Exercise 4.13

Q1

If $\cos^{-1}\frac{x}{2}+\cos^{-1}\frac{y}{3}=\alpha$, then prove that $9x^2-12xy\cos\alpha+4y^2=36\sin^2\alpha$.

Solution

Given

$$\cos^{-1}\frac{x}{2}+\cos^{-1}\frac{y}{3}=\alpha$$

$$\cos\left[\frac{x}{2}\times\frac{y}{3}-\sqrt{1-\left(\frac{x}{2}\right)^2}\sqrt{1-\left(\frac{y}{3}\right)^2}\right]=\alpha \qquad \left\{\text{Since } \cos^{-1}x+\cos^{-1}y=\cos^{-1}\left[xy-\sqrt{1-x^2}\sqrt{1-y^2}\right]\right\}$$

$$\frac{xy}{6}-\frac{\sqrt{4-x^2}}{2}\frac{\sqrt{9-y^2}}{3}=\cos\alpha$$

$$xy-\sqrt{4-x^2}\sqrt{9-y^2}=6\cos\alpha$$

$$xy-6\cos\alpha=\sqrt{4-x^2}\sqrt{9-y^2}$$

Squaring both the sides,

$$(xy-6\cos\alpha)^2=\left(4-x^2\right)\left(9-y^2\right)$$

$$x^2y^2+36\cos^2\alpha-12xy\cos\alpha=36-9x^2-4y^2+x^2y^2$$

$$9x^2+4y^2-x^2y^2-36+x^2y^2+36\cos^2\alpha-12xy\cos\alpha=0$$

$$9x^2+4y^2-12xy\cos\alpha-36\left(1-\cos^2\alpha\right)=0$$

$$9x^2+4y^2-12xy\cos\alpha-36\sin^2\alpha=0$$

$$9x^2+4y^2-12xy\cos\alpha=36\sin^2\alpha$$

Q2

Solve the equation:

$$\cos^{-1}\frac{a}{x}-\cos^{-1}\frac{b}{x}=\cos^{-1}\frac{1}{b}-\cos^{-1}\frac{1}{a}$$

Solution

$$\cos^{-1}\frac{a}{x}-\cos^{-1}\frac{b}{x}=\cos^{-1}\frac{1}{b}-\cos^{-1}\frac{1}{a}$$
$$\cos^{-1}\frac{a}{x}+\cos^{-1}\frac{1}{a}=\cos^{-1}\frac{b}{x}+\cos^{-1}\frac{1}{b}$$
$$\cos^{-1}\left[\frac{1}{x}-\sqrt{1-\frac{a^2}{x^2}}\sqrt{1-\frac{1}{a^2}}\right]=\cos^{-1}\left[\frac{1}{x}-\sqrt{1-\frac{b^2}{x^2}}\sqrt{1-\frac{1}{b^2}}\right]$$
$$\frac{1}{x}-\sqrt{1-\frac{a^2}{x^2}}\sqrt{1-\frac{1}{a^2}}=\frac{1}{x}-\sqrt{1-\frac{b^2}{x^2}}\sqrt{1-\frac{1}{b^2}}$$
$$\sqrt{1-\frac{a^2}{x^2}}\sqrt{1-\frac{1}{a^2}}=\sqrt{1-\frac{b^2}{x^2}}\sqrt{1-\frac{1}{b^2}}$$
$$\left(1-\frac{a^2}{x^2}\right)\left(1-\frac{1}{a^2}\right)=\left(1-\frac{b^2}{x^2}\right)\left(1-\frac{1}{b^2}\right)$$
$$1-\frac{1}{a^2}-\frac{a^2}{x^2}+\frac{1}{x^2}=1-\frac{1}{b^2}-\frac{b^2}{x^2}+\frac{1}{x^2}$$
$$\frac{b^2}{x^2}-\frac{a^2}{x^2}=\frac{1}{a^2}-\frac{1}{b^2}$$
$$\left(b^2-a^2\right)a^2b^2=x^2\left(b^2-a^2\right)$$
$$x^2=a^2b^2$$
$$x=ab$$

Q3

Solve :

$$\cos^{-1}\sqrt{3}x+\cos^{-1}x=\frac{\pi}{2}$$

Solution

$$\cos^{-1}\sqrt{3}x+\cos^{-1}x=\frac{\pi}{2}$$
$$\cos^{-1}\left[\sqrt{3}x^2-\sqrt{1-3x^2}\sqrt{1-x^2}\right]=\frac{\pi}{2}$$
$$\sqrt{3}x^2-\sqrt{1-3x^2}\sqrt{1-x^2}=0$$
$$\sqrt{3}x^2=\sqrt{1-3x^2}\sqrt{1-x^2}$$
$$3x^4=1-x^2-3x^2+3x^4$$
$$4x^2-1=0$$
$$x^2=\frac{1}{4}$$
$$x=\pm\frac{1}{2}$$

Q4

Prove that:

$$\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}=\cos^{-1}\frac{33}{65}$$

Solution

$$\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}$$
$$=\cos^{-1}\left[\frac{4}{5}\times\frac{12}{13}-\sqrt{1-\frac{16}{25}}\sqrt{1-\frac{144}{169}}\right]$$
$$=\cos^{-1}\left[\frac{48}{65}-\frac{3}{5}\times\frac{5}{13}\right]$$
$$=\cos^{-1}\left[\frac{48}{65}-\frac{15}{65}\right]$$
$$=\cos^{-1}\left[\frac{33}{65}\right]$$