I_1 = \sqrt{x^2 + 2x + 3}$$

$$\text{Now in } I_1 = \int \frac{1}{\sqrt{x^2 + 2x + 3}} dx = \int \frac{dx}{\sqrt{x^2 + 2x + 3}}$$

$$I_2 = \log[(x+1)^2 + \sqrt{(x+1)^2 + 2}]$$

Now $I = I_1 + I_2$

$$\Rightarrow I = \sqrt{x^2 + 2x + 3} + \log(x + 1 + \sqrt{x^2 + 2x + |3}) + c$$

18. Evaluate :

$$\int \frac{dx}{x(x^5 + 3)}$$

Solution:

$$I=\int\frac{dx}{x(x^5+3)}$$

$$I=\int\frac{x^4dx}{x(x^5+3)}$$

Let $x^5=t\Rightarrow 5x^4dx=dt$

$$I=\frac{1}{5}\int\frac{dt}{t(t+3)}$$

$$I=\frac{1}{5}.\frac{1}{3}\int\left(\frac{1}{t}-\frac{1}{t+3}\right)dt$$

$$I=\frac{1}{15}\{\log t-\log(t+3)\}+c$$

$$I=\frac{1}{15}\log\int\left(\frac{t}{t+3}\right)+c$$

$$I=\frac{1}{15}\log\int\left(\frac{x^5}{x^5+3}\right)+c$$

19. Evaluate

$$\int_0^{2\pi}\frac{1}{1+e^{\sin x}}dx$$

Solution:

$$I=\int_0^{2\pi}\frac{1}{1+e^{\sin x}}dx \qquad ....(1)$$

$$I=\int_0^{2\pi}\frac{1}{1+e^{\sin(2\pi-x)}}dx$$

$$I=\int_0^{2\pi}\frac{1}{1+e^{-\sin x}}dx$$

$$I=\int_0^{2\pi}\frac{e^{\sin x}}{e^{\sin x}+1}dx$$

Adding (1) & (2) we get

$$\Rightarrow\ 21=\int_0^{2\pi}\left(\frac{1+e^{\sin x}}{1+e^{\sin x}}\right)dx$$

$$\Rightarrow\ 21=[x]_0^{2\pi}$$

$$\Rightarrow\ 21=2\pi$$

$$I=\pi$$

20. If $\vec{a}=\vec{i}-\vec{j}+7k$ and $\vec{b}=5\vec{i}+\vec{j}+\lambda\vec{k}$, then find the value of λ, so that $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ are perpendicular vectors.

Solution:

$\vec{a} = \vec{i} - \vec{j} + 7k$

$\vec{b} = 5\vec{i} + \vec{j} + \lambda\vec{k}$

$\vec{a} + \vec{b} = 6\vec{i} - 2\vec{j} + (7+\lambda)\vec{k}$

$\vec{a} - \vec{b} = 4\vec{i} - 0\vec{j} + (7-\lambda)\vec{k}$

given $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ are perpendicular

$\therefore \quad (\vec{a} + \vec{b}).(\vec{a} - \vec{b}) = 0$

$\{6\vec{i} - 2\vec{j} + (7+\lambda)\vec{k}\} \times \{4\vec{i} - 0\vec{j} + (7-\lambda)\vec{k}\} = 0$

$6(-4) + 0(-2) + (7+\lambda)(7-\lambda) = 0$

$-24 + 49 - \lambda^2 = 0$

$\lambda^2 = 25 \Rightarrow \boxed{\lambda = \pm 5}$

21. Show that the lines

$\vec{r} = 3\vec{i} + 2\vec{j} - 4\vec{k} + \lambda(\vec{i} + 2\vec{j} + 2\vec{k});$

$\vec{r} = 3\vec{i} + 2\vec{j} + \mu(3\vec{i} + 2\vec{j} + 6\vec{k})$

are intersecting. Hence find their point of intersection.

OR

Find the vector equation of the plane through the points (2, 1, –1) and (–1, 3, 4) and perpendicular to the plane x – 2y + 4z = 10.

Sol. If the given lines are intersecting then the shortest distance between the lines is zero and also they have same

common point $\vec{r} = 3\vec{i} + 2\vec{j} - 4\vec{k} + \lambda(\vec{i} + 2\vec{j} + 2\vec{k})$

$$\Rightarrow \frac{x-3}{1} = \frac{y-2}{2} = \frac{z+4}{2} (= \lambda)(\text{Let})$$

Let P is $(\lambda + 3, 2\lambda + 2, 2\lambda - 4)$

Also, $\vec{r} = 5\vec{i} - 2\vec{j} + \mu(3\vec{i} + 2\vec{j} + 6\vec{k})$

$$\Rightarrow \frac{x-5}{3} = \frac{y+2}{2} = \frac{z-0}{6} (= \mu)(\text{Let})$$

Let Q is $(\mu + 5, 2\mu - 2, 6\mu)$

If lines are intersecting then P and Q will be same.

$\lambda + 3 = 3\mu + 5 \qquad ....(1)$

$2\lambda + 2 = 2\mu - 2 \qquad ....(2)$

$2\lambda - 4 = 6\mu \qquad ....(3)$

Solve(2) & (3)

$$\lambda + 1 = \mu - 1$$
$$2\lambda - 2 = 3\mu$$
$$-\quad + \quad -$$
$$3 = 2\mu - 1$$
$$4 = 2\mu$$
$$\boxed{\mu = -2}$$

put $\mu = -2$ ....(3)

$$2\lambda - 4 = 6(-2)$$
$$2\lambda = -12 + 4$$
$$2\lambda = -8$$
$$\boxed{\lambda = -4}$$

put μ & λ in (1)

$$\lambda + 3 = 3\mu + 5$$
$$-4 + 3 = 3(-2) + 5$$
$$-1 = -1$$

$\therefore$ from $\lambda = -4$ then P is $(-1, -6, -12)$

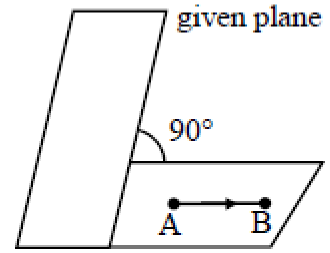
from $\mu = -2$ then Q is $(-1, -6, -12)$

as P and Q are same

$\therefore$ lines are intersecting lines and their point of intersection is (–1, –6, –12).

OR

A(2, 1, –1) ; B(–1, 3, 4)



$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$\overrightarrow{AB} = -3\hat{i} + 2\hat{j} - 5k$$

given plane $x - 2y + 4z = 10$

$$\therefore \vec{n}_1 = \hat{i} - 2\hat{j} + 4k$$

The required plane is perpendicular to given plane.

Therefore n r of required plane will be perpendicular to $\vec{n}_1$ and AB.

$\therefore\ \vec{n} \parallel (\vec{n}_1 \times \overrightarrow{AB})$

$\vec{n}_1 = \hat{i} - 2\hat{j} + 4k$

$\overrightarrow{AB} = -3\hat{i} + 2\hat{j} - 5k$

$\therefore\ \vec{n}_1 \times \overrightarrow{AB} = -18\hat{i} - 17\hat{j} - 4k$

$\therefore$ required plane is

$\vec{r}.\vec{n} = \vec{a}.\vec{n}$

$\vec{r}.(-18\hat{i} - 17\hat{j} - 4k) = (2\hat{i} + \hat{j} - k).(-18\hat{i} - 17\hat{j} - 4k)$

$\vec{r}.(-18\hat{i} - 17\hat{j} - 4k) = -36 - 17 + 4$

$\vec{r}.(-18\hat{i} - 17\hat{j} - 4k) = -49$

$\boxed{18x + 17y + 4z = 49}$

22. The probabilities of two students A and B coming to the school in time are $\frac{3}{7}$ and $\frac{5}{7}$ respectively.

Assuming that the events, 'A coming in time' and 'B coming in time' are independent, find the probability of only one of them coming to the school in time. Write at least one advantage of coming to school in time.

Sol. If P(A come in school time) = 3/7

P (B come in school time) = 5/7

P (A not come in school time) = 4/7

P (B not come in school time) = 2/7

P (only one of them coming school in time)

$= P(A) \times P(\vec{B}) + P(\vec{A}).P(B)$

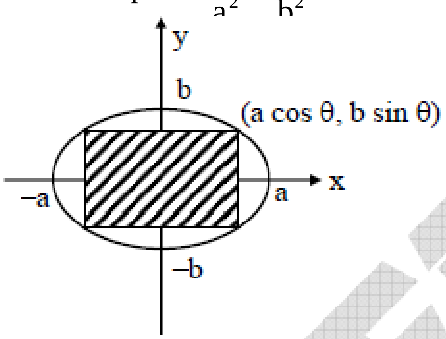
$= \frac{3}{7} \times \frac{2}{7} + \frac{5}{7} \times \frac{4}{7} = \frac{26}{49}$

23. Find the area of the greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

OR

Find the equations of tangents to the curve 3x2 – y2 = 8, which pass through the point $\left(\frac{4}{3}, 0\right)$

Sol. Given ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



Area of reactangle

$A = 2a = \cos\theta.\ 2b\sin\theta$

$A = 2ab.\sin 2\theta$

$\therefore\ A_{max} = 2ab$

OR

Let a point (x_1, y_1)

$3x^2 - y^2 = 8 \Rightarrow 6x - 2y.y' = 0$

$\Rightarrow\ y' = \dfrac{3x}{y}$

$\therefore$ Tangent $y - y_1 = \dfrac{3x_1}{y_1}(x - x_1)$

It passing throgh $\left(\dfrac{4}{3}, 0\right)$

$-y_1 = \dfrac{3x_1}{y_1}\left(\dfrac{4}{3} - x_1\right)$

$\Rightarrow\ -y_1^2 = 4x_1 - 3x_1^2 \quad \Rightarrow\ y_1^2 = 4x_1 - 3x_1^2$

$\Rightarrow\ 3x_1^2 - 8 = 3x_1^2 - 4x_1$

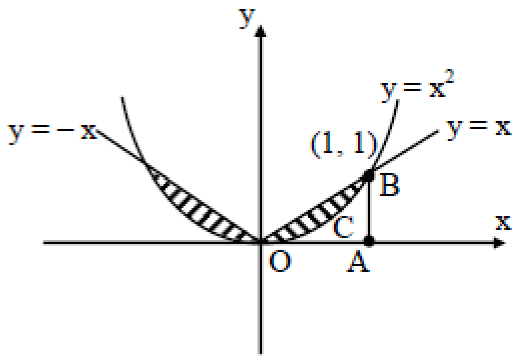
$\therefore\ x_1 = 2$

So $12 - y^2 = 8$

$\Rightarrow\ y^2 = 4 \quad \Rightarrow\ y_1 = \pm 2$

24. Find the area of the region bounded by the parabola $y = x^2$ and $y = |x|$.

Sol.



Required area = 2[area of ΔOAB – Area of curve OCBA]

$A = 2\left[\dfrac{1}{2}(1)(1) - \int_0^1 x^2 dx\right]$

$A = 2\left[\dfrac{1}{2} - \dfrac{1}{3}\right] \quad \Rightarrow A = 2\left[\dfrac{1}{6}\right] = \dfrac{1}{3}$

25. Find the particular solution of the differential equation (tan–1y – x)dy = $(1 + y^2)$dx, given that when x = 0, y=0

Sol. $(\tan^{-1}y - x)dy = (1 + y^2)dx$

$$\Rightarrow \frac{dx}{dy} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

$$IF = e^{\int \frac{1}{1+y^2}dy}$$

$$IF = e^{\tan^{-1}y}$$

$$x.\,IF = \int Q.IF\,dy + c$$

$$\Rightarrow x.\,e^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{(1+y^2)}.e^{\tan^{-1}y}dy + c$$

Put $\tan^{-1}y = t$

$$\Rightarrow x.\,e^{\tan^{-1}y} = (t.e^t) - (e^t) + c$$

$$\Rightarrow x.\,e^{\tan^{-1}y} = \tan^{-1}y.\,e^{\tan^{-1}y} - e^{\tan^{-1}y} + c$$

26. Find the equation of the plane passing through the line of intersection of the planes $\vec{r}.(\hat{i}+3\hat{j})-6=0$ and $\vec{r}.(3\hat{j}-\hat{j}-4k)=0,$ whose perpendicular distance from origin is unity.

OR

Find the vector equation of the line passing through the point (1, 2, 3) and parallel to the planes $\vec{r}.(3\hat{i}-\hat{j}-2k)=5$ and $\vec{r}.(3\hat{i}+\hat{j}+k)=6$

Sol. P_1 is $\vec{r}.(3\hat{i}+\hat{j})-6=0$

P_1 is $x+3y-6=0$

P_2 is $\vec{r}.(3\hat{j}-\hat{j}-4k)=0$

P_2 is $3x-y-4z=0$

Equation of plane passing through intersection of P_1 and P_2 is $P_1 + \lambda P_2 = 0$

$$(x+3y-6)+\lambda(3x-y-4z)=0$$

$$(1+3\lambda)x+(3-\lambda)y+(-4\lambda)z+(-6)=0$$

Its distance from (0, 0, 0) is 1.

$$\left|\frac{0+0+0-6}{\sqrt{(1+3\lambda)^2+(3-\lambda)^2+(-4\lambda)^2}}\right|$$

$$36 = (1+3\lambda)^2 + (3-\lambda)^2 + (-4\lambda)^2$$

$$36 = 1 + 9\lambda^2 + 6\lambda + 9 + \lambda^2 - 6\lambda + 16\lambda^2$$

$$36 = 26\lambda^2 + 10 \Rightarrow 26\lambda^2 = 26 \Rightarrow \lambda^2 = 1 \Rightarrow \boxed{\lambda = \pm 1}$$

Hence required plane is

For $\lambda = 1$, $(x + 3y - 6) + 1\,(3x - y - 4z) = 0$

$4x + 2y - 4z - 6 = 0$

For $\lambda = -1$, $(x + 3y - 6) - 1(3x - y - 4z) = 0$
$-2x + 4y + 4z - 6 = 0$

OR

P_1 is $\vec{r}.(\hat{i} - \hat{j} + 2k) = 5$

$\therefore\quad \vec{n}_1 = \hat{i} - \hat{j} + 2k$

P_2 is $\vec{r}.(3\hat{i} - \hat{j} + 2k) = 6$

$\vec{n}_2 = 3\hat{i} - \hat{j} + k$

The line parallel to plane P_1 & P_2 will be perpendicular to $\vec{n}_1$ & $\vec{n}_2$

$\therefore\quad \vec{b} \parallel (\vec{n}_1 \times \vec{n}_2)$

$\vec{n}_1 = \hat{i} - \hat{j} + 2k$

$\vec{n}_2 = 3\hat{i} - \hat{j} + k$

$\vec{n}_1 \times \vec{n}_2 = -3\hat{i} + 5\hat{j} + 4k$

$\therefore\quad \vec{b} = -3\hat{i} + 5\hat{j} + 4k$

Point is (1, 2, 3)

$\therefore\quad \vec{a} = \hat{i} - \hat{j} + 3k$

$\therefore\quad$ required line is $\vec{r} = \vec{a} \,|+ \lambda\vec{b}$

$\vec{r} = (\hat{i} + 2\hat{j} + 3k) + \lambda(-3\hat{i} + 5\hat{j} + 4|\ k)$

27. In a hockey match, both teams A and B scored same number of goals up to the end of the game, so to decide the winner, the referee asked both the captains to throw a die alternately and decided that the team, whose captain gets a six first, will be declared the winner. If the captain of team A was asked to start, find the irrespective probabilities of winning the match and state whether the decision of the referee was fair or not.

Sol. P(6 get) = 1/6

P(6 not get) = $P\overline{(6get)}$ = 5/6

$$P.(A\ win) = P(Aget6) + P(\overline{6get}).P(\overline{6get})P(6get) + P(\overline{6get}).P(\overline{6get}).P(\overline{6get}).\,.P(\overline{6get})P(6get) + \ldots.. + \infty$$

$$P.(A\ win) = \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6} \times \frac{1}{6} + \ldots + \infty$$

$$= \frac{1}{6} + \left(\frac{5}{6}\right)^2 \times \frac{1}{6} + \left(\frac{5}{6}\right)^4 \times \frac{1}{6} + \ldots + \infty$$

$$\boxed{\because S_\infty = \frac{a}{1-r}}$$

$$= \frac{\left(\frac{1}{6}\right)}{\left(1 - \frac{25}{36}\right)} = \frac{36}{11 \times 6} = \frac{6}{11}$$

Similarly winning for B

P(B win) = 1 – P(A win)

$$=1-\frac{6}{11}=\frac{5}{11}$$

28. A manufacturer considers that men and women workers are equally efficient and so he pays them at the same rate. He has 30 and 17 units of workers (male and female) and capital respectively, which he uses to produce two types of goods A and B. To produce one unit of A, 2 workers and 3 units of capital are required while 3workers and 1 unit of capital is required to produce one unit of B. If A and B are priced at j– 100 and j– 120per unit respectively, how should he use his resources to maximise the total revenue ? Form the above as an LPP and solve graphically. Do you agree with this view of the manufacturer that men and women workers are equally efficient and so should be paid at the same rate ?

Sol. if $z_{max.} = 100x + 120y$

	typeA	typeB	
worker	2	3	30
capitl	3	1	17

Subject to,

$2x+3y\le30$

$3x+y\le17$

$x\ge0$

Let object of type A = x

Object of type B = y

pts	coordinate	$Z^{max}=100x+ 120y$
O	(0,0)	Z=0
A	$\left(\frac{17}{3},0\right)$	$Z=\frac{1700}{3}$
E	(3.8)	Z=300+960=1260
C	(0.10)	Z=1200

maximum revenue = 1260.

29. The management committee of a residential colony decided to award some of its members (say x) for honesty, some (say y) for helping others and some other (say z) for supervising the workers to keep the colony neat and clean. The sum of all the awardees is 12. Three times the sum of awardees for cooperation

and supervision added to two times the number of awardees for honesty is 33. If the sum of the number of awardees for honesty and supervision is twice the number of awardees for helping others, using matrix method, find the number of awardees of each category. Apart from these value, namely, honesty, cooperation and supervision, suggest one more value which the management of the colony must include for awards.

Sol. Given

x + y + z = 12 …..(1)

3(y + z) + 2x = 33 …..(2)

(x + z) = 2y …..(3)

x + y + z = 12

2x + 3y + 3z = 33

x – 2y + z =

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 1 & -2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 33 \\ 0 \end{bmatrix}$$

AX = B

A–1(AX) = A–1(B)

I · X = A–1·B

X = A–1.B

$$X = \frac{(Adj.A).B}{|A|}$$

$$|A| = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 1 & -2 & 1 \end{bmatrix}$$

| A | = 1(3 + 6) – 1(2 – 3) + 1 (–4 – 3)

| A | = 9 + 1 – 7 = 3

$|A| \neq 0$

$$(Adj.\ A) = \begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}$$

$$(Adj.\ A).B = \begin{bmatrix} 9 & -3 & 0 \\ 1 & 0 & -1 \\ -7 & 3 & 1 \end{bmatrix}_{3\times3} \begin{bmatrix} 12 \\ 33 \\ 0 \end{bmatrix}_{3\times3}$$

$$(Adj.\ A).B = \begin{bmatrix} 9 \\ 12 \\ 15 \end{bmatrix}$$

$$\therefore \quad X = \frac{(Adj.A).B}{|A|}$$

$$X = \frac{1}{3}\begin{bmatrix} 9 \\ 12 \\ 15 \end{bmatrix} \Rightarrow X = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}$$

$x = 3$, $y = 4$, $z = 5$.