Miscellaneous Exercise

Question 1:

Show that the sum of $(m+n)^{th}$ and $(m-n)^{th}$ terms of an A.P. is equal to twice the m^{th} term.

Solution 1:

Let a and d be the first term and the common difference of the A.P. respectively. It is known that the k^{th} term of an A.P. is given by

$a_k = a+(k-1)d$

$\therefore a_{m+n} = a+(m+n-1)d$

$a_{m-n} = a+(m-n-1)d$

$a_m = a+(m-1)d$

$\therefore a_{m+n}+a_{m-n} = a+(m+n-1)d+a+(m-n-1)d$

$= 2a+(m+n-1+m-n-1)d$

$= 2a+(2m-2)d$

$= 2a+2(m-1)d$

$= 2[a+(m-1)d]$

$= 2a_m$

Thus, the sum of $(m+n)^{th}$ and $(m-n)^{th}$ terms of an A.P. is equal to twice the m^{th} term.

Question 2:

Let the sum of three numbers in A.P., is 24 and their product is 440, find the numbers.

Solution 2:

Let the three numbers in A.P. be $a-d, a,$ and $a+d$.

According to the given information,

$(a-d)+(a)+(a+d)=24$ ......(1)

$\Rightarrow 3a=24$

$\therefore a=8$

$(a-d)a(a+d)=440$ ......(2)

$\Rightarrow (8-d)(8)(8+d)=440$

$\Rightarrow (8-d)(8+d)=55$

$\Rightarrow 64-d^2=55$

$\Rightarrow d^2=64-55=9$

$\Rightarrow d^2=\pm 3$

Therefore, when $d=3$, the numbers are 5, 8 and 11 and when $d=-3$, the numbers are 11, 8 and 5.

Thus, the three numbers are 5, 8 and 11.

Question 3:

Let the sum of $n, 2n, 3n$ terms of an A.P. be S_1, S_2 and S_3, respectively, show that $S_3=3(S_2-S_1)$

Solution 3:

Let a and b be the first term and the common difference of the A.P. respectively. Therefore,

$S_1=\frac{n}{2}[2a+(n-1)d]$ .... (1)

$S_2=\frac{2n}{2}[2a+(2n-1)d]=n[2a+(2n-1)d]$ ....(2)

$S_3=\frac{3n}{2}[2a+(3n-1)d]$ ....(3)

From (1) and (2), we obtain

$S_2-S_1=n[2a+(2n-1)d]-\frac{n}{2}[2a+(n-1)d]$

$=n\left\{\frac{4a+4nd-2d-2a-nd+d}{2}\right\}$

$=n\left[\frac{2a+3nd-d}{2}\right]$

$=\frac{n}{2}[2a+(3n-1)d]$

$\therefore\ 3(S_2-S_1)=\frac{3n}{2}[2a+(3n-1)d]=S_3$ [From (3)]

Hence, the given result is proved.

Question 4:
Find the sum of all numbers between 200 and 400 which are divisible by 7.

Solution 4:
The numbers lying between 200 and 400, which are divisible by 7, are 203, 210, 217…. 399
$\therefore$ First term, $a = 203$
Last term, $I = 399$
Common difference, $d = 7$
Let the number of terms of the A.P. be n.

$\therefore a_n = 399 = a + (n-1)d$

$\Rightarrow 399 = 203 + (n-1)7$

$\Rightarrow 7(n-1) = 196$

$\Rightarrow n - 1 = 28$

$\Rightarrow n = 29$

$\therefore S_{29} = \frac{29}{2}(203 + 399)$

$= \frac{29}{2}(602)$

$= (29)(301)$

$= 8729$

Thus, the required sum is 8729.

Question 5:
Find the sum of integers from 1 to 100 that are divisible by 2 or 5.

Solution 5:
The integers from 1 to 100, which are divisible by 2, are 2, 4, 6 ….. 100.
This forms an A.P. with both the first term and common difference equal to 2.

$\Rightarrow 100 = 2 + (n-1)2$

$\Rightarrow n = 50$

$\therefore 2 + 4 + 6 + \ldots\ldots + 100 = \frac{50}{2}\left[2(2) + (50-1)(2)\right]$

$= \frac{50}{2}[4 + 98]$

$= (25)(102)$

$= 2550$

The integers from 1 to 100, which are divisible by 5, 10…. 100.
This forms an A.P. with both the first term and common difference equal to 5.

$\therefore 100 = 5 + (n-1)5$

$\Rightarrow 5n = 100$

$\Rightarrow n = 20$

$\therefore 5+10+....+100=\frac{20}{2}\left[2(5)+(20-1)5\right]$

$=10\left[10+(19)5\right]$

$=10[10+95]=10\times 105$

$=1050$

The integers, which are divisible by both 2 and 5, are 10, 20, ….. 100.
This also forms an A.P. with both the first term and common difference equal to 10.

$\therefore\ 100=10+(n-1)(10)$

$\Rightarrow 100=10n$

$\Rightarrow n=10$

$\therefore\ 10+20+....+100=\frac{10}{2}\left[2(10)+(10-1)(10)\right]$

$=5[20+90]=5(110)=550$

$\therefore$ Required sum $=2550+1050-550=3050$

Thus, the sum of the integers from 1 to 100, which are divisible by 2 or 5, is 3050.

Question 6:
Find the sum of all two digit numbers which when divided by 4, yields 1 as remainder.

Solution 6:
The two-digit numbers, which when divided by 4, yield 1 as remainder, are 13, 17, …97.
This series forms an A.P. with first term 13 and common difference 4.
Let n be the number of terms of the A.P.
It is known that the n^{th} term of an A.P. is given by, $a_n=a+(n-1)d$

$\therefore 97=13+(n-1)(4)$

$\Rightarrow 4(n-1)=84$

$\Rightarrow n-1=21$

$\Rightarrow n=22$

Sum of n terms of an A.P. is given by

$S_n=\frac{n}{2}\left[2a+(n-1)d\right]$

$\therefore\ S_{22}=\frac{22}{2}\left[2(13)+(22-1)(4)\right]$

$=11[26+84]$

$=1210$

Thus, the required sum is 1210.

Question 7:

If f is a function satisfying $f(x+y)=f(x).f(y)$ for all $x, y \in N$, such that $f(1)=3$ and $\sum_{x=1}^{n} f(x)=120$ find the value of n.

Solution 7:

It is given that,

$f(x+y)=f(x)\times f(y)$ for all $x, y \in N$ ……. (1)

$f(1)=3$

Taking $x=y=1$ in (1),

We obtain $f(1+1)=f(2)=f(1)f(1)=3\times3=9$

Similarly,

$f(1+1+1)=f(3)=f(1+2)=f(1)f(2)=3\times9=27$

$f(4)=f(1+4)=f(1)f(3)=3\times27=81$

$\therefore$ $f(1), f(2), f(3), ...,$ that is 3, 9, 27,….., forms a G.P. with both the first term and common ratio equal to 3.

It is known that, $S_n=\dfrac{a(r^n-1)}{r-1}$

It is given that, $\sum_{k=1}^{n} f(x)=120$

$\therefore\ 120=\dfrac{3(3^n-1)}{3-1}$

$\Rightarrow 120=\dfrac{3}{2}(3^n-1)$

$\Rightarrow 3^n-1=80$

$\Rightarrow 3^n=81=3^4$

$\therefore n=4$

Thus, the value of n is 4.

Question 8:

The sum of some terms of G.P. is 315 whose first term and the common ratio are 5 and 2, respectively. Find the last term and the number of terms.

Solution 8:

Let the sum of n terms of the G.P. be 315.

It is known that, $S_n=\dfrac{a(r^n-1)}{r-1}$

It is given that the first term a is 5 and common ratio r is 2.

$$\therefore 315 = \frac{5\left(2^n - 1\right)}{2-1}$$
$$\Rightarrow 2^n - 1 = 63$$
$$\Rightarrow 2^n = 64 = (2)^6$$
$$\Rightarrow n = 6$$
$$\therefore \quad \text{Last term of the G.P.} = 6^{th} \text{ term} = ar^{6-1} = (5)(2)^5 = (5)(32)$$
$$= 160$$
Thus, the last term of the G.P. is 160.

Question 9:

The first term of a G.P. is 1. The sum of the third term and fifth term is 90. Find the common ratio of G.P.

Solution 9:

Let a and r be the first term and the common ratio of the G.P. respectively.

$\therefore a = 1 \qquad a_3 = ar^2 = r^2 \qquad a_5 = ar^4 = r^4$

$$\therefore r^2 + r^4 = 90$$
$$\Rightarrow r^4 + r^2 - 90 = 0$$
$$\Rightarrow r^2 = \frac{-1 + \sqrt{1+360}}{2} = \frac{-1 \pm \sqrt{361}}{2} = -10 \text{ or } 9$$
$\therefore r = \pm 3$ [Taking real roots]

Thus, the common ratio of the G.P. is ± 3.

Question 10:

The sum of the three numbers in G.P. is 56. If we subtract 1, 7, 21 from these numbers in that order, we obtain an arithmetic progression. Find the numbers.

Solution 10:

Let the three numbers in G.P. be $a, ar,$ and ar^2.

From the given condition,

$$a + ar + ar^2 = 56$$
$$\Rightarrow a\left(1 + r + r^2\right) = 56 \qquad ......(1)$$
$a - 1,\ ar - 7,\ ar^2 - 21$ forms an A.P.

$$\therefore (ar - 7) - (a - 1) = \left(ar^2 - 21\right) - (ar - 7) \text{ b}$$
$$\Rightarrow ar - a - 6 = ar^2 - ar - 14$$
$$\Rightarrow ar^2 - 2ar + a = 8$$
$$\Rightarrow ar^2 - ar - ar + a = 8$$
$$\Rightarrow a\left(r^2 + 1 - 2r\right) = 8$$
$$\Rightarrow a\left(r^2 - 1\right)^2 = 8 \qquad ......(2)$$
From (1) and (2), we get

$\Rightarrow 7\left(r^2-2r+1\right)=1+r+r^2$
$\Rightarrow 7r^2-14r+7-1-r-r^2=0$
$\Rightarrow 6r^2-15r+6=0$
$\Rightarrow 6r^2-12r-3r+6=0$
$\Rightarrow 6r\left(r-2\right)-3\left(r-2\right)=0$
$\Rightarrow \left(6r-3\right)\left(r-2\right)=0$
When $r=2,\ a=8$
Therefore, when $r=2$, the three numbers in G.P. are 8, 16 and 32.
When, $r=\frac{1}{2}$, the three numbers in G.P. are 32, 16 and 8.
Thus, in either case, the three required numbers are 8, 16 and 32.

Question 11:
A G.P. consists of an even number of terms. If the sum of all the terms is 5 times the sum of terms occupying odd places, then find its common ratio.

Solution 11:
Let the G.P. be $T_1, T_2, T_3, T_4 \ldots T_{2n}$.
Number of terms $=2n$
According to the given condition,
$T_1+T_2+T_3+\ldots+T_{2n}=5\left[T_1+T_3+\ldots+T_{2n-1}\right]$
$\Rightarrow T_1+T_2+T_3+\ldots+T_{2n}-5\left[T_1+T_3+\ldots+T_{2n-1}\right]=0$
$\Rightarrow T_2+T_4+\ldots+T_{2n}=4\left[T_1+T_3+\ldots+T_{2n-1}\right]$
Let the G.P. be $a, ar, ar^2, ar^3 \ldots$

$$\therefore \frac{ar\left(r^n-1\right)}{r-1}=\frac{4\times a\left(r^n-1\right)}{r-1}$$

$\Rightarrow ar=4a$
$\Rightarrow r=4$
Thus, the common ratio of the G.P. is 4.

Question 12:
The sum of the first four terms of an A.P. is 56. The sum of the last four terms is 112. If its first term is 11, then find the number of terms.

Solution 12:
Let the A.P. be $a, a+d, a+2d, a+3d \ldots a+\left(n-2\right)d, a+\left(n-1\right)d$.
Sum of first four terms $=a+\left(a+d\right)+\left(a+2d\right)+\left(a+3d\right)=4a+6d$
Sum of last four terms
$=\left[a+\left(n-4\right)d\right]+\left[a+\left(n-3\right)d\right]+\left[a+\left(n-2\right)d\right]+\left[a+\left(n-1\right)d\right]$

$= 4a + (4n - 10)d$

According to the given condition,

$4a + 6d = 56$

$\Rightarrow 4(11) + 6d = 56$ [Since $a = 11(\text{given})$]

$= 6d = 12$

$= d = 2$

$\therefore 4a + (4n - 10)d = 112$

$\Rightarrow 4(11) + (4n - 10)2 = 112$

$\Rightarrow (4n - 10)2 = 68$

$\Rightarrow 4n - 10 = 34$

$\Rightarrow 4n = 44$

$\Rightarrow n = 11$

Thus, the number of terms of the A.P. is 11.

Question 13:

If $\frac{a+bx}{a-bx} = \frac{b+cx}{b-cx} = \frac{c+dx}{c-dx} (x \neq 0)$ then show that a, b, c and d are in G.P.

Solution 13:

It is given that,

$$\frac{a+bx}{a-bx} = \frac{b+cx}{b-cx}$$

$\Rightarrow (a+bx)(b-cx) = (b+cx)(a-bx)$

$\Rightarrow ab - acx + b^2x - bcx^2 = ab - b^2x + acx - bcx^2$

$\Rightarrow 2b^2x = 2acx$

$\Rightarrow b^2 = ac$

$\Rightarrow \frac{b}{a} = \frac{c}{b}$ ......(1)

Also, $\frac{b+cx}{b-cx} = \frac{c+dx}{c-dx}$

$\Rightarrow (b+cx)(c-dx) = (b-cx)(c+dx)$

$\Rightarrow bc - bdx + c^2x - cdx^2 = bc + bdx - c^2x - cdx^2$

$\Rightarrow 2c^2x = 2bdx$

$\Rightarrow c^2 = bd$

$\Rightarrow \frac{c}{d} = \frac{d}{c}$ ......(2)

From (1) and (2), we obtain

$$\frac{b}{a} = \frac{c}{b} = \frac{d}{c}$$

Thus, a, b, c and d are in G.P.

Question 14:

Let S be the sum, P the product and R the sum of reciprocals of n terms in a G.P. Prove that $P^2R^n = S^n$

Solution 14:

Let the G.P. be $a, ar, ar^2, ar^3 .... ar^{n-1}$

According to the given information,

$$S = \frac{a\left(r^n - 1\right)}{r-1}$$

$$P = a^n \times r^{1+2+...+n-1}$$

$$= a^n r^{\frac{n(n-1)}{2}} \qquad [\because \text{ Sum of first n natural numbers is } n\frac{(n+1)}{2}]$$

$$R = \frac{1}{a} + \frac{1}{ar} + ... + \frac{1}{ar^{n-1}}$$

$$= \frac{r^{n-1} + r^{n-2} + ...r + 1}{ar^{n-1}}$$

$$= \frac{1\left(r^n - 1\right)}{(r-1)} \times \frac{1}{ar^{n-1}} \qquad \left[\because 1, r, ...r^{n-1} \text{ forms a G.P}\right]$$

$$= \frac{r^n - 1}{ar^{n-1}(r-1)}$$

$$\therefore P^2R^n = a^{2n} r^{n(n-)} \frac{\left(r^n - 1\right)^n}{a^n r^{n(n-1)} (r-1)^n}$$

$$= \frac{a^n \left(r^n - 1\right)^n}{(r-1)^n}$$

$$= \left[\frac{a\left(r^n - 1\right)}{(r-1)}\right]^n$$

$$= S^n$$

Hence, $P^2R^n = S^n$

Question 15:

The p^{th}, q^{th} and r^{th} terms of an A.P. are a, b, c respectively. Show that $(q-r)a + (r-p)b + (p-q)c = 0$

Solution 15:

Let t and d be the first term and the common difference of the A.P. respectively.

The n^{th} term of an A.P. is given by, $a_n = t + (n-1)d$

Therefore,

$$a_p = t + (p-1)d = a \quad ....(1)$$

$$a_q = t + (q-1)d = b \quad ....(2)$$

$$a_r = t + (r-1)d = c \quad ....(3)$$

Subtracting equation (2) from (1), we obtain

$$(p-1-q+1)d = a-b$$

$$\Rightarrow (p-q)d = a-b$$

$$\therefore d = \frac{a-b}{p-q} \quad ......(4)$$

Subtracting equation (3) from (2), we obtain

$$(q-1-r+1)d = b-c$$

$$\Rightarrow (q-r)d = b-c$$

$$\Rightarrow d = \frac{b-c}{q-r} \quad ......(5)$$

Equating both the values of d obtained in (4) and (5), we obtain

$$\frac{a-b}{p-q} = \frac{b-c}{q-r}$$

$$\Rightarrow (a-b)(q-r) = (b-c)(p-q)$$

$$\Rightarrow aq - bq - ar + br = bp - bq - cp + cq$$

$$\Rightarrow bp - cp + cq - aq + ar - br = 0$$

$$\Rightarrow (-aq + ar) + (bp - br) + (-cp + cq) = 0 \quad \text{(By rearranging terms)}$$

$$\Rightarrow -a(q-r) - b(r-p) - c(p-q) = 0$$

$$\Rightarrow a(q-r) + b(r-p) + c(p-q) = 0$$

Thus, the given result is proved.

Question 16:

If $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in A.P., prove that a, b, c are in A.P.

Solution 16:

It is given that $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in A.P.

$$\therefore b\left(\frac{1}{c}+\frac{1}{a}\right) - a\left(\frac{1}{b}+\frac{1}{c}\right) = c\left(\frac{1}{a}+\frac{1}{b}\right) - b\left(\frac{1}{c}+\frac{1}{a}\right)$$

$$\Rightarrow \frac{b(a+c)}{ac} - \frac{a(b+c)}{bc} = \frac{c(a+b)}{ab} - \frac{b(a+c)}{ac}$$

$$\Rightarrow \frac{b^2a+b^2c-a^2b-a^2c}{abc}=\frac{c^2a+c^2b-b^2a-b^2c}{abc}$$
$$\Rightarrow b^2a-a^2b+b^2c-a^2c=c^2a-b^2a+c^2b-b^2c$$
$$\Rightarrow ab(b-a)+c(b^2-a^2)=a(c^2-b^2)+bc(c-b)$$
$$\Rightarrow ab(b-a)+c(b-a)(b+a)=a(c-b)(c+b)+bc(c-b)$$
$$\Rightarrow (b-a)(ab+cb+ca)=(c-b)(ac+ab+bc)$$
$$\Rightarrow b-a=c-b$$
Thus, a,b and c are in A.P.

Question 17:

If a,b,c,d are in G.P., prove that $(a^n+b^n),(b^n+c^n),(c^n+d^n)$ are in G.P.

Solution 17:

It is given that a,b,c and d are in G.P.

$\therefore b^2=ac$ ......(1)

$c^2=bd$ ....(2)

$ad=bc$ ......(3)

It has to be proved that $(a^n+b^n),(b^n+c^n),(c^n+d^n)$ are in G.P. i.e.,

$(b^n+c^n)^2=(a^n+b^n),(c^n+d^n)$

Consider L.H.S.

$(b^n+c^n)^2=b^{2n}+2b^nc^n+c^{2n}$

$=(b^2)^n+2b^nc^n+(c^2)^n$

$=(ac)^n+2b^nc^n+(bd)^n$ [Using (1) and (2)]

$=a^nc^n+b^nc^n+b^nc^n+b^nd^n$

$=a^nc^n+b^nc^n+a^nd^n+b^nd^n$ [Using (3)]

$=c^n(a^n+b^n)+d^n(a^n+b^n)$

$=(a^n+b^n)(c^n+d^n)=$ R.H.S

$\therefore (b^n+c^n)^2=(a^n+b^n)(c^n+d^n)$

Thus, (a^n+b^n), (b^n+c^n), and (c^n+d^n) are in G.P.

Question 18:

If a and b are the roots of $x^2-3x+p=0$ and c,d are roots of $x^2-12x+q=0$, where a,b,c,d form a G.P. Prove that $(q+p):(q-p)=17:15$.

Solution 18:

It is given that a and b are the roots of $x^2-3x+p=0$

$\therefore a+b=3$ and $ab=p$ ....(1)

Also, c and d are the roots of $x^2-12x+q=0$

$\therefore c+d=12$ and $cd=q$ ......(2)

It is given that a,b,c,d are in G.P.

Let $a=x,\ b=xr,\ c=xr^2,\ d=xr^3$

From (1) and (2),

We obtain $x+xr=3 \Rightarrow x(1+r)=3$

$xr^2+xr^3=12$

$\Rightarrow xr^2(1+r)=12$

On dividing, we obtain

$$\frac{xr^2(1+r)}{x(1+r)}=\frac{12}{3}$$

$\Rightarrow r^2=4$

$\Rightarrow r=\pm 2$

When $r=2,\ x=\dfrac{3}{1+2}=\dfrac{3}{3}=1$

When $r=-2,\ x=\dfrac{3}{1-2}=\dfrac{3}{-1}=-3$

Case I: When $r=2$ and $x=1$, $ab=x^2r=2$, $cd=x^2r^5=32$

$$\therefore \frac{q+p}{q-p}=\frac{32+2}{32-2}=\frac{34}{30}=\frac{17}{15}$$

i.e., $(q+p):(q-p)=17:15$

Case II:

When $r=-2,\ x=-3,\ ab=x^2r=-18,\ cd=x^2r^5=-288$

$$\therefore \frac{q+p}{q-p}=\frac{-288-18}{-288+18}=\frac{-306}{-270}=\frac{17}{15}$$

i.e., $(q+p):(q-p)=17:15$

Thus, in both the cases, we obtain $(q+p):(q-p)=17:15$.

Question 19:

The ratio of the A.M and G.M. of two positive numbers a and b, is $m:n$. Show that

$$a:b=\left(m+\sqrt{m^2-n^2}\right):\left(m-\sqrt{m^2-n^2}\right)$$

Solution 19:

Let the two numbers be a and b.

A.M $= \frac{a+b}{2}$ and G.M. $= \sqrt{ab}$

According to the given condition,

$$\frac{a+b}{2\sqrt{ab}} = \frac{m}{n}$$

$$\Rightarrow \frac{(a+b)^2}{4(ab)} = \frac{m^2}{n^2}$$

$$\Rightarrow (a+b)^2 = \frac{4abm^2}{n^2}$$

$$\Rightarrow (a+b) = \frac{2\sqrt{ab}m}{n} \qquad (1)$$

Using this in the identity $(a-b)^2 = (a+b)^2 - 4ab$, we obtain

$$(a-b)^2 = \frac{4abm^2}{n^2} - 4ab = \frac{4ab(m^2-n^2)}{n^2}$$

$$\Rightarrow (a-b) = \frac{2\sqrt{ab}\sqrt{m^2-n^2}}{n} \qquad ....(2)$$

Adding (1) and (2), we obtain

$$2a = \frac{2\sqrt{ab}}{n}\left(m+\sqrt{m^2-n^2}\right)$$

$$\Rightarrow a = \frac{\sqrt{ab}}{n}\left(m+\sqrt{m^2-n^2}\right)$$

Substituting the value of a in (1), we obtain

$$b = \frac{2\sqrt{ab}}{n}m - \frac{\sqrt{ab}}{n}\left(m+\sqrt{m^2-n^2}\right)$$

$$= \frac{\sqrt{ab}}{n}m - \frac{\sqrt{ab}}{n}\sqrt{m^2-n^2}$$

$$= \frac{\sqrt{ab}}{n}\left(m-\sqrt{m^2-n^2}\right)$$

$$\therefore a:b = \frac{a}{b} = \frac{\frac{\sqrt{ab}}{n}\left(m+\sqrt{m^2-n^2}\right)}{\frac{\sqrt{ab}}{n}\left(m-\sqrt{m^2-n^2}\right)} = \frac{\left(m+\sqrt{m^2-n^2}\right)}{\left(m-\sqrt{m^2-n^2}\right)}$$

Thus, $a:b = \left(m+\sqrt{m^2-n^2}\right):\left(m-\sqrt{m^2-n^2}\right)$

Question 20:

If a,b,c are in A.P; b,c,d are in G.P. and $\frac{1}{c},\frac{1}{d},\frac{1}{e}$ are in A.P. prove that a,c,e are in G.P.

Solution 20:

It is given that a,b,c are in A.P.

$\therefore b-a=c-b \qquad ......(1)$

It is given that b,c,d are in G.P.

$\therefore c^2=bd \qquad ......(2)$

Also, $\frac{1}{c},\frac{1}{d},\frac{1}{e}$ are in A.P.

$\frac{1}{d}-\frac{1}{c}=\frac{1}{e}-\frac{1}{d}$

$\frac{2}{d}=\frac{1}{c}+\frac{1}{e} \qquad ......(3)$

It has to be proved that a,c,e are in G.P. i.e., $c^2=ae$

From (1), we obtain

$2b=a+c$

$\Rightarrow b=\frac{a+c}{2}$

From (2), we obtain

$d=\frac{c^2}{b}$

Substituting these values in (3), we obtain

$\frac{2b}{c^2}=\frac{1}{c}+\frac{1}{e}$

$\Rightarrow \frac{2(a+c)}{2c^2}=\frac{1}{c}+\frac{1}{e}$

$\Rightarrow \frac{a+c}{c^2}=\frac{e+c}{ce}$

$\Rightarrow \frac{a+c}{c}=\frac{e+c}{e}$

$\Rightarrow (a+c)e=(e+c)c$

$\Rightarrow ae+ce=ec+c^2$

$\Rightarrow c^2=ae$

Thus, a,c and e are in G.P.

Question 21:

Find the sum of the following series up to n terms:

(i) $5+55+555+....$ (ii) $.6+.66+.666+...$

Solution 21:

(i) $5+55+555+....$

Let $S_n=5+55+555+....\text{to } n$ terms

$=\frac{5}{9}[9+99+999+.....\text{to } n \text{ terms}]$

$=\frac{5}{9}\left[(10-1)+(10^2-1)+(10^3-1)+...\text{to } n \text{ terms}\right]$

$=\frac{5}{9}\left[(10+10^2+10^3+\text{to } n \text{ terms})-(1+1+...\text{to } n \text{ terms})\right]$

$=\frac{5}{9}\left[\frac{10(10^n-1)}{10-1}-n\right]$

$=\frac{5}{9}\left[\frac{10(10^n-1)}{9}-n\right]$

$=\frac{50}{81}(10^n-1)-\frac{5n}{9}$

(ii) $.6+.66+.666+...$

Let $S_n=06.+0.66+0.666+....\text{to } n \text{ terms}$

$=6[0.1+0.11+0.111+....\text{to } n \text{ terms}]$

$=\frac{6}{9}[0.9+0.99+0.999+....\text{to } n \text{ terms}]$

$=\frac{6}{9}\left[\left(1-\frac{1}{10}\right)+\left(1-\frac{1}{10^2}\right)+\left(1-\frac{1}{10^3}\right)+....\text{to } n \text{ terms}\right]$

$=\frac{2}{3}\left[(1+1+...\ n \text{ terms})-\frac{1}{10}\left(1+\frac{1}{10}+\frac{1}{10^2}+...n \text{ terms}\right)\right]$

$=\frac{2}{3}\left[n-\frac{1}{10}\left(\frac{1-\left(\frac{1}{10}\right)^n}{1-\frac{1}{10}}\right)\right]$

$=\frac{2}{3}n-\frac{2}{30}\times\frac{10}{9}(1-10^{-n})$

$=\frac{2}{3}n-\frac{2}{27}(1-10^{-n})$

Question 22:

Find the 20th term of the series $2\times4+4\times6+6\times8+.....+n$ terms .

Solution 22:

The given series is $2\times4+4\times6+6\times8+.....n$ terms

$\therefore n^{th}$ term $=a_n=2n\times(2n+2)=4n^2+4n$

$a_{20}=4(20)^2+4(20)=4(400)+80=1600+80=1680$

Thus, the 20th term of the series is 1680.

Question 23:
Find the sum of the first n terms of the series: $3+7+13+21+31+.....$

Solution 23:
The given series is $3+7+13+21+31+.....$

$$S=3+7+13+21+31+....+a_{n-1}+a_n$$

$$S=3+7+13+21+....+a_{n-2}+a_{n-1}+a_n$$

On subtracting both the equations, we obtain

$$S-S=\left[3+\left(7+13+21+31+...+a_{n-1}+a_n\right)+\right]-\left[\left(3+7+13+21+31+....+a_{n-1}\right)+a_n\right]$$

$$S-S=3+\left[\left(7-3\right)+\left(13-7\right)+\left(21-13\right)+....+\left(a_n-a_{n-1}\right)\right]-a_n$$

$$0=3+\left[4+6+8+....\left(n-1\right)\text{terms}\right]-a_n$$

$$a_n=3+\left[4+6+8+.....\left(n-1\right)\text{terms}\right]$$

$$\Rightarrow a_n=3+\left(\frac{n-1}{2}\right)\left[2\times4+\left(n-1-1\right)2\right]$$

$$=3+\left(\frac{n-1}{2}\right)\left[8+\left(n-2\right)2\right]$$

$$=3+\frac{\left(n-1\right)}{2}\left(2n+4\right)$$

$$=3+\left(n-1\right)\left(n+2\right)$$

$$=3+\left(n^2+n-2\right)$$

$$=n^2+n+1$$

$$\therefore\ \sum_{k=1}^{n}a_k=\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}k+\sum_{k=1}^{n}1$$

$$=\frac{n\left(n+1\right)\left(2n+1\right)}{6}+\frac{n\left(n+1\right)}{2}+n$$

$$=n\left[\frac{\left(n+1\right)\left(2n+1\right)+3\left(n+1\right)+6}{6}\right]$$

$$=n\left[\frac{2n^2+3n+1+3n+3+6}{6}\right]$$

$$=n\left[\frac{2n^2+6n+10}{6}\right]$$

$$=\frac{n}{3}\left[n^2+3n+5\right]$$

Question 24:

If S_1, S_2, S_3 are the sum of first n natural numbers, their squares and their cubes, respectively, show that $9S_2^2 = S_3(1+8S_1)$.

Solution 24:

From the given information,

$$S_1 = \frac{n(n+1)}{2}$$

$$S_3 = \frac{n^2(n+1)^2}{4}$$

Here, $S_3(1+8S_1) = \frac{n^2(n+1)^2}{4}\left[1+\frac{8n(n+1)}{2}\right]$

$$= \frac{n^2(n+1)^2}{4}\left[1+4n^2+4n\right]$$

$$= \frac{n^2(n+1)^2}{4}(2n+1)^2$$

$$= \frac{\left[n(n+1)(2n+1)\right]^2}{4} \qquad ......(1)$$

Also, $9S_2^2 = 9\frac{\left[n(n+1)(2n+1)\right]^2}{(6)^2}$

$$= \frac{9}{36}\left[n(n+1)(2n+1)\right]^2$$

$$= \frac{\left[n(n+1)(2n+1)\right]^2}{4} \qquad ......(2)$$

Thus, from (1) and (2), we obtain $9S_2^2 = S_3(1+8S_1)$.

Question 25:

Find the sum of the following series up to n terms:

$$\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+....$$

Solution 25:

The n^{th} term of the given series is $\frac{1^3+2^3+3^3+.....+n^3}{1+3+5+....+(2n-1)} = \frac{\left[\frac{n(n+1)}{2}\right]^2}{1+3+5+.....+(2n-1)}$

Here, $1,3,5.....(2n-1)$ is an A.P. with first term a, last term $(2n-1)$ and number of terms as n

$$\therefore 1+3+5+....+(2n-1)=\frac{n}{2}[2\times 1+(n-1)2]=n^2$$

$$\therefore a_n=\frac{n^2(n+1)^2}{4n^2}=\frac{(n+1)^2}{4}=\frac{1}{4}n^2+\frac{1}{2}n+\frac{1}{4}$$

$$\therefore S_n=\sum_{k=1}^{n}a_k=\sum_{k=1}^{n}\left(\frac{1}{4}K^2+\frac{1}{2}K+\frac{1}{4}\right)$$

$$=\frac{1}{4}\frac{n(n+1)(2n+1)}{6}+\frac{1}{2}\frac{n(n+1)}{2}+\frac{1}{4}n$$

$$=\frac{n[(n+1)(2n+1)+6(n+1)+6]}{24}$$

$$=\frac{n[2n^2+3n+1+6n+6+6]}{24}$$

$$=\frac{n(2n^2+9n+13)}{24}$$

Question 26:

Show that $\frac{1\times 2^2+2\times 3^2+....+n\times(n+1)^2}{1^2\times 2+2^2\times 3+.....+n^2\times(n+1)}=\frac{3n+5}{3n+1}$

Solution 26:

n^{th} term of the numerator $=n(n+1)^2=n^3+2n^2+n$

n^{th} term of the denominator $=n^2(n+1)=n^3+n^2$

$$\frac{1\times 2^2+2\times 3^2+....+n\times(n+1)^2}{1^2+2+2^2\times 3+....+n^2\times(n+1)}=\frac{\sum_{k=1}^{n}a_K}{\sum_{k=1}^{n}a_K}=\frac{\sum_{k=1}^{n}(K^3+2K^2+K)}{\sum_{k=1}^{n}(K^3+K^2)} \qquad \text{......... (1)}$$

Here, $\sum_{k=1}^{n}(K^3+2K^2+K)$

$$=\frac{n^2(n+1)^2}{4}+\frac{2n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}$$

$$=\frac{n(n+1)}{2}\left[\frac{n(n+1)}{2}+\frac{2}{3}(2n+1)+1\right]$$

$$=\frac{n(n+1)}{2}\left[\frac{3n^2+3n+8n+4+6}{6}\right]$$

$$=\frac{n(n+1)}{12}[3n^2+11n+10]$$

$$=\frac{n(n+1)}{12}\left[3n^2+6n+5n+10\right]$$

$$=\frac{n(n+1)}{12}\left[3n(n+2)+5(n+2)\right]$$

$$=\frac{n(n+1)(n+2)(3n+5)}{12} \qquad ......(2)$$

Also, $\sum_{K=1}^{n}\left(K^3+K^2\right)=\frac{n^2(n+1)^2}{4}+\frac{n(n+1)(2n+1)}{6}$

$$=\frac{n(n+1)}{2}\left[\frac{n(n+1)}{2}+\frac{2n+1}{3}\right]$$

$$=\frac{n(n+1)}{2}\left[\frac{3n^2+3n+4n+2}{6}\right]$$

$$=\frac{n(n+1)}{12}\left[3n^2+7n+2\right]$$

$$=\frac{n(n+1)}{12}\left[3n^2+6n+n+2\right]$$

$$=\frac{n(n+1)}{12}\left[3n(n+2)+1(n+2)\right]$$

$$=\frac{n(n+1)(n+2)(3n+1)}{12} \qquad ......(3)$$

From (1), (2) and (3), we obtain

$$\frac{1\times 2^2+2\times 3^2+.....+n\times(n+1)^2}{1^2\times 2+2^2\times 3+....+n^2(n+1)}=\frac{\frac{n(n+1)(n+2)(3n+5)}{12}}{\frac{n(n+1)(n+2)(3n+1)}{12}}$$

$$=\frac{n(n+1)(n+2)(3n+5)}{n(n+1)(n+2)(3n+1)}=\frac{3n+5}{3n+1}$$

Thus, the given result is proved.

Question 27:

A farmer buys a used tractor for Rs. 12000. He pays Rs. 6000 cash and agrees to pay the balance in annual installments of Rs. 500 plus 12% interest on the unpaid amount. How much will be the tractor cost him?

Solution 27:

It is given farmer pays Rs. 6000 in cash.

Therefore, unpaid amount = Rs. 12000 – Rs. 6000 = Rs. 6000

According to the given condition, the interest paid annually is

12% of 6000, 12% of 5500, 12% of 5000 ….. 12% of 500

Thus, total interest to be paid

= 12% of 6000 + 12% of 5500 + 12% of 5000 + ….. + 12% of 500
= 12% of (6000 + 5500 + 5000 + …. + 500)
= 12% of (500 + 1000 + 1500 + ….. + 6000)
Now, the series 500, 1000, 1500 …. 6000 is an A.P. with both the first term and common difference equal to 500.
Let the number of terms of the A.P. be n.

$\therefore 6000 = 500 + (n-1)500$

$\Rightarrow 1 + (n-1) = 12$

$\Rightarrow n = 12$

$\therefore$ Sum of the A.P

$= \frac{12}{2}\left[2(500) + (12-1)(500)\right] = 6[1000 + 5500] = 6(6500) = 39000$

Thus, total interest to be paid
= 12% of (500 + 1000 + 1500 + ….. + 6000)
= 12% of 39000 = Rs. 4680
Thus, cost of tractor = (Rs. 12000 + Rs. 4680) = Rs. 16680.

Question 28:
Shamshad Ali buys a scooter for Rs. 22000. He pays Rs. 4000 cash and agrees to pay the balance in annual installment of Rs. 1000 plus 10% interest on the unpaid amount. How much will the scooter cost him?

Solution 28:
It is given that Shamshad Ali buys a scooter for Rs. 22000 and pays Rs. 4000 in cash.
$\therefore$ Unpaid amount = Rs. 22000 – Rs. 4000 = Rs. 18000
According to the given condition, the interest paid annually is
10% of 18000, 10% of 17000, 10% of 16000 ….. 10% of 1000
Thus, total interest to be paid
= 10% of 18000 + 10% of 17000 + 10% of 16000 + ….. + 10% of 1000
= 10% of (18000 + 17000 + 16000 + ….. + 1000)
= 10% of (1000 + 2000 + 3000 + ….. + 18000)
Here, 1000, 2000, 3000 …. 18000 forms an A.P. with first term and common difference both equal to 1000.
Let the number of terms be n.

$\therefore 18000 = 1000 + (n-1)(1000)$

$\Rightarrow n = 18$

$\therefore 1000 + 2000 + ... + 18000 = \frac{18}{2}\left[2(1000) + (18-1)(1000)\right]$

$= 9[2000 + 17000]$

$= 171000$

$\therefore$ Total interest paid = 10% of (18000 + 17000 + 16000 + …. + 1000)
= 10% of Rs.171000 = Rs. 17100
$\therefore$ Cost of scooter = Rs. 22000 + Rs. 17100 = Rs. 39100.

Question 29:

A person writes a letter to four of his friends. He asks each one of them to copy the letter and mail to four different persons with instruction that they move the chain similarly. Assuming that the chain is not broken and that it costs 50 paise to mail one letter. Find the amount spent on the postage when 8^{th} set of letter is mailed.

Solution 29:

The numbers of letters mailed forms a G.P.: $4, 4^2, \ldots 4^8$

First term = 4

Common ratio = 4

Number of terms = 8

It is known that the sum of n terms of a G.P. is given by

$$S_n = \frac{a\left(r^n - 1\right)}{r - 1}$$

$$\therefore S_8 = \frac{4\left(4^8 - 1\right)}{4 - 1} = \frac{4\left(65536 - 1\right)}{3} = \frac{4\left(65535\right)}{3} = 4\left(21845\right) = 87380$$

It is given that the cost to mail one letter is 50 paisa.

$\therefore$ Cost of mailing 87380 letters $= \text{Rs.}87380 \times \frac{50}{100} = \text{Rs.}43690$

Thus, the amount spent when 8^{th} set of letter is mailed is Rs. 43690.

Question 30:

A man deposited Rs. 10000 in a bank at the rate of 5% simple interest annually. Find the amount in 15^{th} year since he deposited the amount and also calculate the total amount after 20 years.

Solution 30:

It is given that the man deposited Rs. 10000 in a bank at the rate of 5% simple interest annually.

$$= \frac{5}{100} \times \text{Rs.}10000 = \text{Rs.}500$$

$\therefore$ Interest in first year $10000 + \underbrace{500 + 500 + \ldots + 500}_{14\text{ times}}$

$\therefore$ Amount in 15^{th} year

$= \text{Rs.}10000 + 14 \times \text{Rs.}500$

$= \text{Rs.}10000 + \text{Rs.}7000$

$= \text{Rs.}17000$

Amount after 20 years $= \text{Rs.}10000 + \underbrace{500 + 500 + \ldots + 500}_{20\text{times}}$

$= \text{Rs.}10000 + 20 \times \text{Rs.}500$

$= \text{Rs.}10000 + \text{Rs.}10000$

$= \text{Rs.}20000.$

Question 31:
A manufacturer reckons that the value of a machine, which costs him Rs. 15625, will depreciate each year by 20%. Find the estimated value at the end of 5 years.

Solution 31:
Cost of machine = Rs. 15625
Machine depreciates by 20% every year.
Therefore, its value after every year is 80% of the original cost i.e., $\frac{4}{5}$ of the original cost.

$\therefore$ Value at the end of 5 years $=15625\times\underbrace{\frac{4}{5}\times\frac{4}{5}\times....\times\frac{4}{5}}_{5\text{ times}}=5\times1024=5120$

Thus, the value of the machine at the end of 5 years is Rs. 5120.

Question 32:
150 workers were engaged to finish a job in a certain number of days. 4 workers dropped out on second day, 4 more workers dropped out on third day and so on. It took 8 more days to finish the work. Find the number of days in which the work was completed.

Solution 32:
Let x be the number of days in which 150 workers finish the work.
According to the given information,
$150x=150+146+142+....(x+8)\text{terms}$

The series $150+146+142+....(x+8)\text{terms}$ is an A.P. with first term 146, common difference -4 and number of terms as $(x+8)$

$\Rightarrow 150x=\frac{(x+8)}{2}\left[2(150)+(x+8-1)(-4)\right]$

$\Rightarrow 150x=(x+8)\left[150+(x+7)(-2)\right]$

$\Rightarrow 150x=(x+8)(150-2x-14)$

$\Rightarrow 150x=(x+8)(136-2x)$

$\Rightarrow 75x=(x+8)(68-x)$

$\Rightarrow 75x=68x-x^2+544-8x$

$\Rightarrow x^2+75x-60x-544=0$

$\Rightarrow x^2+15x-544=0$

$\Rightarrow x^2+32x-17x-544=0$

$\Rightarrow x(x+32)-17(x+32)=0$

$\Rightarrow (x-17)(x+32)=0$

$\Rightarrow x=17$ or $x=-32$

However, x cannot be negative.

$\therefore x = 17$

Therefore, originally, the number of days in which the work was completed is 17. Thus, required number of days $= (17+8) = 25$.