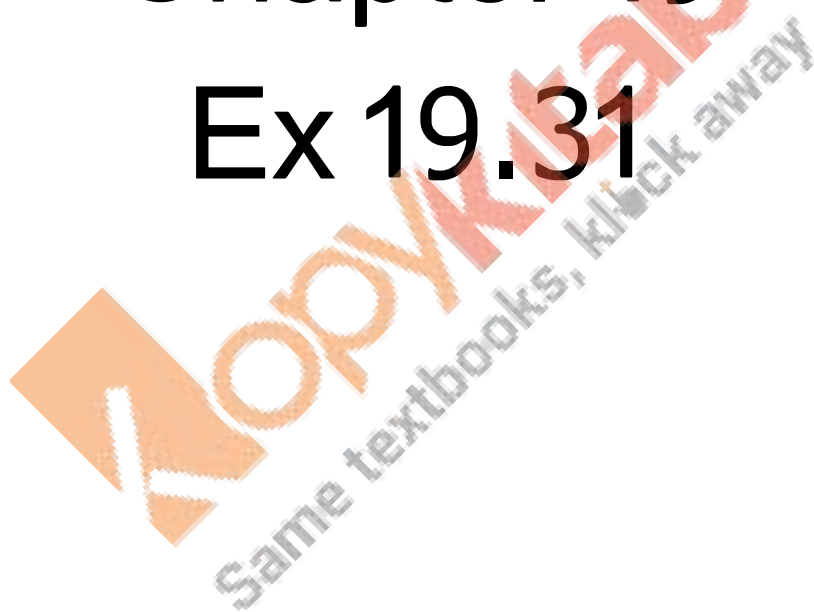


RD Sharma
Solutions Class
12 Maths
Chapter 19
Ex 19.31



Indefinite Integrals Ex 19.31 Q1

$$\begin{aligned} I &= \int \frac{x^2 + 1}{x^4 + x^2 + 1} dx \\ &= \int \frac{1 + \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx \end{aligned}$$

Dividing numerator and denominator by x^2

$$= \int \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x - \frac{1}{x}\right)^2 + 3} dx$$

$$\text{Let } x - \frac{1}{x} = t \Rightarrow \left(1 + \frac{1}{x^2}\right) dx = dt$$

$$\begin{aligned} \therefore I &= \int \frac{dt}{t^2 + 3} \\ &= \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{t}{\sqrt{3}} \right) + c \end{aligned}$$

$$\therefore I = \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{x^2 - 1}{\sqrt{3}x} \right) + c$$

Indefinite Integrals Ex 19.31 Q2

$$\int \sqrt{\cot \theta} d\theta$$

$$\text{Let } \cot \theta = x^2$$

$$\Rightarrow -\operatorname{cosec}^2 \theta d\theta = 2x dx$$

$$\begin{aligned} \Rightarrow d\theta &= \frac{-2x}{\operatorname{cosec}^2 \theta} dx \\ &= \frac{-2x}{1 + \cot^2 \theta} dx \\ &= \frac{-2x}{1 + x^4} dx \end{aligned}$$

$$\begin{aligned} \therefore I &= -\int \frac{2x^2}{1 + x^4} dx \\ &= -\int \frac{2}{\frac{1}{x^2} + x^2} dx \end{aligned}$$

Dividing numerator and denominator by x^2

$$\begin{aligned} &= -\int \frac{1 + \frac{1}{x^2} + 1 - \frac{1}{x^2}}{x^2 + \frac{1}{x^2}} dx \\ &= -\int \frac{\left(1 + \frac{1}{x^2}\right) dx}{\left(x - \frac{1}{x}\right)^2 + 2} - \int \frac{\left(1 - \frac{1}{x^2}\right) dx}{\left(x + \frac{1}{x}\right)^2 - 2} \end{aligned}$$

$$\text{Let } x - \frac{1}{x} = t \Rightarrow \left(1 + \frac{1}{x^2}\right) dx = dt$$

$$\text{and } x + \frac{1}{x} = z \Rightarrow \left(1 - \frac{1}{x^2}\right) dx = dz$$

$$\begin{aligned} \Rightarrow I &= -\int \frac{dt}{t^2 + 2} - \int \frac{dz}{z^2 - 2} \\ &= -\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{t}{\sqrt{2}} \right) - \frac{1}{2\sqrt{2}} \log \left| \frac{z - \sqrt{2}}{z + \sqrt{2}} \right| + c \\ &= -\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x^2 - 1}{\sqrt{2}x} \right) - \frac{1}{2\sqrt{2}} \log \left| \frac{x^2 + 1 - \sqrt{2}x}{x^2 + 1 + \sqrt{2}x} \right| + c \\ I &= -\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\cot \theta - 1}{\sqrt{2} \cot \theta} \right) - \frac{1}{2\sqrt{2}} \log \left| \frac{\cot \theta + 1 - \sqrt{2} \cot \theta}{\cot \theta + 1 + \sqrt{2} \cot \theta} \right| + c \end{aligned}$$

$$\text{Let } I = \int \frac{x^2 + 9}{x^4 + 81} dx$$

Dividing numerator and denominator by x^2

$$\begin{aligned} I &= \int \frac{1 + \frac{9}{x^2}}{x^2 + \frac{81}{x^2}} dx \\ &= \int \frac{1 + \frac{9}{x^2}}{\left(x - \frac{9}{x}\right)^2 + 18} dx \end{aligned}$$

$$\text{Let } \left(x - \frac{9}{x}\right) = t \Rightarrow \left(1 + \frac{9}{x^2}\right) dx = dt$$

$$\begin{aligned} \therefore I &= \int \frac{dt}{t^2 + 18} \\ \Rightarrow I &= \frac{1}{3\sqrt{2}} \tan^{-1} \left(\frac{t}{3\sqrt{2}} \right) + c \end{aligned}$$

Thus,

$$I = \frac{1}{3\sqrt{2}} \tan^{-1} \left(\frac{x^2 - 9}{3\sqrt{2}x} \right) + c$$

$$\text{Let } I = \int \frac{1}{x^4 + x^2 + 1} dx$$

Dividing numerator and denominator by x^2

$$I = \int \frac{\frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx$$

$$= \frac{1}{2} \int \frac{1 + \frac{1}{x^2} - 1 + \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx$$

$$= \frac{1}{2} \left[\int \frac{\left(1 + \frac{1}{x^2}\right) dx}{\left(x - \frac{1}{x}\right)^2 + 3} - \int \frac{\left(1 - \frac{1}{x^2}\right) dx}{\left(x + \frac{1}{x}\right)^2 - 1} \right]$$

$$\text{Let } \left(x - \frac{1}{x}\right) = t$$

$$\Rightarrow \left(1 + \frac{1}{x^2}\right) dx = dt$$

$$\text{and } \left(x + \frac{1}{x}\right) = z$$

$$\Rightarrow \left(1 - \frac{1}{x^2}\right) dx = dz$$

$$\therefore I = \frac{1}{2} \int \frac{dt}{t^2 + 3} - \frac{1}{2} \int \frac{dz}{z^2 - 1}$$

$$\Rightarrow I = \frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{t}{\sqrt{3}} \right) - \frac{1}{4} \log \left| \frac{z-1}{z+1} \right| + c$$

$$\Rightarrow I = \frac{1}{2\sqrt{3}} \tan^{-1} \left(\frac{x^2 - 1}{\sqrt{3}x} \right) - \frac{1}{4} \log \left| \frac{x^2 + 1 - x}{x^2 + 1 + x} \right| + c$$

$$\text{Let } I = \int \frac{x^2 - 3x + 1}{x^4 + x^2 + 1} dx$$

Dividing numerator and denominator by x^2

$$\begin{aligned} \therefore I &= \int \frac{1 - \frac{3}{x} + \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx \\ &= \int \frac{\left(1 + \frac{1}{x^2}\right) dx}{\left(x - \frac{1}{x}\right)^2 + 3} - \int \frac{3x}{x^4 + x^2 + 1} dx \end{aligned}$$

$$\text{Let } \left(x - \frac{1}{x}\right) = t$$

$$\Rightarrow \left(1 + \frac{1}{x^2}\right) dx = dt \quad [\text{For Ist part}]$$

$$\text{Let } x^2 = z$$

$$\Rightarrow 2x dx = dz \quad [\text{For IInd part}]$$

$$\therefore I = \int \frac{dt}{t^2 + 3} - \frac{3}{2} \int \frac{dz}{z^2 + z + 1}$$

$$\Rightarrow = \int \frac{dt}{t^2 + 3} - \frac{3}{2} \int \frac{dz}{\left(z + \frac{1}{2}\right)^2 + \frac{3}{4}}$$

$$\Rightarrow = \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{t}{\sqrt{3}} \right) - \frac{3}{2} \times \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{z + \frac{1}{2}}{\frac{\sqrt{3}}{2}} \right) + c$$

$$\Rightarrow I = \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{x^2 - 1}{\sqrt{3}x} \right) - \sqrt{3} \tan^{-1} \left(\frac{2z + 1}{\sqrt{3}} \right) + c$$

Hence,

$$I = \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{x^2 - 1}{\sqrt{3}x} \right) - \sqrt{3} \tan^{-1} \left(\frac{2x^2 + 1}{\sqrt{3}} \right) + c$$

$$\text{Let } I = \int \frac{x^2 + 1}{x^4 - x^2 + 1} dx$$

Dividing numerator and denominator by x^2

$$\begin{aligned} \therefore I &= \int \frac{\left(1 + \frac{1}{x^2}\right)}{x^2 - 1 + \frac{1}{x^2}} dx \\ &= \int \frac{\left(1 + \frac{1}{x^2}\right) dx}{\left(x - \frac{1}{x}\right)^2 + 1} \end{aligned}$$

$$\text{Let } \left(x - \frac{1}{x}\right) = t$$

$$\Rightarrow \left(1 + \frac{1}{x^2}\right) dx = dt$$

$$\begin{aligned} \Rightarrow I &= \int \frac{dt}{t^2 + 1} \\ &= \tan^{-1} t + c \end{aligned}$$

$$\therefore I = \tan^{-1} \left(\frac{x^2 - 1}{x}\right) + c$$

Indefinite Integrals Ex 19.31 Q7

$$\text{Let } I = \int \frac{x^2 - 1}{x^4 + 1} dx$$

Dividing numerator and denominator by x^2

$$\begin{aligned} \therefore I &= \int \frac{\left(1 - \frac{1}{x^2}\right)}{x^2 + \frac{1}{x^2}} dx \\ &= \int \frac{\left(1 - \frac{1}{x^2}\right)}{\left(x + \frac{1}{x}\right)^2 - 2} dx \end{aligned}$$

$$\text{Let } \left(x + \frac{1}{x}\right) = t$$

$$\Rightarrow \left(1 - \frac{1}{x^2}\right) dx = dt$$

$$\begin{aligned} \therefore I &= \int \frac{dt}{t^2 - 2} \\ &= \frac{1}{2\sqrt{2}} \log \left| \frac{t - \sqrt{2}}{t + \sqrt{2}} \right| + c \end{aligned}$$

So,

$$I = \frac{1}{2\sqrt{2}} \log \left| \frac{x^2 + 1 - \sqrt{2}x}{x^2 + 1 + \sqrt{2}x} \right| + c$$

Indefinite Integrals Ex 19.31 Q8

$$\text{Let } I = \int \frac{x^2 + 1}{x^4 + 7x^2 + 1} dx$$

Dividing numerator and denominator by x^2

$$\begin{aligned}\therefore I &= \int \frac{\left(1 + \frac{1}{x^2}\right)}{x^2 + 7 + \frac{1}{x^2}} dx \\ &= \int \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x - \frac{1}{x}\right)^2 + 9} dx\end{aligned}$$

$$\text{Let } \left(x - \frac{1}{x}\right) = t$$

$$\Rightarrow \left(1 + \frac{1}{x^2}\right) dx = dt$$

$$\begin{aligned}\therefore I &= \int \frac{dt}{t^2 + 9} \\ &= \frac{1}{3} \tan^{-1} \left| \frac{t}{3} \right| + c\end{aligned}$$

Hence,

$$I = \frac{1}{3} \tan^{-1} \left(\frac{x^2 - 1}{3x} \right) + c$$

$$\begin{aligned}\text{Let } I &= \int \frac{(x-1)^2}{x^4+x^2+1} dx \\ &= \int \frac{x^2-2x+1}{x^4+x^2+1} dx\end{aligned}$$

Dividing numerator and denominator by x^2

$$\begin{aligned}\therefore I &= \int \frac{1 - \frac{2}{x} + \frac{1}{x^2}}{x^2 + 1 + \frac{1}{x^2}} dx \\ &= \int \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x - \frac{1}{x}\right)^2 + 3} dx - \int \frac{2x}{x^4 + x^2 + 1} dx\end{aligned}$$

$$\text{Let } \left(x - \frac{1}{x}\right) = t$$

$$\Rightarrow \left(1 + \frac{1}{x^2}\right) dx = dt \quad \text{[For 1st part]}$$

$$\text{Let } x^2 = z$$

$$\Rightarrow 2x dx = dz \quad \text{[For 2nd part]}$$

$$\begin{aligned}\therefore I &= \int \frac{dt}{t^2+3} - \int \frac{dz}{z^2+z+1} \\ &= \int \frac{dt}{t^2+3} - \int \frac{dz}{\left(z + \frac{1}{2}\right)^2 + \frac{3}{4}}\end{aligned}$$

$$\Rightarrow I = \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{t}{\sqrt{3}} \right) - \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{z + \frac{1}{2}}{\frac{\sqrt{3}}{2}} \right) + c$$

$$\Rightarrow I = \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{x^2 - 1}{\sqrt{3}x} \right) - \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2z + 1}{\sqrt{3}} \right) + c$$

Hence,

$$I = \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{x^2 - 1}{\sqrt{3}x} \right) - \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2x^2 + 1}{\sqrt{3}} \right) + c$$

$$\text{Let } I = \int \frac{1}{x^4 + 3x^2 + 1} dx$$

Dividing numerator and denominator by x^2

$$\begin{aligned} \therefore I &= \int \frac{\frac{1}{x^2}}{x^2 + 3 + \frac{1}{x^2}} dx \\ &= \frac{1}{2} \int \frac{\left(1 + \frac{1}{x^2}\right) - \left(1 - \frac{1}{x^2}\right)}{x^2 + 3 + \frac{1}{x^2}} dx \\ &= \frac{1}{2} \int \frac{\left(1 + \frac{1}{x^2}\right)}{\left(x - \frac{1}{x}\right)^2 + 5} dx - \frac{1}{2} \int \frac{\left(1 - \frac{1}{x^2}\right)}{\left(x + \frac{1}{x}\right)^2 + 1} dx \end{aligned}$$

$$\text{Let } \left(x - \frac{1}{x}\right) = t$$

$$\Rightarrow \left(1 + \frac{1}{x^2}\right) dx = dt$$

$$\text{and } x + \frac{1}{x} = z$$

$$\Rightarrow \left(1 - \frac{1}{x^2}\right) dx = dz$$

$$\begin{aligned} \therefore I &= \frac{1}{2} \int \frac{dt}{t^2 + 5} - \frac{1}{2} \int \frac{dz}{z^2 + 1} \\ &= \frac{1}{2\sqrt{5}} \tan^{-1} \left(\frac{t}{\sqrt{5}} \right) - \frac{1}{2} \tan^{-1}(z) + c \end{aligned}$$

Hence,

$$I = \frac{1}{2\sqrt{5}} \tan^{-1} \left(\frac{x^2 - 1}{\sqrt{5}x} \right) - \frac{1}{2} \tan^{-1} \left(\frac{x^2 + 1}{x} \right) + c$$

Consider the integral

$$I = \int \frac{1}{\sin^4 x + \sin^2 x \cos^2 x + \cos^4 x} dx$$

Divide both the numerator and the denominator by $\cos^4 x$, we have,

$$I = \int \frac{\frac{1}{\cos^4 x}}{\frac{\sin^4 x + \sin^2 x \cos^2 x + \cos^4 x}{\cos^4 x}} dx$$

$$= \int \frac{\sec^4 x}{\tan^4 x + \tan^2 x + 1} dx$$

$$= \int \frac{\sec^2 x \times \sec^2 x}{\tan^4 x + \tan^2 x + 1} dx$$

$$= \int \frac{(\tan^2 x + 1) \times \sec^2 x}{\tan^4 x + \tan^2 x + 1} dx$$

Substituting $\tan x = t$; $\sec^2 x dx = dt$

Thus,

$$I = \int \frac{(1+t^2) dt}{t^4 + t^2 + 1}$$

$$= \int \frac{\left(1 + \frac{1}{t^2}\right) dt}{\left(t^2 + \frac{1}{t^2} + 1\right)}$$

$$= \int \frac{\left(1 + \frac{1}{t^2}\right) dt}{\left(t^2 + \frac{1}{t^2} - 2 + 2 + 1\right)}$$

$$= \int \frac{\left(1 + \frac{1}{t^2}\right) dt}{\left(t - \frac{1}{t}\right)^2 + 3}$$

$$= \int \frac{\left(1 + \frac{1}{t^2}\right) dt}{\left(t - \frac{1}{t}\right)^2 + 3}$$

Substituting $z = t - \frac{1}{t}$; $dz = \left(1 + \frac{1}{t^2}\right) dt$

$$I = \int \frac{dz}{z^2 + 3}$$

$$\Rightarrow I = \int \frac{dz}{z^2 + (\sqrt{3})^2}$$

$$I = \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{z}{\sqrt{3}} \right) + C$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{t - \frac{1}{t}}{\sqrt{3}} \right) + C$$

$$= \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{\tan x - \frac{1}{\tan x}}{\sqrt{3}} \right) + C$$

$$I = \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{\tan x - \cot x}{\sqrt{3}} \right) + C$$