Exercise 4.7

Q1

Evaluate $\sin^{-1}\left(\sin\frac{5\pi}{6}\right)$

Solution

We know that,

$$\sin^{-1}(\sin\theta) = \begin{cases} -\pi-\theta, & \text{if } \theta \in \left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right] \\ \theta, & \text{if } \theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \\ \pi-\theta, & \text{if } \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \\ -2\pi+\theta, & \text{if } \theta \in \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right] \end{cases}$$

$\therefore \sin^{-1}\left(\sin\frac{5\pi}{6}\right)$

$= \pi - \frac{5\pi}{6}$ $\left[\text{since } \frac{5\pi}{6} \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]\right]$

$= \frac{\pi}{6}$

Hence,

$\sin^{-1}\left(\sin\frac{5\pi}{6}\right) = \frac{\pi}{6}$.

Q2

Evaluate the following :

$\sin^{-1}\left(\sin\frac{7\pi}{6}\right)$

Solution

$\sin^{-1}\left(\sin\frac{7\pi}{6}\right)$

$= \sin^{-1}\left(\sin\left(\pi+\frac{\pi}{6}\right)\right)$

$= \sin^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right)$

$= -\frac{\pi}{6}$

Q3

Evaluate the following :

$\sin^{-1}\left(\sin\frac{5\pi}{6}\right)$

Solution

$\sin^{-1}\left(\sin\frac{5\pi}{6}\right)$

$= \sin^{-1}\left(\sin\left(\pi - \frac{\pi}{6}\right)\right)$

$= \sin^{-1}\left(\sin\left(\frac{\pi}{6}\right)\right)$

$= \frac{\pi}{6}$

Q4

Evaluate the following :

$\sin^{-1}\left(\sin\frac{13\pi}{7}\right)$

Solution

$\sin^{-1}\left(\sin\frac{13\pi}{7}\right)$

$= \sin^{-1}\left(\sin\left(2\pi - \frac{\pi}{7}\right)\right)$

$= \sin^{-1}\left(\sin\left(-\frac{\pi}{7}\right)\right)$

$= -\frac{\pi}{7}$

Q5

Evaluate the following :

$\sin^{-1}\left(\sin\frac{17\pi}{8}\right)$

Solution

$\sin^{-1}\left(\sin\frac{17\pi}{8}\right)$

$= \sin^{-1}\left(\sin\left(2\pi+\frac{\pi}{8}\right)\right)$

$= \sin^{-1}\left(\sin\left(\frac{\pi}{8}\right)\right)$

$= \frac{\pi}{8}$

Q6

Evaluate the following :

$\sin^{-1}\left\{\left(\sin-\frac{17\pi}{8}\right)\right\}$

Solution

$\sin^{-1}\left(\sin-\frac{17\pi}{8}\right)$

$= \sin^{-1}\left(\sin\left(-2\pi-\frac{\pi}{8}\right)\right)$

$= \sin^{-1}\left(-\sin\left(\frac{\pi}{8}\right)\right)$

$= -\frac{\pi}{8}$

Q7

Evaluate the following :

$\sin^{-1}(\sin 3)$

Solution

$\sin^{-1}(\sin 3)$

$= \pi - 3 \quad \ldots\ldots\ldots\left(\because \sin^{-1}(\sin\theta) = \pi-\theta,\ \text{if } \theta \in \left[\frac{\pi}{2},\frac{3\pi}{2}\right]\right)$

Q8

Evaluate the following :

$\sin^{-1}(\sin 4)$

Solution

$\sin^{-1}(\sin 4)$

$= \pi - 4 \quad \ldots\ldots\ldots \left(\because \sin^{-1}(\sin\theta) = \pi - \theta, \text{ if } \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]\right)$

Q9

Evaluate the following :

$\sin^{-1}(\sin 12)$

Solution

$\sin^{-1}(\sin 12)$

$= -4\pi + 12 \quad \ldots\ldots\ldots \left(\because \sin^{-1}(\sin\theta) = -4\pi + \theta, \text{ if } \theta \in \left[\frac{7\pi}{2}, \frac{9\pi}{2}\right]\right)$

Q10

Evaluate $\sin^{-1}(\sin 2)$

Solution

We know that,

$$\sin^{-1}(\sin\theta) = \begin{cases} -\pi - \theta, & \text{if } \theta \in \left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right] \\ \theta, & \text{if } \theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \\ \pi - \theta, & \text{if } \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \\ -2\pi + \theta, & \text{if } \theta \in \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right] \end{cases}$$

$\therefore \sin^{-1}(\sin 2)$

$= \pi - 2 \qquad \left[\text{since } 2 \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]\right]$

Hence,

$\sin^{-1}(\sin 2) = \pi - 2.$

Q11

Evaluate $\cos^{-1}\left\{\cos\left(\frac{-\pi}{4}\right)\right\}$

Solution

We know that,

$$\cos^{-1}(\cos\theta) = \begin{cases} -\theta & , \text{ if } \theta \in [-\pi, 0] \\ \theta & , \text{ if } \theta \in [0, \pi] \\ 2\pi - \theta & , \text{ if } \theta \in [\pi, 2\pi] \\ -2\pi + \theta & , \text{ if } \theta \in [2\pi, 3\pi] \end{cases}$$

$$\therefore \cos^{-1}\left\{\cos\left(-\frac{\pi}{4}\right)\right\}$$

$$= -\left(-\frac{\pi}{4}\right) \qquad \left\{\text{since } -\frac{\pi}{4} \in [-\pi, 0]\right\}$$

$$= \frac{\pi}{4}$$

Hence,

$$\cos^{-1}\left\{\cos\left(-\frac{\pi}{4}\right)\right\} = \frac{\pi}{4}.$$

Q12

Evaluate $\cos^{-1}\left\{\cos\left(\frac{4\pi}{3}\right)\right\}$

Solution

We know that,

$$\cos^{-1}(\cos\theta) = \begin{cases} -\theta & , \text{ if } \theta \in [-\pi, 0] \\ \theta & , \text{ if } \theta \in [0, \pi] \\ 2\pi - \theta & , \text{ if } \theta \in [\pi, 2\pi] \\ -2\pi + \theta & , \text{ if } \theta \in [2\pi, 3\pi] \end{cases}$$

$$\therefore \cos^{-1}\left\{\cos\left(\frac{4\pi}{3}\right)\right\}$$

$$= 2\pi - \frac{4\pi}{3} \qquad \left\{\text{since } \frac{4\pi}{3} \in [\pi, 2\pi]\right\}$$

$$= \frac{2\pi}{3}$$

Hence,

$$\cos^{-1}\left\{\cos\left(\frac{4\pi}{3}\right)\right\} = \frac{2\pi}{3}$$

Q13

Evaluate the following :

$\cos^{-1}(\cos 3)$

Solution

$\cos^{-1}(\cos 3)$

$= 3 \quad \ldots\ldots\ldots\left(\because \cos^{-1}(\cos\theta) = \theta, \text{ if } \theta \in [0, \pi]\right)$

Q14

Evaluate the following :

$\cos^{-1}(\cos 4)$

Solution

$\cos^{-1}(\cos 4)$

$= 2\pi - 4 \quad \ldots\ldots\ldots\left(\because \cos^{-1}(\cos\theta) = 2\pi - \theta, \text{ if } \theta \in [\pi, 2\pi]\right)$

Q15

Evaluate the following :

$\cos^{-1}(\cos 5)$

Solution

$\cos^{-1}(\cos 5)$

$= 2\pi - 5 \quad \ldots\ldots\ldots\left(\because \cos^{-1}(\cos\theta) = 2\pi - \theta, \text{ if } \theta \in [\pi, 2\pi]\right)$

Q16

Evaluate the following :

$\cos^{-1}(\cos 12)$

Solution

$\cos^{-1}(\cos 12)$

$= 4\pi - 12 \quad \ldots\ldots\ldots\left(\because \cos^{-1}(\cos\theta) = 4\pi - \theta, \text{ if } \theta \in [3\pi, 4\pi]\right)$

Q17

Evaluate the following :

$\tan^{-1}\left(\tan\frac{\pi}{3}\right)$

Solution

$\tan^{-1}\left(\tan\frac{\pi}{3}\right)$

$= \frac{\pi}{3}$ ....$\left(\because \tan^{-1}(\tan\theta) = \theta, \text{ if } \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

Q18

Evaluate the following :

$\tan^{-1}\left(\tan\frac{6\pi}{7}\right)$

Solution

$\tan^{-1}\left(\tan\frac{6\pi}{7}\right)$

$= \frac{6\pi}{7} - \pi$ ....$\left(\because \tan^{-1}(\tan\theta) = \theta - \pi, \text{ if } \theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)\right)$

$= -\frac{\pi}{7}$

Q19

Evaluate $\tan^{-1}\left\{\tan\frac{7\pi}{6}\right\}$

Solution

$\tan^{-1}\left\{\tan\frac{7\pi}{6}\right\}$

We know that,

$$\tan^{-1}(\tan\theta) = \begin{cases} \pi-\theta &, \text{ if } \theta \in \left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right] \\ \theta &, \text{ if } \theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \\ \theta-\pi &, \text{ if } \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \\ \theta-2\pi &, \text{ if } \theta \in \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right] \end{cases}$$

$\therefore \tan^{-1}\left\{\tan\frac{7\pi}{6}\right\}$

$= \frac{7\pi}{6} - \pi \qquad \left\{\text{since } \frac{7\pi}{6} \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]\right\}$

$= \frac{\pi}{6}$

Hence,

$\tan^{-1}\left(\tan\frac{7\pi}{6}\right) = \frac{\pi}{6}$

Q20

Evaluate the following

$\tan^{-1}\left(\tan\frac{9\pi}{4}\right)$

Solution

$\tan^{-1}\left(\tan\frac{9\pi}{4}\right)$

$= \frac{9\pi}{4} - 2\pi \quad \ldots\ldots\ldots \left(\because \tan^{-1}(\tan\theta) = \theta - 2\pi, \text{ if } \theta \in \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)\right)$

$= \frac{\pi}{4}$

Q21

Evaluate the following :

$\tan^{-1}(\tan 1)$

Solution

$\tan^{-1}(\tan 1)$

$= 1 \quad \ldots\ldots\ldots \left(\because \tan^{-1}(\tan\theta) = \theta, \text{ if } \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

Q22

Evaluate the following :

$\tan^{-1}(\tan 2)$

Solution

$\tan^{-1}(\tan 2)$

$= 2 - \pi \quad \ldots\ldots\ldots \left(\because \tan^{-1}(\tan\theta) = \theta - \pi, \text{ if } \theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)\right)$

Q23

Evaluate the following :

$\tan^{-1}(\tan 4)$

Solution

$\tan^{-1}(\tan 4)$

$= 4 - \pi \quad \ldots\ldots\ldots \left(\because \tan^{-1}(\tan\theta) = \theta - \pi, \text{ if } \theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)\right)$

Q24

Evaluate the following :

$\tan^{-1}(\tan 12)$

Solution

$\tan^{-1}(\tan 12)$

$= 12 - 4\pi \quad \ldots\ldots\ldots \left(\because \tan^{-1}(\tan\theta) = \theta - 4\pi, \text{ if } \theta \in \left(\frac{7\pi}{2}, \frac{9\pi}{2}\right)\right)$

Q25

Evaluate the following :

$\sec^{-1}\left(\sec\frac{\pi}{3}\right)$

Solution

$$\sec^{-1}\left(\sec\frac{\pi}{3}\right)$$
$$= \sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$$
$$= \frac{\pi}{3}$$

Q26

Evaluate the following :

$$\sec^{-1}\left(\sec\frac{2\pi}{3}\right)$$

Solution

$$\sec^{-1}\left(\sec\frac{2\pi}{3}\right)$$
$$= \sec^{-1}\left(\sec\left(\frac{\pi}{2}+\frac{\pi}{6}\right)\right)$$
$$= \sec^{-1}\left(-\operatorname{cosec}\left(\frac{\pi}{6}\right)\right)$$
$$= \sec^{-1}(-2)$$
$$= \frac{2\pi}{3}$$

Q27

Evaluate the following :

$$\sec^{-1}\left(\sec\frac{5\pi}{4}\right)$$

Solution

$$\sec^{-1}\left(\sec\frac{5\pi}{4}\right)$$
$$= \sec^{-1}\left(\sec\left(\pi+\frac{\pi}{4}\right)\right)$$
$$= \sec^{-1}\left(-\sec\left(\frac{\pi}{4}\right)\right)$$
$$= \sec^{-1}(-\sqrt{2})$$
$$= \frac{3\pi}{4}$$

Q28

Evaluate the following :

$\sec^{-1}\left(\sec\frac{7\pi}{3}\right)$

Solution

$\sec^{-1}\left(\sec\frac{7\pi}{3}\right)$

$= \sec^{-1}\left(\sec\left(2\pi + \frac{\pi}{3}\right)\right)$

$= \sec^{-1}\left(\sec\left(\frac{\pi}{3}\right)\right)$

$= \sec^{-1}(2)$

$= \frac{\pi}{3}$

Q29

Evaluate the following :

$\sec^{-1}\left(\sec\frac{9\pi}{5}\right)$

Solution

$\sec^{-1}\left(\sec\frac{9\pi}{5}\right)$

$= \sec^{-1}\left(\sec\left(2\pi - \frac{\pi}{5}\right)\right)$

$= \sec^{-1}\left(\sec\left(\frac{\pi}{5}\right)\right)$

$= \frac{\pi}{5}$

Q30

Evaluate the following :

$\sec^{-1}\left\{\sec\left(-\frac{7\pi}{3}\right)\right\}$

Solution

$\sec^{-1}\left\{\sec\left(-\frac{7\pi}{3}\right)\right\}$

$= \sec^{-1}\left\{\sec\left(-2\pi-\frac{\pi}{3}\right)\right\}$

$= \sec^{-1}\left(\sec\left(\frac{\pi}{3}\right)\right)$

$= \frac{\pi}{3}$

Q31

Evaluate the following :

$\sec^{-1}\left(\sec\frac{13\pi}{4}\right)$

Solution

$\sec^{-1}\left(\sec\frac{13\pi}{4}\right)$

$= \sec^{-1}\left\{\sec\left(3\pi+\frac{\pi}{4}\right)\right\}$

$= \sec^{-1}\left(-\sec\left(\frac{\pi}{4}\right)\right)$

$= \sec^{-1}(-1)$

$= \frac{3\pi}{4}$

Q32

Evaluate the following :

$\sec^{-1}\left(\sec\frac{25\pi}{6}\right)$

Solution

$\sec^{-1}\left(\sec\frac{25\pi}{6}\right)$

$= \sec^{-1}\left\{\sec\left(4\pi+\frac{\pi}{6}\right)\right\}$

$= \sec^{-1}\left(\sec\left(\frac{\pi}{6}\right)\right)$

$= \frac{\pi}{6}$

Q33

Evaluate the following :

$$\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{\pi}{4}\right)$$

Solution

$$\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{\pi}{4}\right)$$
$$= \operatorname{cosec}^{-1}\left(\sqrt{2}\right)$$
$$= \frac{\pi}{4}$$

Q34

Evaluate the following :

$$\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{3\pi}{4}\right)$$

Solution

$$\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{3\pi}{4}\right)$$
$$= \operatorname{cosec}^{-1}\left(-\sqrt{2}\right)$$
$$= \operatorname{cosec}^{-1}\left(\sqrt{2}\right)$$
$$= \frac{\pi}{4}$$

Q35

Evaluate the following :

$$\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{6\pi}{5}\right)$$

Solution

$$\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{6\pi}{5}\right)$$
$$= \operatorname{cosec}^{-1}\left(\operatorname{cosec}\left(\pi + \frac{\pi}{5}\right)\right)$$
$$= \operatorname{cosec}^{-1}\left(-\operatorname{cosec}\left(\frac{\pi}{5}\right)\right)$$
$$= -\frac{\pi}{5}$$

Q36

Evaluate the following :

$$\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{11\pi}{6}\right)$$

Solution

$$\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{11\pi}{6}\right)$$

$$= \operatorname{cosec}^{-1}\left(\operatorname{cosec}\left(2\pi-\frac{\pi}{6}\right)\right)$$

$$= \operatorname{cosec}^{-1}\left(\operatorname{cosec}\left(-\frac{\pi}{6}\right)\right)$$

$$= \operatorname{cosec}^{-1}\left(\operatorname{cosec}\left(\frac{\pi}{6}\right)\right)$$

$$= \frac{\pi}{6}$$

Q37

Evaluate the following :

$$\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{13\pi}{6}\right)$$

Solution

$$\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{13\pi}{6}\right)$$

$$= \operatorname{cosec}^{-1}\left(\operatorname{cosec}\left(2\pi+\frac{\pi}{6}\right)\right)$$

$$= \operatorname{cosec}^{-1}\left(\operatorname{cosec}\left(\frac{\pi}{6}\right)\right)$$

$$= \frac{\pi}{6}$$

Q38

Evaluate the following :

$$\operatorname{cosec}^{-1}\left\{\operatorname{cosec}\left(-\frac{9\pi}{4}\right)\right\}$$

Solution

$\operatorname{cosec}^{-1}\left\{\operatorname{cosec}\left(-\frac{9\pi}{4}\right)\right\}$

$= \operatorname{cosec}^{-1}\left(\operatorname{cosec}\left(-2\pi - \frac{\pi}{4}\right)\right)$

$= \operatorname{cosec}^{-1}\left(-\operatorname{cosec}\left(\frac{\pi}{4}\right)\right)$

$= -\frac{\pi}{4}$

Q39

Evaluate the following :

$\cot^{-1}\left(\cot\frac{\pi}{3}\right)$

Solution

$\cot^{-1}\left(\cot\frac{\pi}{3}\right)$

$= \cot^{-1}\left(\sqrt{3}\right)$

$= \frac{\pi}{3}$

Q40

Evaluate the following :

$\cot^{-1}\left(\cot\frac{4\pi}{3}\right)$

Solution

$\cot^{-1}\left(\cot\frac{4\pi}{3}\right)$

$= \cot^{-1}\left(\cot\left(\pi + \frac{\pi}{3}\right)\right)$

$= \cot^{-1}\left(\cot\left(\frac{\pi}{3}\right)\right)$

$= \cot^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$= \frac{\pi}{3}$

Q41

Evaluate the following :

$$\cot^{-1}\left(\cot\frac{9\pi}{4}\right)$$

Solution

$$\cot^{-1}\left(\cot\frac{9\pi}{4}\right)$$
$$= \cot^{-1}\left(\cot\left(2\pi + \frac{\pi}{4}\right)\right)$$
$$= \cot^{-1}\left(\cot\left(\frac{\pi}{4}\right)\right)$$
$$= \frac{\pi}{4}$$

Q42

Evaluate the following :

$$\cot^{-1}\left(\cot\frac{19\pi}{6}\right)$$

Solution

$$\cot^{-1}\left(\cot\frac{19\pi}{6}\right)$$
$$= \cot^{-1}\left(\cot\left(3\pi + \frac{\pi}{6}\right)\right)$$
$$= \cot^{-1}\left(\cot\left(\frac{\pi}{6}\right)\right)$$
$$= \frac{\pi}{6}$$

Q43

Evaluate the following :

$$\cot^{-1}\left\{\cot\left(-\frac{8\pi}{3}\right)\right\}$$

Solution

$\cot^{-1}\left\{\cot\left(-\frac{8\pi}{3}\right)\right\}$

$\cot^{-1}\left\{-\cot\left(\frac{8\pi}{3}\right)\right\}$

$= \cot^{-1}\left\{-\cot\left(3\pi - \frac{\pi}{3}\right)\right\}$

$= \cot^{-1}\left(-\cot\left(-\frac{\pi}{3}\right)\right)$

$= \cot^{-1}\left(\cot\left(\frac{\pi}{3}\right)\right)$

$= \frac{\pi}{3}$

Q44

Evaluate the following :

$\cot^{-1}\left\{\cot\left(\frac{21\pi}{4}\right)\right\}$

Solution

$\cot^{-1}\left\{\cot\left(\frac{21\pi}{4}\right)\right\}$

$\cot^{-1}\left\{\cot\left(5\pi + \frac{\pi}{4}\right)\right\}$

$= \cot^{-1}\left\{\cot\left(\frac{\pi}{4}\right)\right\}$

$= \frac{\pi}{4}$

Q45

Write $\cot^{-1}\frac{a}{\sqrt{x^2-a^2}}$, $|x| > a$ in the simplest form.

Solution

$\cot^{-1}\frac{a}{\sqrt{x^2-a^2}}, \ |x| > a$

Let, $x = a\sec\theta$

$$\cot^{-1}\left(\frac{a}{\sqrt{a^2\sec^2\theta - a^2}}\right)$$
$$= \cot^{-1}\left(\frac{a}{\sqrt{a^2\left(\sec^2\theta - 1\right)}}\right)$$
$$= \cot^{-1}\frac{1}{\sqrt{\tan^2\theta}} \qquad \left\{\text{Since, } \sec^2\theta - 1 = \tan^2\theta\right\}$$
$$= \cot^{-1}\left(\cot\theta\right)$$
$$= \theta$$
$$= \sec^{-1}\left(\frac{x}{a}\right)$$

Hence,

$$\cot^{-1}\left(\frac{a}{\sqrt{x^2-a^2}}\right) = \sec^{-1}\left(\frac{x}{a}\right)$$

Q46

Write $\tan^{-1}\left\{x+\sqrt{1+x^2}\right\}$ in the simplest form

Solution

$\tan^{-1}\left\{x+\sqrt{1+x^2}\right\}$

Let, $x = \cot\theta$

$\tan^{-1}\left[\cot + \sqrt{1+\cot^2\theta}\right]$

$= \tan^{-1}\left[\cot\theta + \sqrt{\operatorname{cosec}^2\theta}\right]$ $\left\{\text{Since, } 1+\cot^2\theta = \operatorname{cosec}^2\theta\right\}$

$= \tan^{-1}\left[\cot\theta + \operatorname{cosec}\theta\right]$

$= \tan^{-1}\left[\frac{1+\cos\theta}{\sin\theta}\right]$ $\left\{\text{Since, } \cot\theta = \frac{\cos\theta}{\sin\theta}, \ \operatorname{cosec}\theta = \frac{1}{\sin\theta}\right\}$

$= \tan^{-1}\left\{\frac{2\cos^2\frac{\theta}{2}}{2\cos\frac{\theta}{2}\sin\frac{\theta}{2}}\right\}$ $\left\{\text{Since, } 1+\cos\theta = 2\cos^2\frac{\theta}{2}, \ \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right\}$

$= \tan^{-1}\left\{\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right\}$

$= \tan^{-1}\left\{\cot\frac{\theta}{2}\right\}$ $\left\{\text{Since, } \cot\theta = \frac{\cos\theta}{\sin\theta}\right\}$

$= \tan^{-1}\left\{\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right\}$ $\left\{\text{Since, } \cot\theta = \tan\left(\frac{\pi}{2}-\theta\right)\right\}$

$= \frac{\pi}{2}-\frac{\theta}{2}$

$= \frac{\pi}{2}-\frac{1}{2}\cot^{-1}x$ $\left\{\text{Since, } \cot\theta = x \Rightarrow \theta = \cot^{-1}x\right\}$

Hence,

$\tan^{-1}\left[x+\sqrt{1+x^2}\right] = \frac{\pi}{2}-\frac{1}{2}\cot^{-1}x$

Q47

Write $\tan^{-1}\left\{\sqrt{1+x^2}-x\right\}$, $x \in R$ in the simplest form.

Solution

$\tan^{-1}\left\{\sqrt{1+x^2}-x\right\},\ x \in R$

Let, $x = \cot\theta$

$$\tan^{-1}\left\{\sqrt{1+\cot^2\theta}-\cot\theta\right\}$$

$$= \tan^{-1}\left(\operatorname{cosec}\theta - \cot\theta\right) \qquad \left\{\text{Since, } 1+\cot^2\theta = \operatorname{cosec}^2\theta\right\}$$

$$= \tan^{-1}\left\{\frac{1-\cos\theta}{\sin\theta}\right\} \qquad \left\{\text{Since, } \operatorname{cosec}\theta = \frac{1}{\sin\theta},\ \cot\theta = \frac{\cos\theta}{\sin\theta}\right\}$$

$$= \tan^{-1}\left\{\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right\} \qquad \left\{\text{Since, } 1-\cos\theta = 2\sin^2\frac{\theta}{2},\ \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right\}$$

$$= \tan^{-1}\left\{\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right\}$$

$$= \tan^{-1}\left\{\tan\frac{\theta}{2}\right\} \qquad \left\{\text{Since, } \tan\theta = \frac{\sin\theta}{\cos\theta}\right\}$$

$$= \frac{\theta}{2}$$

$$= \frac{1}{2}\cot^{-1}x \qquad \left\{\text{Since, } \cot\theta = x \Rightarrow \theta = \cot^{-1}x\right\}$$

Hence,

$$\tan^{-1}\left\{\sqrt{1+x^2}-x\right\} = \frac{1}{2}\cot^{-1}x$$

Q48

Write $\tan^{-1}\left\{\frac{\sqrt{1+x^2}-1}{x}\right\},\ x \neq 0$ in the simplest form.

Solution

$\tan^{-1}\left\{\frac{\sqrt{1+x^2}-1}{x}\right\}, x \neq 0$

Let, $x = \tan\theta$

$$\tan^{-1}\left\{\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right\}$$

$$= \tan^{-1}\left\{\frac{\sec\theta-1}{\tan\theta}\right\} \quad \left\{\text{Since, } 1+\tan^2\theta = \sec^2\theta\right\}$$

$$= \tan^{-1}\left\{\frac{1-\cos\theta}{\sin\theta}\right\} \quad \left\{\text{Since, } \sec\theta = \frac{1}{\cos\theta},\ \tan\theta = \frac{\sin\theta}{\cos\theta}\right\}$$

$$= \tan^{-1}\left\{\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right\} \quad \left\{\text{Since, } 1-\cos\theta = 2\sin^2\frac{\theta}{2},\ \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right\}$$

$$= \tan^{-1}\left\{\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right\}$$

$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right) \quad \left\{\text{Since, } \frac{\sin\theta}{\cos\theta} = \tan\theta\right\}$$

$$= \frac{\theta}{2}$$

$$= \frac{1}{2}\tan^{-1}x \quad \left\{\text{Since, } \tan\theta = x \Rightarrow \theta = \tan^{-1}x\right\}$$

Hence,

$$\tan^{-1}\left\{\frac{\sqrt{1+x^2}-1}{x}\right\} = \frac{1}{2}\tan^{-1}x$$

Q49

Write $\tan^{-1}\left\{\frac{\sqrt{1+x^2}+1}{x}\right\}, x \neq 0$ in the simplest form.

Solution

$$\tan^{-1}\left\{\frac{\sqrt{1+x^2}+1}{x}\right\},\ x \neq 0$$

Let, $x = \tan\theta$

$$\tan^{-1}\left\{\frac{\sqrt{1+\tan^2\theta}+1}{\tan\theta}\right\}$$

$$= \tan^{-1}\left\{\frac{\sec\theta+1}{\tan\theta}\right\} \quad \left\{\text{Since, } 1+\tan^2\theta = \sec^2\theta\right\}$$

$$= \tan^{-1}\left\{\frac{1+\cos\theta}{\sin\theta}\right\} \quad \left\{\text{Since, } \sec\theta = \frac{1}{\cos\theta},\ \tan\theta = \frac{\sin\theta}{\cos\theta}\right\}$$

$$= \tan^{-1}\left\{\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right\} \quad \left\{\text{Since, } 1+\cos\theta = 2\cos^2\frac{\theta}{2},\ 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \sin\theta\right\}$$

$$= \tan^{-1}\left\{\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right\}$$

$$= \tan^{-1}\left\{\cot\frac{\theta}{2}\right\} \quad \left\{\text{Since, } \cot\theta = \frac{\cos\theta}{\sin\theta}\right\}$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right\} \quad \left\{\text{Since, } \cot\theta = \tan\left(\frac{\pi}{2}-\theta\right)\right\}$$

$$= \frac{\pi}{2}-\frac{\theta}{2}$$

$$= \frac{\pi}{2}-\frac{1}{2}\tan^{-1}x \quad \left\{\text{Since, } \tan\theta = x \Rightarrow \theta = \tan^{-1}x\right\}$$

$$\tan^{-1}\left\{\frac{\sqrt{1+x^2}+1}{x}\right\} = \frac{\pi}{2}-\frac{1}{2}\tan^{-1}x$$

Q50

Write $\tan^{-1}\sqrt{\frac{a-x}{a+x}},\ -a < x < a$ in the simplest form.

Solution

$\tan^{-1}\sqrt{\frac{a-x}{a+x}},\ -a < x < a$

Let, $x = a\cos\theta$

$$\tan^{-1}\sqrt{\frac{a-a\cos\theta}{a+a\cos\theta}}$$

$$= \tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

$$= \tan^{-1}\sqrt{\frac{\frac{2\sin^2\theta}{2}}{\frac{2\cos^2\theta}{2}}} \qquad \left\{\text{Since, } 1-\cos\theta = \frac{2\sin^2\theta}{2},\ 1+\cos\theta = \frac{2\cos^2\theta}{2}\right\}$$

$$= \tan^{-1}\left(\frac{\frac{\sin\theta}{2}}{\frac{\cos\theta}{2}}\right)$$

$$= \tan^{-1}\left(\frac{\tan}{2}\right) \qquad \left\{\text{Since, } \frac{\sin}{\cos} = \tan\right\}$$

$$= \frac{\theta}{2}$$

$$= \frac{1}{2}\cos^{-1}\left(\frac{x}{a}\right) \qquad \left\{\text{Since, } x = a\cos\theta \Rightarrow \theta = \cos^{-1}\left(\frac{x}{a}\right)\right\}$$

Hence,

$$\tan^{-1}\sqrt{\frac{a-x}{a+x}} = \frac{1}{2}\cos^{-1}\left(\frac{x}{a}\right)$$

Q51

Write $\tan^{-1}\left\{\frac{x}{a+\sqrt{a^2-x^2}}\right\},\ -a < x < a$ in the simplest form.

Solution

$$\tan^{-1}\left\{\frac{x}{a+\sqrt{a^2-x^2}}\right\}, \quad -a<x<a$$

Let, $\quad x = a\sin\theta$

$$\tan^{-1}\left\{\frac{a\sin\theta}{1+\sqrt{a^2-a^2\sin^2\theta}}\right\}$$

$$= \tan^{-1}\left\{\frac{a\sin\theta}{a+a\sqrt{1-\sin^2\theta}}\right\}$$

$$= \tan^{-1}\left\{\frac{a\sin\theta}{a(1+\cos\theta)}\right\} \qquad \left\{\text{Since, } 1-\sin^2\theta=\cos^2\theta\right\}$$

$$= \tan^{-1}\left\{\frac{\sin\theta}{1+\cos\theta}\right\}$$

$$= \tan^{-1}\left\{\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\right\} \qquad \left\{\text{Since, } \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2},\ 1+\cos\theta = 2\cos^2\frac{\theta}{2}\right\}$$

$$= \tan^{-1}\left\{\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right\}$$

$$= \tan^{-1}\left\{\tan\frac{\theta}{2}\right\} \qquad \left\{\text{Since, } \frac{\sin\theta}{\cos\theta} = \tan\theta\right\}$$

$$= \frac{\theta}{2}$$

$$= \frac{1}{2}\sin^{-1}x \qquad \left\{\text{Since, } x = a\sin\theta \Rightarrow \theta = \sin^{-1}\left(\frac{x}{a}\right)\right\}$$

Hence,

$$\tan^{-1}\left\{\frac{x}{a+\sqrt{a^2-x^2}}\right\} = \frac{1}{2}\sin^{-1}\frac{x}{a}$$

Q52

Write each of the following in the simplest form:

$$\sin-1\left\{\frac{x+\sqrt{1-x2}}{\sqrt{2}}\right\}, -\frac{1}{2} < x < \frac{1}{\sqrt{2}}$$

Solution

$\sin^{-1}\left\{\dfrac{x+\sqrt{1-x^2}}{\sqrt{2}}\right\}$

Let, $x = \sin\theta$

$$\sin^{-1}\left\{\frac{\sin\theta+\sqrt{1-\sin^2\theta}}{\sqrt{2}}\right\}$$

$$= \sin^{-1}\left\{\frac{\sin\theta+\cos\theta}{\sqrt{2}}\right\} \qquad \{\text{Since, } 1-\sin^2\theta = \cos^2\theta\}$$

$$= \sin^{-1}\left\{\frac{1}{\sqrt{2}}\sin\theta+\frac{1}{\sqrt{2}}\cos\theta\right\}$$

$$= \sin^{-1}\left\{\sin\theta\cos\frac{\pi}{4}+\cos\theta\sin\frac{\pi}{4}\right\}$$

$$= \sin^{-1}\left\{\sin\left(\theta+\frac{\pi}{4}\right)\right\} \qquad \{\text{Since, } \sin x\cos y+\cos x\sin y = \sin(x+y)\}$$

$$= \theta+\frac{\pi}{4}$$

$$= \frac{\pi}{4}+\sin^{-1}x \qquad \{\text{Since, } \sin\theta = x \Rightarrow \theta = \sin^{-1}x\}$$

Hence,

$$\sin^{-1}\left\{\frac{x+\sqrt{1-x^2}}{\sqrt{2}}\right\} = \frac{\pi}{4}+\sin^{-1}x$$

Q53

Write $\sin^{-1}\left\{\dfrac{\sqrt{1+x}+\sqrt{1-x}}{2}\right\}$, $0 < x < 1$ in the simplest form.

Solution

$\sin^{-1}\left\{\frac{\sqrt{1+x}+\sqrt{1-x}}{2}\right\}, \quad 0 < x < 1$

Let, $x = \cos 2\theta$

$$\sin^{-1}\left\{\frac{\sqrt{1+\cos^2\theta}+\sqrt{1-\cos^2\theta}}{2}\right\}$$

$$= \sin^{-1}\left\{\frac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{2}\right\} \quad \left\{\text{Since, } 1+\cos^2\theta = 2\cos^2\theta, \; 1-\cos^2\theta = 2\sin^2\theta\right\}$$

$$= \sin^{-1}\left\{\frac{\sqrt{2}\cos\theta}{2}+\frac{\sqrt{2}\sin\theta}{2}\right\}$$

$$= \sin^{-1}\left\{\frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta\right\}$$

$$= \sin^{-1}\left\{\sin\frac{\pi}{4}\cos\theta+\cos\frac{\pi}{4}\sin\theta\right\}$$

$$= \sin^{-1}\left\{\sin\left(\frac{\pi}{4}+\theta\right)\right\} \quad \left\{\text{Since, } \sin x\cos y+\cos x\sin y = \sin(x+y)\right\}$$

$$= \frac{\pi}{4}+\theta$$

$$= \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x \quad \left\{\text{Since, } \cos 2\theta = x \Rightarrow \theta = \frac{1}{2}\cos^{-1}x\right\}$$

Hence,

$$\sin^{-1}\left\{\frac{\sqrt{1+x}+\sqrt{1-x}}{2}\right\} = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x$$

Q54

Write $\sin\left\{2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right\}$ in the simplest form.

Solution

$$\sin\left\{2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right\}$$

$$= \sin\left\{\sin^{-1}\left(\frac{2\sqrt{\frac{1-x}{1+x}}}{1+\left(\sqrt{\frac{1-x}{1+x}}\right)^2}\right)\right\} \qquad \left\{\text{Since, } 2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}\right\}$$

$$= \sin\left\{\sin^{-1}\left(\frac{2\sqrt{\frac{1-x}{1+x}}}{\frac{1+x+1-x}{1+x}}\right)\right\}$$

$$= 2\sqrt{\frac{1-x}{1+x}} \times \frac{1+x}{2}$$

$$= \sqrt{1-x}\sqrt{1+x}$$

$$= \sqrt{1-x^2}$$

Hence,

$$\sin\left\{2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right\} = \sqrt{1-x^2}$$