Exercise 9.3

Question 1:

Find the 20th and n^{th} terms of the G.P. $\frac{5}{2}, \frac{5}{4}, \frac{5}{8}, ...$

Solution 1:

The given G.P. is $\frac{5}{2}, \frac{5}{4}, \frac{5}{8}, ...$

Here, $a =$ First term $= \frac{5}{2}$

$r =$ Common ratio $= \frac{5/4}{5/2} = \frac{1}{2}$

$$a_{20} = ar^{20-1} = \frac{5}{2}\left(\frac{1}{2}\right)^{19} = \frac{5}{(2)(2)^{19}} = \frac{5}{(2)^{20}}$$

$$a_n = ar^{n-1} = \frac{5}{2}\left(\frac{1}{2}\right)^{n-1} = \frac{5}{(2)(2)^{n-1}} = \frac{5}{(2)^n}$$

Question 2:
Find the 12th term of a G.P. whose 8th term is 192 and the common ratio is 2.

Solution 2:
Common ratio, r = 2
Let be the first term of the G.P.

$$\therefore a_8 = ar^{8-1} = ar^7 \Rightarrow ar^7 = 192 \Rightarrow a(2)^7 = 192 \Rightarrow a(7)^7 = (2)^6(3)$$

$$\Rightarrow a = \frac{(2)^6 \times 3}{(2)^7} = \frac{3}{2}$$

$$\therefore a_{12} = ar^{12-1} = \left(\frac{3}{2}\right)(2)^{11} = (3)(2)^{10} = 3072.$$

Question 3:
The 5th, 8th and 11th terms of a G.P. are p, q and s, respectively. Show that $q^2 = ps$.

Solution 3:
Let be the first term and r be the common ratio of the G.P. According to the given condition,

$$a_5 = ar^{5-1} = ar^4 = p \text{ ....}(1)$$

$$a_8 = ar^{8-1} = ar^7 = q \text{ ....}(2)$$

$$a_{11} = ar^{11-1} = ar^{10} = s \text{ ....}(3)$$

Dividing equation (2) by (1), we obtain

$$\frac{ar^7}{ar^4} = \frac{q}{p}$$

$$r^3 = \frac{q}{p} \qquad \text{.......}(4)$$

Dividing equation (3) by (2), we obtain

$$\frac{ar^{10}}{ar^7} = \frac{s}{q}$$

$$\Rightarrow r^3 = \frac{s}{q} \qquad \text{......}(5)$$

Equating the values of r^3 obtained in (4) and (5), we obtain

$$\frac{q}{p} = \frac{s}{q}$$

$$\Rightarrow q^2 = ps$$

Thus, the given result is proved.

Question 4:

The 4^{th} term of a G.P. is square of its second term, and the first term is -3. Determine its 7^{th} term.

Solution 4:

Let be the first term and r be the common ratio of the G.P.

$\therefore a = -3$

It is known that, $a_n = ar^{n-1}$

$\therefore a_4 = ar^3 = (-3)r^3$

$a_2 = a r^1 = (-3)r$

According to the given condition,

$(-3)r^3 = [(-3)r]^2$

$\Rightarrow -3r^3 = 9r^2 \Rightarrow r = -3 a_7 = a r^{7-1} = ar^6 = (-3)(-3)^6 = -(3)^7 = -2187$

Thus, the seventh term of the G.P. is -2187.

Question 5:

Which term of the following sequences:

(a) $2, 2\sqrt{2}, 4.....$ is 128?

(b) $\sqrt{3}, 3, 3\sqrt{3}, ...$ is 729?

(c) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, ...$ is $\frac{1}{19683}$?

Solution 5:

(a) The given sequence is $2, 2\sqrt{2}, 4.....$ is 128?

Here, $a = 2$ and $r = (2\sqrt{2})/2 = \sqrt{2}$

Let the n^{th} term of the given sequence be 128.

$a_n = a r^{n-1}$

$\Rightarrow (2)(\sqrt{2})^{n-1} = 128$

$\Rightarrow (2)(2)^{\frac{n-1}{2}} = (2)^7$

$\Rightarrow (2)^{\frac{n-1}{2}+1} = (2)^7$

$\therefore \frac{n-1}{2} + 1 = 7$

$\Rightarrow \frac{n-1}{2} = 6$

$\Rightarrow n - 1 = 12$

$\Rightarrow n = 13$

Thus, the 13^{th} term of the given sequence is 128.

(b) The given sequence is $\sqrt{3}, 3, 3\sqrt{3}, \ldots\ldots$

$a = \sqrt{3}$ and $r = \dfrac{3}{\sqrt{3}} = \sqrt{3}$

Let the n^{th} term of the given sequence be 729.

$a_n = a r^{n-1}$

$\therefore a r^{n-1} = 729$

$\Rightarrow (\sqrt{3})(\sqrt{3})^{n-1} = 729$

$\Rightarrow (3)^{1/2} (3)^{\frac{n-1}{2}} = (3)^6$

$\Rightarrow (3)^{\frac{1}{2}+\frac{n-1}{2}} = (3)^6$

$\therefore \dfrac{1}{2} + \dfrac{n-1}{2} = 6$

$\Rightarrow \dfrac{1+n-1}{2} = 6$

$\Rightarrow n = 12$

Thus, the 12th term of the given sequence is 729.

(c) The given sequence is $\dfrac{1}{3}, \dfrac{1}{9}, \dfrac{1}{27} \ldots$

Here, $a = \dfrac{1}{3}$ and $r = \dfrac{1}{9} \div \dfrac{1}{3} = \dfrac{1}{3}$

Let the n^{th} term of the given sequence be $\dfrac{1}{19683}$.

$a_n = a r^{n-1}$

$\therefore a r^{n-1} = \dfrac{1}{19683}$

$\Rightarrow \left(\dfrac{1}{3}\right)\left(\dfrac{1}{3}\right)^{n-1} = \dfrac{1}{19683}$

$\Rightarrow \left(\dfrac{1}{3}\right)^{n} = \left(\dfrac{1}{3}\right)^{9}$

$\Rightarrow n = 9$

Thus, the 9th term of the given sequence is $\dfrac{1}{19683}$.

Question 6:

For what values of x, the numbers $\dfrac{2}{7}, x, -\dfrac{7}{2}$ are in G.P.?

Solution 6:

The given numbers are $\frac{-2}{7}, x, \frac{-7}{2}$

Common ratio $= \frac{x}{-2/7} = \frac{-7x}{2}$

Also, common ratio $= \frac{-7/2}{x} = \frac{-7}{2x}$

$\therefore \frac{-7x}{2} = \frac{-7}{2x}$

$\Rightarrow x^2 = \frac{-2\times7}{-2\times7} = 1$

$\Rightarrow x = \sqrt{1}$

$\Rightarrow x = \pm1$

Thus, for $x = \pm1$, the given numbers will be in G.P.

Question 7:

Find the sum up to 20 terms in the geometric progression 0.15, 0.015, 0.0015….

Solution 7:

The given G.P. is 0.15, 0.015, 0.00015 …

Here, $a = 0.15$ and $r = \frac{0.015}{0.15} = 0.1$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$\therefore S_{20} = \frac{0.15\left[1-(0.1)^{20}\right]}{1-0.1}$$

$$= \frac{0.15}{0.9}\left[1-(0.1)^{20}\right]$$

$$= \frac{15}{90}\left[1-(0.1)^{20}\right]$$

$$= \frac{1}{6}\left[1-(0.1)^{20}\right]$$

Question 8:

Find the sum of   terms in the geometric progression $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, \ldots$

Solution 8:

The given G.P. is $\sqrt{7}, \sqrt{21}, 3\sqrt{7}, \ldots$

Here, $a = \sqrt{7}$ and $r = \frac{\sqrt{21}}{7} = \sqrt{3}$

$$S_n = \frac{a\left(1-r^n\right)}{1-r}$$

$$\Rightarrow S_n = \frac{\sqrt{7}\left[1-\left(\sqrt{3}\right)^n\right]}{1-\sqrt{3}}$$

$$\Rightarrow S_n = \frac{\sqrt{7}\left[1-\left(\sqrt{3}\right)^n\right]}{1-\sqrt{3}} \times \frac{1+\sqrt{3}}{1+\sqrt{3}}$$

$$\Rightarrow S_n = \frac{\sqrt{7}\left(\sqrt{3}+1\right)\left[1-\left(\sqrt{3}\right)^n\right]}{1-3}$$

$$\Rightarrow S_n = \frac{-\sqrt{7}\left(\sqrt{3}+1\right)\left[1-\left(\sqrt{3}\right)^n\right]}{2}$$

$$\Rightarrow \frac{\sqrt{7}\left(1+\sqrt{3}\right)}{2}\left[(3)^{\frac{n}{2}}-1\right]$$

Question 9:

Find the sum of terms in the geometric progression $1, -a, a^2, -a^3 \ldots\ldots (\text{if } a \neq -1)$

Solution 9:

The given G.P. is $1, -a, a^2, -a^3 \ldots\ldots$

Here, first term $= a_1 = 1$

Common ratio $= r = -a$

$$S_n = \frac{a_1\left(1-r^n\right)}{1-r}$$

$$\therefore S_n = \frac{1\left[1-(-a)^n\right]}{1-(-a)} = \frac{\left[1-(-a)^n\right]}{1+a}$$

Question 10:

Find the sum of terms in the geometric progression $x^3, x^5, x^7 \ldots. (\text{if } x \neq \pm 1)$

Solution 10:

The given G.P. is $x^3, x^5, x^7 \ldots.$

Here, $a = x^3$ and $r = x^2$

$$S_n = \frac{a\left(1-r^n\right)}{1-r} = \frac{x^3\left[1-\left(x^2\right)^n\right]}{1-x^2} = \frac{x^3\left(1-x^{2n}\right)}{1-x^2}$$

Question 11:

Evaluate $\sum_{k=1}^{11}\left(2+3^{k}\right)$

Solution 11:

$$\sum_{k=1}^{11}\left(2+3^{k}\right)=\sum_{k=1}^{11}(2)+\sum_{k=1}^{11}\left(3^{k}\right)=22+\sum_{k=1}^{11}3^{k} \qquad ....(1)$$

$$\sum_{k=1}^{11}3^{k}=3^{1}+3^{2}+3^{3}+......+3^{11}$$

The terms of this sequence $3,3^{2},3^{3}\ldots\ldots$ forms a G.P.

$$S_n=\frac{a\left(r^{n}-1\right)}{r-1}$$

$$\Rightarrow S_n=\frac{3\left[(3)^{11}-1\right]}{3-1}$$

$$\Rightarrow S_n=\frac{3}{2}\left(3^{11}-1\right)$$

$$\therefore \sum_{k=1}^{11}3^{k}=\frac{3}{2}\left(3^{11}-1\right)$$

Substituting this value in equation (1), we obtain

$$\sum_{k=1}^{11}\left(2+3^{k}\right)=22+\frac{3}{2}\left(3^{11}-1\right)$$

Question 12:

The sum of first three terms of a G.P. is $\frac{39}{10}$ and their product is 1. Find the common ratio and the terms.

Solution 12:

Let $\frac{a}{r},\ a,\ ar$ be the first three terms of the G.P.

$$\frac{a}{r}+a+ar=\frac{39}{10} \qquad ....(1)$$

$$\left(\frac{a}{r}\right)(a)(ar)=1 \qquad ....(2)$$

From (2), we

Obtain $a^{3}=1$

$\Rightarrow a=1$ (Considering real roots only)

Substituting $a=1$ in equation (1), we obtain

$$\frac{1}{r}+1+r=\frac{39}{10}$$

$$\Rightarrow 1+r+r^2=\frac{39}{10}r$$

$$\Rightarrow 10+10r+10r^2-39r=0$$

$$\Rightarrow 10r^2-29r+10=0$$

$$\Rightarrow 10r^2-25r-4r+10=0$$

$$\Rightarrow 5r(2r-5)-2(2r-5)=0$$

$$\Rightarrow (5r-2)(2r-5)=0$$

$$\Rightarrow r=\frac{2}{5} \text{ or } \frac{5}{2}$$

Thus, the three terms of G.P. are $\frac{5}{2}, 1$ and $\frac{2}{5}$.

Question 13:

How many terms of G.P. $3, 3^2, 3^3 \ldots$ are needed to give the sum 120?

Solution 13:

The given G.P. is $3, 3^2, 3^3 \ldots$

Let terms of this G.P. be required to obtain in the sum as 120.

$$S_n=\frac{a\left(1-r^n\right)}{1-r}$$

Here, $a=3$ and $r=3$

$$\therefore S_n=120=\frac{3\left(3^n-1\right)}{3-1}$$

$$\Rightarrow 120=\frac{3\left(3^n-1\right)}{2}$$

$$\Rightarrow \frac{120\times 2}{3}=3^n-1$$

$$\Rightarrow 3^n-1=80$$

$$\Rightarrow 3^n=81$$

$$\Rightarrow 3^n=3^4$$

$$\therefore n=4$$

Thus, four terms of the given G.P. are required to obtain the sum as 120.

Question 14:

The sum of first three terms of a G.P. is 16 and the sum of the next three terms is 128. Determine the first term, the common ratio and the sum to n terms of the G.P.

Solution 14:

Let the G.P. be $a, ar, ar^2, ar^3, ...$ According to the given condition,

$a+ar+ar^2=16$ and $ar^3+ar^4+ar^5=128$

$\Rightarrow a\left(1+r+r^2\right)=16 \qquad ......(1)$

$ar^3\left(1+r+r^2\right)=128 \qquad ......(2)$

Dividing equation (2) by (1), we obtain

$$\frac{ar^3\left(1+r+r^3\right)}{a\left(1+r+r^2\right)}=\frac{128}{16}$$

$\Rightarrow r^3=8$

$\therefore r=2$

Substituting $r=2$ in (1), we obtain $a(1+2+4)=16$

$\Rightarrow a(7)=16$

$\Rightarrow a=\frac{16}{7}$

$$S_n=\frac{a\left(r^n-1\right)}{r-1}$$

$$\Rightarrow S_n=\frac{16}{7}\frac{\left(2^n-1\right)}{2-1}=\frac{16}{7}\left(2^n-1\right)$$

Question 15:

Given a G.P. with $a=729$ and 7th term 64, determine S_7.

Solution 15:

$a=729 \quad a_7=64$

Let r be the common ratio of the G.P. It is known that,

$a_n=ar^{n-1}$

$a_7=ar^{7-1}=(729)r^6$

$\Rightarrow 64=729r^6$

$\Rightarrow r^6=\left(\frac{2}{3}\right)^6$

$\Rightarrow r=\frac{2}{3}$

Also, it is known that,

$$S_n=\frac{a\left(1-r^n\right)}{1-r}$$

$$\therefore S_7 = \frac{729\left(1-\left(\frac{2}{3}\right)^7\right)}{1-\frac{2}{3}}$$

$$= 3\times 729\left[1-\left(\frac{2}{3}\right)^7\right]$$

$$= (3)^7\left[\frac{(3)^7-(2)^7}{(3)^7}\right]$$

$$= (3)^7-(2)^7$$

$$= 2187-128$$

$$= 2059$$

Question 16:

Find a G.P. for which sum of the first two terms is -4 and the fifth term is 4 times the third term.

Solution 16:

Let   be the first term and r be the common ratio of the G.P.

According to the given conditions,

$$A_2 = -4 = \frac{a\left(1-r^2\right)}{1-r} \quad \text{......(1)}$$

$$a_5 = 4\times a_3$$

$$\Rightarrow ar^4 = 4ar^2 \Rightarrow r^2 = 4$$

$$\therefore r = \pm 2$$

From (1), we obtain

$$-4 = \frac{a\left[1-(2)^2\right]}{1-2} \text{ for } r = 2$$

$$\Rightarrow -4 = \frac{a(1-4)}{-1}$$

$$\Rightarrow -4 = a(3)$$

$$\Rightarrow a = \frac{-4}{3}$$

Also, $-4 = \frac{a\left[1-(-2)^2\right]}{1-(-2)}$ for $r = -2$

$$\Rightarrow -4 = \frac{a(1-4)}{1+2}$$

$\Rightarrow -4 = \frac{a(-3)}{3}$

$\Rightarrow a = 4$

Thus, the required G.P. is $\frac{-4}{3}, \frac{-8}{3}, \frac{-16}{3}, \ldots$ or $4, -8, -16, -32\ldots$

Question 17:

If the 4th, 10th and 16th terms of a G.P. are x, y, and z, respectively. Prove that x, y, z are in G.P.

Solution 17:

Let be the first term and r be the common ratio of the G.P.

According to the given condition,

$a_4 = ar^3 = x \quad \ldots\ldots(1)$

$a_{10} = ar^9 = y \quad \ldots\ldots(2)$

$a_{16} = ar^{15} = z \quad \ldots\ldots(3)$

Dividing (2) by (1), we obtain

$\frac{y}{x} = \frac{ar^9}{ar^3} \Rightarrow \frac{y}{x} = r^6$

Dividing (3) by (2), we obtain

$\frac{z}{y} = \frac{ar^{15}}{ar^9} \Rightarrow \frac{z}{y} = r^6$

$\therefore \frac{y}{x} = \frac{z}{y}$

Thus, x, y, z are in G.P.

Question 18:

Find the sum to terms of the sequence, 8, 88, 888, 8888 ….

Solution 18:

The given sequence is 8, 88, 888, 8888 ….

This sequence is not a G.P. However, it can be changed to G.P. by writing the terms as $S_n = 8 + 88 + 888 + 8888 + \ldots\ldots$ to n terms

$= \frac{8}{9}[9 + 99 + 999 + 9999 + \ldots\ldots\text{to } n \text{ terms}]$

$= \frac{8}{9}\left[(10-1) + (10^2 - 1) + (10^3 - 1) + (10^4 - 1) + \ldots\ldots\text{to } n \text{ terms}\right]$

$= \frac{8}{9}\left[(10 + 10^2 + \ldots\ldots\ n \text{ terms}) - (1 + 1 + 1 + \ldots\ldots n \text{ terms})\right]$

$$=\frac{8}{9}\left[\frac{10\left(10^{n}-1\right)}{10-1}-n\right]$$

$$=\frac{8}{9}\left[\frac{10\left(10^{n}-1\right)}{9}-n\right]$$

$$=\frac{80}{81}\left(10^{n}-1\right)-\frac{8}{9}n$$

Question 19:

Find the sum of the products of the corresponding terms of the sequences 2, 4, 8, 16, 32 and 128, 32, 8, 2, 1/2.

Solution 19:

Required sum $=2\times128+4\times32+8\times8+16\times2+32\times\frac{1}{2}$

$$=64\left[4+2+1+\frac{1}{2}+\frac{1}{2^{2}}\right]$$

Here, $4,2,1,\frac{1}{2},\frac{1}{2^{2}}$ is a G.P.

First term, $a=4$

Common ratio, $r=\frac{1}{2}$

It is known that, $S_{n}=\frac{a\left(1-r^{n}\right)}{1-r}$

$$\therefore S_{5}=\frac{4\left[1-\left(\frac{1}{2}\right)^{5}\right]}{1-\frac{1}{2}}=\frac{4\left[1-\frac{1}{32}\right]}{\frac{1}{2}}=8\left(\frac{32-1}{32}\right)=\frac{31}{4}$$

$$\therefore \quad \text{Required sum }=64\left(\frac{31}{4}\right)=(16)(31)=496$$

Question 20:

Show that the products of the corresponding terms of the sequences form $a, ar, ar^{2},\ldots..ar^{n-1}$ and A, AR, AR^{2}, AR^{n-1} a G.P., and find the common ratio.

Solution 20:

It has to be proved that the sequence: $aA, arAR, ar^{2}AR^{2}$, $\ldots\ldots ar^{n-1}AR^{n-1}$, forms a G.P.

$$\frac{\text{Second term}}{\text{First term}}=\frac{ar\,AR}{aA}=rR$$

$$\frac{\text{Third term}}{\text{Second term}} = \frac{ar^2 AR^2}{arAR} = rR$$

Thus, the above sequence forms a G.P. and the common ratio is rR.

Question 21:

Find four numbers forming a geometric progression in which third term is greater than the first term by 9, and the second term is greater than the 4^{th} by 18.

Solution 21:

Let a be the first term and r be the common ratio of the G.P.

$a_1 = a,\ a_2 = ar,\ a_3 = ar^2,\ a_4 = ar^3$

By the given condition,

$a_3 = a_1 + 9 \Rightarrow ar^2 = a + 9 \qquad .....(1)$

$a_4 = a_4 + 18 \Rightarrow ar = ar^3 + 18 \quad .....(2)$

From (1) and (2), we obtain

$a(r^2 - 1) = 9.......(3)$

$ar(1 - r^2) = 18.......(4)$

Dividing (4) by (3), we obtain

$$\frac{ar(1-r^2)}{a(r^2-1)} = \frac{18}{9}$$

$\Rightarrow -r = 2$

$\Rightarrow r = -2$

Substituting the value of r in (1), we obtain

$4a = a + 9$

$\Rightarrow 3a = 9$

$\therefore a = 3$

Thus, the first four numbers of the G.P. are $3,\ 3(-2),\ 3(-2)^2$, and $3(-2)^3$

i.e., $3, -6, 12$ and -24.

Question 22:

If p^{th}, q^{th} and r^{th} terms of a G.P. are a, b and c, respectively. Prove that $a^{q-r} \cdot b^{r-p} \cdot c^{p-q} = 1$.

Solution 22:

Let A be the first term and R be the common ratio of the G.P.

According to the given information,

$AR^{p-1} = a$

$AR^{q-1} = b$

$AR^{r-1} = c$

$a^{q-r} \cdot b^{r-p} \cdot c^{p-q}$

$= A^{q-r} \times R^{(p-1)(q-r)} \times A^{r-p} \times R^{(q-1)(r-p)} \times A^{p-q} \times R^{(r-1)(p-q)}$

$= A^{q-r+r-p+p-q} \times R^{(pr-pr-q+r)+(rq-r+p-pq)+(pr-p-qr+q)}$

$= A^0 \times R^0$

$= 1$

Thus, the given result is proved.

Question 23:

If the first and the n^{th} term of a G.P. are   and b, respectively, and if P is the product of   terms, prove that $P^2 = (ab)^n$.

Solution 23:

The first term of the G.P is   and the last term is b.

Therefore, the G.P. is $a, ar, ar^2, ar^3 .... ar^{n-1}$, where r is the common ratio.

$b = ar^{n-1}$ ....(1)

$P =$ Product of   terms

$= (a)(ar)(ar^2)......(ar^{n-1})$

$= (a \times a \times ....a)(r \times r^2 \times ....r^{n-1})$

$= a^n\, r^{1+2+....(n-1)}$ ....(2)

Here, $1, 2, ...(n-1)$ is an A.P.

$\therefore 1 + 2 + ... + (n-1)$

$= \frac{n-1}{2}\left[2 + (n-1-1) \times 1\right] = \frac{n-1}{2}\left[2 + n - 2\right] = \frac{n(n-1)}{2}$

$P = a^n\, r^{\frac{n(n-1)}{2}}$

$\therefore P^2 = a^{2n}\, r^{n(n-1)}$

$= \left[a^2\, r^{(n-1)}\right]^n$

$= \left[a \times ar^{n-1}\right]^n$

$= (ab)^n$ [Using (1)]

Thus, the given result is proved.

Question 24:

Show that the ratio of the sum of first   terms of a G.P. to the sum of terms from $(n+1)^{th}$ to $(2n)^{th}$ term is $\frac{1}{r^n}$.

Solution 24:

Let   be the first term and r be the common ratio of the G.P.

Sum of first   terms $=\dfrac{a\left(1-r^{n}\right)}{(1-r)}$

Since there are   terms from $(n+1)^{th}$ to $(2n)^{th}$ term,

Sum of terms from $(n+1)^{th}$ to $(2n)^{th}$ term

$$S_n=\frac{a_{n+1}\left(1-r^{n}\right)}{1-r}$$

$$a^{n+1}=ar^{n+1-1}=ar^{n}$$

Thus, required ratio $=\dfrac{a\left(1-r^{n}\right)}{(1-r)}\times\dfrac{(1-r)}{ar^{n}\left(1-r^{n}\right)}=\dfrac{1}{r^{n}}$

Thus, the ratio of the sum of first   terms of a G.P. to the sum of terms from $(n+1)^{th}$ to $(2n)^{th}$ term is $\dfrac{1}{r^{n}}$.

Question 25:

If a, b, c and   are in G.P. show that:

$$\left(a^{2}+b^{2}+c^{2}\right)\left(b^{2}+c^{2}+d^{2}\right)=(ab+bc+cd)^{2}$$

Solution 25:

If a, b, c and   are in G.P. Therefore,

$bc=ad$ ....(1)

$b^{2}=ac$........(2)

$c^{2}=bd$ ....(3)

It has to be proved that,

$$\left(a^{2}+b^{2}+c^{2}\right)\left(b^{2}+c^{2}+d^{2}\right)=(ab+bc+cd)^{2}$$

R.H.S.

$=(ab+bc+cd)^{2}$

$=(ab+ad+cd)^{2}$ [Using (1)]

$=[ab+d(a+c)]^{2}$

$=a^{2}b^{2}+2abd(a+c)+d^{2}(a+c)^{2}$

$=a^{2}b^{2}+2a^{2}bd+2acbd+d^{2}\left(a^{2}+2ac+c^{2}\right)$

$=a^{2}b^{2}+2a^{2}c^{2}+2b^{2}c^{2}+d^{2}a^{2}+2d^{2}b^{2}+d^{2}c^{2}$ [Using (1) and (2)]

$=a^{2}b^{2}+a^{2}c^{2}+a^{2}c^{2}+b^{2}c^{2}+b^{2}c^{2}+d^{2}a^{2}+d^{2}b^{2}+d^{2}b^{2}+d^{2}c^{2}$

$= a^2b^2 + a^2c^2 + a^2d^2 + b^2 \times b^2 + b^2c^2 + b^2d^2 + c^2b^2 + c^2 \times c^2 + c^2d^2$

[Using (2) and (3) and rearranging terms]

$= a^2\left(b^2 + c^2 + d^2\right) + b^2\left(b^2 + c^2 + d^2\right) + c^2\left(b^2 + c^2 + d^2\right)$

$= \left(a^2 + b^2 + c^2\right)\left(b^2 + c^2 + d^2\right) = \text{L.H.S}$

$\therefore$ L.H.S = R.H.S.

$\therefore \left(a^2 + b^2 + c^2\right)\left(b^2 + c^2 + d^2\right) = \left(ab + bc + cd\right)^2$.

Question 26:

Insert two numbers between 3 and 81 so that the resulting sequence is G.P.

Solution 26:

Let G_1 and G_2 be two numbers between 3 and 81 such that the series, 3, $G_1, G_2, 81$, forms a G.P.

Let be the first term and r be the common ratio of the G.P.

$\therefore\ 81 = (3)(r)^3$

$\Rightarrow r^3 = 27$

$\therefore r = 3$ (Talking real roots only)

For $r = 3$,

$G_1 = ar = (3)(3) = 9$

$G_2 = ar^2 = (3)(3)^2 = 27$

Thus, the required two numbers are 9 and 27.

Question 27:

Find the value of so that $\dfrac{a^{n+1} + b^{n+1}}{a^n + b^n}$ may be the geometric mean between and b.

Solution 27:

M. of and b is $\sqrt{ab}$

By the given condition: $\dfrac{a^{n+1} + b^{n+1}}{a^n + b^n} = \sqrt{ab}$

Squaring both sides, we obtain

$$\frac{\left(a^{n+1} + b^{n+1}\right)^2}{\left(a^n + b^n\right)^2} = ab$$

$\Rightarrow a^{2n+2} + 2a^{n+1}b^{n+1} + b^{2n+2} = (ab)\left(a^{2n} + 2a^nb^n + b^{2n}\right)$

$\Rightarrow a^{2n+2} + 2a^{n+1}b^{n+1} + b^{2n+2} = a^{2n+1}b + 2a^{n+1}b^{n+1} + ab^{2n+1}$

$\Rightarrow a^{2n+2} + b^{2n+2} = a^{2n+1}b + ab^{2n+1}$

$\Rightarrow a^{2n+2} - a^{2n+1}b = ab^{2n+1} - b^{2n+2}$

$$\Rightarrow a^{2n+1}(a-b)=b^{2n+1}(a-b)$$

$$\Rightarrow \left(\frac{a}{b}\right)^{2n+1}=1=\left(\frac{a}{b}\right)^{0}$$

$$\Rightarrow 2n+1=0$$

$$\Rightarrow n=\frac{-1}{2}$$

Question 28:

The sum of two numbers is 6 times their geometric mean, show that numbers are in the ratio $\left(3+2\sqrt{2}\right):\left(3-2\sqrt{2}\right)$

Solution 28:

Le the two numbers be   and b .

G.M. $=\sqrt{ab}$

According to the given condition,

$$a+b=6\sqrt{ab} \qquad ......(1)$$

$$\Rightarrow (a+b)^2=36(ab)$$

Also,

$$(a-b)^2=(a+b)^2-4ab=36ab-4ab=32ab$$

$$\Rightarrow a-b=\sqrt{32}\sqrt{ab}$$

$$=4\sqrt{2}\sqrt{ab} \qquad ......(2)$$

Adding (1) and (2), we obtain

$$2a=\left(6+4\sqrt{2}\right)\sqrt{ab}$$

$$a=\left(3+2\sqrt{2}\right)\sqrt{ab}$$

Substituting the value of   in (1), we obtain

$$b=6\sqrt{ab}-\left(3+2\sqrt{2}\right)\sqrt{ab}$$

$$\Rightarrow b=\left(3-2\sqrt{2}\right)\sqrt{ab}$$

$$\frac{a}{b}=\frac{\left(3+2\sqrt{2}\right)\sqrt{ab}}{\left(3-2\sqrt{2}\right)\sqrt{ab}}=\frac{3+2\sqrt{2}}{3-2\sqrt{2}}$$

Thus, the required ratio is $\left(3+2\sqrt{2}\right):\left(3-2\sqrt{2}\right)$.

Question 29:

If A and G be A.M. and G.M., respectively between two positive numbers, prove that the numbers are $A \pm \sqrt{(A+G)(A-G)}$

Solution 29:

It is given that A and G are A.M. and G.M. between two positive numbers.

Let these two positive numbers be and b.

$$\therefore AM = A = \frac{a+b}{2} \quad .....(1)$$

$$GM = G = \sqrt{ab} \quad ....(2)$$

From (1) and (2), we obtain

$$a+b = 2A \quad .....(3)$$

$$ab = G^2 \quad ....(4)$$

Substituting the value of and b from (3) and (4) in the identity

$(a-b)^2 = (a+b)^2 - 4ab$,

We obtain

$$(a-b)^2 = 4A^2 - 4G^2 = 4(A^2 - G^2)$$

$$(a-b)^2 = 4(A+G)(A-G)$$

$$(a-b) = 2\sqrt{(A+G)(A-G)} \quad ......(5)$$

From (3) and (5), we obtain

$$2a = 2A + 2\sqrt{(A+G)(A-G)}$$

$$\Rightarrow a = A + \sqrt{(A+G)(A-G)}$$

Substituting the value of in (3), we obtain

$$b = 2A - A - \sqrt{(A+G)(A-G)} = A - \sqrt{(A+G)(A-G)}$$

Thus, the two numbers are $A \pm \sqrt{(A+G)(A-G)}$.

Question 30:

The number of bacteria in a certain culture doubles every hour. If there were 30 bacteria present in the culture originally, how many bacteria will be present at the end of 2nd hour, 4th hour and n^{th} hour?

Solution 30:

It is given that the number of bacteria doubles every hour. Therefore, the number of bacteria after every hour will form a G.P.

Here, $a = 30$ and $r = 2$ $\quad \therefore a_3 = ar^2 = (30)(2)^2 = 120$

Therefore, the number of bacteria at the end of 2nd hour will be 120.

$a_5 = ar^4 = (30)(2)^4 = 480$

The number of bacteria at the end of 4th hour will be 480.

$a_{n+1} = ar^n = (30)2^n$

Thus, number of bacteria at the end of n^{th} hour will be $30(2)^n$.

Question 31:

What will Rs. 500 amounts to in 10 years after its deposit in a bank which pays annual interest rate of 10% compounded annually?

Solution 31:

The amount deposited in the bank is Rs. 500.

At the end of first year, amount $= \text{Rs.}500\left(1+\frac{1}{10}\right) = \text{Rs.}500(1.1)$

At the end of 2nd year, amount = Rs. 500 (1.1) (1.1)

At the end of 3rd year, amount = Rs. 500 (1.1) (1.1) (1.1) and so on

$\therefore$ Amount at the end of 10 years = Rs. 500 (1.1) (1.1) ….. (10 times)

$= \text{Rs.}500(1.1)^{10}$.

Question 32:

If A.M. and G.M. of roots of a quadratic equation are 8 and 5, respectively, then obtain the quadratic equation.

Solution 32:

Let the root of the quadratic equation be and b.

According to the given condition,

A.M. $= \frac{a+b}{2} = 8 \Rightarrow a+b = 16$ .....(1)

G.M. $= \sqrt{ab} = 5 \Rightarrow ab = 25$ ......(2)

The quadratic equation is given by,

$x^2 - x(\text{Sum of roots}) + (\text{Product of roots}) = 0$

$x^2 - x(a+b) + (ab) = 0$

$x^2 - 16x + 25 = 0$ [Using (1) and (2)]

Thus, the required quadratic equation is $x^2 - 16x + 25 = 0$.