NCERT Class 12 Maths

Solutions

Exercise 9.4

For each of the differential equations in Exercises 1 to 4, find the general solution:

1. $\dfrac{dy}{dx} = \dfrac{1-\cos x}{1+\cos x}$

Sol. The given differential equation is

$$\frac{dy}{dx} = \frac{1-\cos x}{1+\cos x} \quad \text{or} \quad dy = \frac{1-\cos x}{1+\cos x}\,dx.$$

Integrating both sides, $\displaystyle\int dy = \int \frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}\,dx$

$$\text{or } y = \int \tan^2\frac{x}{2}\,dx = \int\left(\sec^2\frac{x}{2} - 1\right)dx = \frac{\tan\frac{x}{2}}{\frac{1}{2}} - x + c$$

or $y = 2 \tan \dfrac{x}{2} - x + c$

which is the required general solution.

2. $\dfrac{dy}{dx} = \sqrt{4 - y^2}$ **(– 2 < *y* < 2)**

Sol. The given D.E. is $\dfrac{dy}{dx} = \sqrt{4 - y^2} \quad \Rightarrow \quad dy = \sqrt{4 - y^2}\ dx$

Separating variables, $\dfrac{dy}{\sqrt{4 - y^2}} = dx$

Integrating both sides, $\displaystyle\int \frac{dy}{\sqrt{2^2 - y^2}}\ dy = \int 1\ dx$

$\therefore \quad \sin^{-1} \dfrac{y}{2} = x + c \qquad \left[\because \displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1} \frac{x}{a}\right]$

$\Rightarrow \quad \dfrac{y}{2} = \sin (x + c)$

$\Rightarrow \quad y = 2 \sin (x + c)$ which is the required general solution.

3. $\dfrac{dy}{dx} + y = 1$ **(*y* ≠ 1)**

Sol. The given differential equation is $\dfrac{dy}{dx} + y = 1$

$\Rightarrow \quad \dfrac{dy}{dx} = 1 - y \quad \Rightarrow \quad dy = (1 - y)\, dx \quad \Rightarrow \quad dy = -\,(y - 1)\, dx$

Separating variables, $\dfrac{dy}{y - 1} = -\, dx$

Integrating both sides, $\displaystyle\int \frac{dy}{y - 1} = -\int 1\ dx$

$\Rightarrow \quad \log |\, y - 1 \,| = -x + c$

$\Rightarrow \quad |\, y - 1 \,| = e^{-x + c} \qquad [\because$ If $\log x = t$, then $x = e^t]$

$\Rightarrow \quad y - 1 = \pm\, e^{-x + c} \quad \Rightarrow \quad y = 1 \pm e^{-x}\, e^c$

$\Rightarrow \quad y = 1 \pm e^c\, e^{-x}$

$\Rightarrow \quad y = 1 + \text{A}e^{-x}$ where $\text{A} = \pm\, e^c$

which is the required general solution.

4. $\sec^2 x \tan y\ dx + \sec^2 y \tan x\ dy = 0$

Sol. The given differential equation is

$$\sec^2 x \tan y\ dx + \sec^2 y \tan x\ dy = 0$$

Dividing by $\tan x \tan y$, we have

$$\frac{\sec^2 x}{\tan x}\ dx + \frac{\sec^2 y}{\tan y}\ dy = 0 \qquad \text{(Variables separated)}$$

Integrating both sides, $\displaystyle\int \frac{\sec^2 x}{\tan x}\ dx + \int \frac{\sec^2 y}{\tan y}\ dy = \log c$

or $\log|\tan x| + \log|\tan y| = \log c$ $\left[\because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)|\right]$

or $\log|(\tan x \tan y)| = \log c$ or $|\tan x \tan y| = c$

$\therefore$ $\tan x \tan y = \pm c = C$ where $C = \pm c$.

$[\because \; |t| = a(a \geq 0) \;\Rightarrow\; t = \pm a]$

which is the required general solution.

For each of the differential equations in Exercises 5 to 7, find the general solution:

5. $(e^x + e^{-x})\,dy - (e^x - e^{-x})\,dx = 0$

Sol. The given D.E. is $(e^x + e^{-x})\,dy = (e^x - e^{-x})\,dx$

or $dy = \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) dx$

Integrating both sides, $\int dy = \int \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) dx$

or $y = \log|e^x + e^{-x}| + c$ $\left[\because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)|\right]$

which is the required general solution.

6. $\frac{dy}{dx} = (1 + x^2)(1 + y^2)$

Sol. The given differential equation is $\frac{dy}{dx} = (1 + x^2)(1 + y^2)$

$\Rightarrow$ $dy = (1 + x^2)(1 + y^2)\,dx$

Separating variables, $\frac{dy}{1 + y^2} = (1 + x^2)\,dx$

Integrating both sides,

$\int \frac{1}{y^2 + 1}\,dy = \int (x^2 + 1)\,dx$ $\Rightarrow$ $\tan^{-1} y = \frac{x^3}{3} + x + c$

which is the required general solution.

7. $y \log y\,dx - x\,dy = 0$

Sol. The given differential equation is $y \log y\,dx - x\,dy = 0$

$\Rightarrow$ $-x\,dy = -y \log y\,dx$

Separating variables, $\frac{dy}{y \log y} = \frac{dx}{x}$ ...(*i*)

Integrating both sides $\int \frac{dy}{y \log y} = \int \frac{dy}{x}$

For integral on left hand side, put $\log y = t$.

$\therefore$ $\frac{1}{y} = \frac{dt}{dy}$ $\Rightarrow$ $\frac{dy}{y} = dt$

$\therefore$ Eqn. (*i*) becomes $\int \frac{dt}{t} = \int \frac{dx}{x}$

$\Rightarrow$ $\log|t| = \log|x| + \log|c|^*$ ...(*ii*)

$= \log|xc|$

$\Rightarrow \quad |t| = |xc|$

$\Rightarrow \quad t = \pm xc$

$[\because \; |x| = |y| \;\; \Rightarrow \;\; x = \pm y]$

$\Rightarrow \quad \log y = \pm xc = ax \quad$ where $a = \pm c$

$\therefore \quad y = e^{ax}$ which is the required general solution.

For each of the differential equations in Exercises 8 to 10, find the general solution:

8. $x^5 \dfrac{dy}{dx} = -y^5$

Sol. The given differential equation is $x^5 \dfrac{dy}{dx} = -y^5$

$\Rightarrow \quad x^5\, dy = -y^5\, dx$

Separating variables, $\dfrac{dy}{(y^5)} = -\dfrac{dx}{(x^5)} \quad \Rightarrow \quad y^{-5}\, dy = -x^{-5}\, dx$

Integrating both sides, $\int y^{-5}\, dy = -\int x^{-5}\, dx$

$$\frac{y^{-4}}{-4} = -\frac{x^{-4}}{-4} + c$$

Multiplying by -4, $\quad y^{-4} = -x^{-4} - 4c$

$\Rightarrow \quad x^{-4} + y^{-4} = -4c \quad \Rightarrow \quad x^{-4} + y^{-4} = C$ where $C = -4c$

which is the required general solution.

9. $\dfrac{dy}{dx} = \sin^{-1} x$

Sol. The given differential equation is $\dfrac{dy}{dx} = \sin^{-1} x$

or $\quad dy = \sin^{-1} x\, dx$

Integrating both sides, $\quad \int 1\, dy = \int \sin^{-1} x\, dx$

or $\quad y = \int \underset{\text{I}}{\sin^{-1} x} \,.\, \underset{\text{II}}{1}\, dx$

Applying product rule,

$$y = (\sin^{-1} x) \int 1\, dx - \int \frac{d}{dx} (\sin^{-1} x) \int 1\, dx\, dx$$

$$= x \sin^{-1} x - \int \frac{1}{\sqrt{1-x^2}}\, x\, dx \qquad \text{...}(i)$$

To evaluate $\int \dfrac{x}{\sqrt{1-x^2}}\, dx = -\dfrac{1}{2} \int \dfrac{-2x}{\sqrt{1-x^2}}\, dx$

Put $1 - x^2 = t$. Differentiate $-2x\, dx = dt$

***Remark. To explain * in eqn. (*ii*)**

If all the terms in the solution of a D.E. involve logs, it is better to use $\log c$ or $\log |c|$ instead of c in the solution.

$$\therefore \int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int \frac{dt}{\sqrt{t}} = -\frac{1}{2}\int t^{-1/2}\,dt$$

$$= -\frac{1}{2}\,\frac{t^{1/2}}{1/2} = -\sqrt{t} = -\sqrt{1-x^2}$$

Putting this value of $\int \frac{x}{\sqrt{1-x^2}}\,dx$ in (*i*), the required general solution is

$$y = x\sin^{-1} x + \sqrt{1-x^2} + c.$$

10. $\boldsymbol{e^x \tan y\, dx + (1 - e^x) \sec^2 y\, dy = 0}$

Sol. The given equation is $e^x \tan y\, dx + (1 - e^x) \sec^2 y\, dy = 0$

Dividing every term by $(1 - e^x) \tan y$, we have

$$\frac{e^x}{1-e^x}\,dx + \frac{\sec^2 y}{\tan y}\,dy = 0 \qquad \text{(Variables separated)}$$

Integrating both sides, $\int \frac{e^x}{1-e^x}\,dx + \int \frac{\sec^2 y}{\tan y}\,dy = c$

or $\quad -\int \frac{-e^x}{1-e^x}\,dx + \log|\tan y| = c$

or $\quad -\log|1-e^x| + \log|\tan y| = c \quad \left[\because \int \frac{f'(x)}{f(x)}\,dx = \log|f(x)|\right]$

or $\quad \log \frac{|\tan y|}{|1-e^x|} = \log c' \quad$ (See Remark at the end of page 612)

or $\quad \frac{|\tan y|}{|1-e^x|} = c'$

or $\quad \tan y = C(1 - e^x). \quad [\because \; |t| = c' \;\Rightarrow\; t = \pm c' = C \text{ (say)}]$

For each of the differential equations in Exercises 11 to 12, find a particular solution satisfying the given condition:

11. $\boldsymbol{(x^3 + x^2 + x + 1)\frac{dy}{dx} = 2x^2 + x,\ y = 1,}$ **when** $\boldsymbol{x = 0}$

Sol. The given differential equation is $(x^3 + x^2 + x + 1)\frac{dy}{dx} = 2x^2 + x$

$\therefore \quad (x^3 + x^2 + x + 1)\,dy = (2x^2 + x)\,dx$

Separating variables $dy = \frac{(2x^2 + x)}{x^3 + x^2 + x + 1}\,dx$

or $$dy = \frac{2x^2 + x}{(x+1)(x^2+1)}\,dx$$

$[\because \; x^3 + x^2 + x + 1 = x^2(x + 1) + (x + 1) = (x + 1)(x^2 + 1)]$

Integrating both sides, we have

$$\int 1 \; dy = \int \frac{2x^2+x}{(x+1)(x^2+1)} \, dx \quad \text{or} \quad y = \int \frac{2x^2+x}{(x+1)(x^2+1)} \, dx \quad ...(i)$$

Let $\frac{2x^2+x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$ (Partial fractions) ...(*ii*)

Multiplying both sides by L.C.M. = $(x + 1)(x^2 + 1)$, we have

$$2x^2 + x = A(x^2 + 1) + (Bx + C)(x + 1)$$

or $\quad 2x^2 + x = Ax^2 + A + Bx^2 + Bx + Cx + C$

Comparing coeff. of x^2 on both sides, we have

$$A + B = 2 \quad ...(iii)$$

Comparing coeff. of x on both sides, we have

$$B + C = 1 \quad ...(iv)$$

Comparing constants $\quad A + C = 0 \quad ...(v)$

Let us solve eqns. (*iii*), (*iv*) and (*v*) for A, B, C

eqn. (*iii*) – eqn. (*iv*) gives to eliminate B,

$$A - C = 1 \quad ...(vi)$$

Adding (*v*) and (*vi*), $\quad 2A = 1 \quad$ or $\quad A = \frac{1}{2}$

From (*v*), $\quad C = -A = -\frac{1}{2}$

Putting $C = -\frac{1}{2}$ in (*iv*), $B - \frac{1}{2} = 1$ or $B = 1 + \frac{1}{2} = \frac{3}{2}$

Putting these values of A, B, C in (*ii*), we have

$$\frac{2x^2+x}{(x+1)(x^2+1)} = \frac{\frac{1}{2}}{x+1} + \frac{\frac{3}{2}x - \frac{1}{2}}{x^2+1}$$

$$= \frac{1}{2}\,\frac{1}{x+1} + \frac{3}{2}\cdot\frac{x}{x^2+1} - \frac{1}{2}\,\frac{1}{x^2+1}$$

$$= \frac{1}{2}\,\frac{1}{x+1} + \frac{3}{4}\cdot\frac{2x}{x^2+1} - \frac{1}{2}\,\frac{1}{x^2+1}$$

Putting this value in (*i*)

$$y = \frac{1}{2}\int \frac{1}{x+1}\,dx + \frac{3}{4}\int \frac{2x}{x^2+1}\,dx - \frac{1}{2}\int \frac{1}{x^2+1}\,dx$$

$$y = \frac{1}{2}\log(x+1) + \frac{3}{4}\log(x^2+1) - \frac{1}{2}\tan^{-1}x + c \quad ...(vii)$$

$$\left[\because \int \frac{2x}{x^2+1}\,dx = \int \frac{f'(x)}{f(x)}\,dx = \log f(x)\right]$$

To find *c*

When $x = 0$, $y = 1$ (given)

Putting $x = 0$ and $y = 1$ in (*vii*),

$$1 = \frac{1}{2}\log 1 + \frac{3}{4}\log 1 - \frac{1}{2}\tan^{-1} 0 + c$$

or $\quad 1 = c$ $\quad [\because \log 1 = 0 \text{ and } \tan^{-1} 0 = 0]$

Putting $c = 1$ in eqn. (*vii*), the required solution is

$$y = \frac{1}{2} \log (x + 1) + \frac{3}{4} \log (x^2 + 1) - \frac{1}{2} \tan^{-1} x + 1.$$

$$y = \frac{1}{4} [2 \log (x + 1) + 3 \log (x^2 + 1)] - \frac{1}{2} \tan^{-1} x + 1$$

$$= \frac{1}{4} [\log (x + 1)^2 + \log (x^2 + 1)^3] - \frac{1}{2} \tan^{-1} x + 1$$

$$= \frac{1}{4} [\log (x + 1)^2 (x^2 + 1)^3] - \frac{1}{2} \tan^{-1} x + 1$$

which is the required particular solution.

12. $\boldsymbol{x (x^2 - 1) \frac{dy}{dx} = 1; y = 0 \text{ when } x = 2.}$

Sol. The given differential equation is $x(x^2 - 1) \dfrac{dy}{dx} = 1$

$$\Rightarrow \quad x(x^2 - 1)\, dy = dx \qquad \Rightarrow \quad dy = \frac{dx}{x(x^2 - 1)}$$

Integrating both sides, $\displaystyle\int 1 \, dy = \int \frac{1}{x(x^2 - 1)}\, dx$

$$\Rightarrow \quad y = \int \frac{1}{x(x+1)(x-1)}\, dx + c \qquad \text{...}(i)$$

Let the integrand $\dfrac{1}{x(x+1)(x-1)} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x-1}$ ...(*ii*)

(By Partial Fractions)

Multiplying by L.C.M. $= x(x + 1)(x - 1)$,

$1 = A(x + 1)(x - 1) + Bx(x - 1) + Cx(x + 1)$

or $\quad 1 = A(x^2 - 1) + B(x^2 - x) + C(x^2 + x)$

or $\quad 1 = Ax^2 - A + Bx^2 - Bx + Cx^2 + Cx$

Comparing coefficients of x^2, x and constant terms on both sides, we have

$\boldsymbol{x^2}$**:** $\quad A + B + C = 0$ ...(*iii*)

$\boldsymbol{x}$**:** $\quad - B + C = 0 \quad \Rightarrow \quad C = B$ ...(*iv*)

Constants $\quad - A = 1 \quad$ or $\quad A = - 1$

Putting $A = - 1$ and $C = B$ from (*iv*) in (*iii*),

$$- 1 + B + B = 0 \quad \text{or} \quad 2B = 1 \qquad \Rightarrow \quad B = \frac{1}{2}$$

$\therefore$ From (*iv*), $C = B = \dfrac{1}{2}$

Putting these values of A, B, C in (*ii*),

$$\frac{1}{x(x+1)(x-1)} = \frac{-1}{x} + \frac{\frac{1}{2}}{x+1} + \frac{\frac{1}{2}}{x-1}$$

$$\therefore \quad \int \frac{1}{x(x+1)(x-1)}\,dx = -\int \frac{1}{x}dx + \frac{1}{2}\int \frac{1}{x+1}dx + \frac{1}{2}\int \frac{1}{x-1}dx$$

$$= -\log|x| + \frac{1}{2}\log|x+1| + \frac{1}{2}\log|x-1|$$

$$= \frac{1}{2}[-2\log|x| + \log|x+1| + \log|x-1|]$$

$$= \frac{1}{2}[-\log|x|^2 + \log|(x+1)(x-1)|]$$

$$\Rightarrow \quad \int \frac{1}{x(x+1)(x-1)}\,dx = \frac{1}{2}\left[\log\frac{|x^2-1|}{|x|^2}\right] = \frac{1}{2}\log\frac{|x^2-1|}{x^2}$$

Putting this value in (*i*),

$$y = \frac{1}{2}\log\left|\frac{x^2-1}{x^2}\right| + c \qquad \ldots(v)$$

To find *c* for the particular solution

Putting $y = 0$, when $x = 2$ (given) in (*v*),

$$0 = \frac{1}{2}\log\frac{3}{4} + c \qquad \Rightarrow \qquad c = \frac{-1}{2}\log\frac{3}{4}$$

Putting this value of *c* in (*v*), the required particular solution is

$$y = \frac{1}{2}\log\left|\frac{x^2-1}{x^2}\right| - \frac{1}{2}\log\frac{3}{4}$$

OR

To evaluate $\int \frac{1}{x(x^2-1)}\,dx = \int \frac{x}{x^2(x^2-1)}\,dx = \frac{1}{2}\int \frac{2x}{x^2(x^2-1)}\,dx$

Put $\quad x^2 = t.$

For each of the differential equations in Exercises 13 to 14, find a particular solution satisfying the given condition:

13. $\cos\left(\frac{dy}{dx}\right) = a\ (a \in \mathbb{R});\ y = 1$ **when** $x = 0$

Sol. The given differential equation is

$$\cos\frac{dy}{dx} = a\ (a \in \mathrm{R});\ y = 1 \text{ when } x = 0$$

$$\therefore \quad \frac{dy}{dx} = \cos^{-1}a \qquad \Rightarrow \qquad dy = (\cos^{-1}a)\,dx$$

Integrating both sides

$$\int 1\,dy = \int (\cos^{-1}a)\,dx \qquad \Rightarrow \qquad y = (\cos^{-1}a)\int 1\,dx$$

$$\Rightarrow \qquad y = (\cos^{-1}a)\,x + c \qquad \ldots(i)$$

To find *c* for particular solution

$y = 1$ when $x = 0$ (given) $\qquad \therefore$ From (*i*), $1 = c$.

Putting $c = 1$ in (*i*), $y = x\cos^{-1}a + 1$

$$\Rightarrow \quad y - 1 = x\cos^{-1}a \qquad \Rightarrow \qquad \frac{y-1}{x} = \cos^{-1}a$$

$\Rightarrow \quad \cos\left(\frac{y-1}{x}\right) = a$ which is the required particular solution.

14. $\frac{dy}{dx} = y \tan x$**; $y = 1$ when $x = 0$**

Sol. The given differential equation is $\frac{dy}{dx} = y \tan x$

$\Rightarrow \quad dy = y \tan x \, dx$

Separating variables, $\frac{dy}{y} = \tan x \, dx$

Integrating both sides $\int \frac{1}{y} \, dy = \int \tan x \, dx$

$\Rightarrow \quad \log |y| = \log |\sec x| + \log |c|$

$\Rightarrow \quad \log |y| = \log |c \sec x| \quad \Rightarrow \quad |y| = |c \sec x|$

$\therefore \quad y = \pm c \sec x$

or $\quad y = C \sec x \qquad ...(i)$

where $C = \pm c$

To find C for particular solution

Putting $y = 1$ and $x = 0$ in (*i*), $1 = C \sec 0 = C$

Putting $C = 1$ in (*i*), the required particular solution is $y = \sec x$.

15. Find the equation of a curve passing through the point (0, 0) and whose differential equation is $y' = e^x \sin x$.

Sol. The given differential equation is $y' = e^x \sin x$

$\Rightarrow \quad \frac{dy}{dx} = e^x \sin x \quad \Rightarrow \quad dy = e^x \sin x \, dx$

Integrating both sides, $\int 1 \, dy = \int e^x \sin x \, dx$

or $\quad y = I + C \qquad ...(i)$

where $I = \int \underset{I}{e^x} \underset{II}{\sin x} \, dx \qquad ...(ii)$

$$\left[\text{Applying Product Rule } \int I \, . \, II \, dx = I \int II \, dx - \int \left(\frac{d}{dx}(I) \int II \, dx\right) dx\right]$$

$$= e^x (-\cos x) - \int e^x (-\cos x) \, dx$$

$\Rightarrow \quad I = -e^x \cos x + \int \underset{I}{e^x} \underset{II}{\cos x} \, dx$

Again applying product rule,

$$I = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$$

$\Rightarrow \quad I = e^x (-\cos x + \sin x) - I \qquad$ [By (*ii*)]

Transposing $2I = e^x (\sin x - \cos x)$

$\therefore \quad I = \frac{e^x}{2} (\sin x - \cos x)$

Putting this value of I in (*i*), the required solution is

$$y = \frac{1}{2} e^x (\sin x - \cos x) + c \qquad ...(iii)$$

To find *c*. Given that required curve (*i*) passes through the point (0, 0).

Putting $x = 0$ and $y = 0$ in (*iii*),

$$0 = \frac{1}{2}(-1) + c \quad \text{or} \quad 0 = \frac{-1}{2} + c \quad \therefore \quad c = \frac{1}{2}$$

Putting $c = \frac{1}{2}$ in (*iii*), the required equation of the curve is

$$y = \frac{1}{2} e^x (\sin x - \cos x) + \frac{1}{2}$$

L.C.M. = 2 $\therefore$ $2y = e^x (\sin x - \cos x) + 1$ or $2y - 1 = e^x(\sin x - \cos x)$ which is the required equation of the curve.

16. For the differential equation $xy \frac{dy}{dx} = (x + 2)(y + 2)$, find the solution curve passing through the point (1, – 1).

Sol. The given differential equation is $xy \frac{dy}{dx} = (x + 2)(y + 2)$

$\Rightarrow$ $$xy\, dy = (x + 2)(y + 2)\, dx$$

Separating variables $$\frac{y}{y+2}\, dy = \frac{x+2}{x}\, dx$$

Integrating both sides, $$\int \frac{y}{y+2}\, dy = \int \frac{x+2}{x}\, dx$$

$\Rightarrow$ $$\int \frac{y+2-2}{y+2}\, dy = \int \left(\frac{x}{x} + \frac{2}{x}\right) dx$$

$\Rightarrow$ $$\int \left(\frac{y+2}{y+2} - \frac{2}{y+2}\right) dy = \int \left(1 + \frac{2}{x}\right) dx$$

$\Rightarrow$ $$\int \left(1 - \frac{2}{y+2}\right) dy = \int \left(1 + \frac{2}{x}\right) dx$$

$\Rightarrow$ $y - 2 \log | y + 2 | = x + 2 \log | x | + c$

$\Rightarrow$ $y - x = \log (y + 2)^2 + \log x^2 + c$ $\qquad | \because \; | x |^2 = x^2$

$\Rightarrow$ $y - x = \log ((y + 2)^2 x^2) + c$ $\qquad ...(i)$

To find *c*. Curve (*i*) passes through the point (1, – 1).

Putting $x = 1$ and $y = -1$ in (*i*), $-1 - 1 = \log (1) + c$

or $-2 = c$ $\qquad (\because \; \log 1 = 0)$

Putting $c = -2$ in (*i*), the particular solution curve is

$$y - x = \log ((y + 2)^2 x^2) - 2$$

or $y - x + 2 = \log ((y + 2)^2 x^2)$.

17. Find the equation of the curve passing through the point (0, – 2) given that at any point (*x*, *y*) on the curve the product of the slope of its tangent and *y*-coordinate of the point is equal to the *x*-coordinate of the point.

Sol. Let P(x, y) be any point on the required curve.

According to the question,

(Slope of the tangent to the curve at P(x, y)) × y = x

$$\Rightarrow \quad \frac{dy}{dx} \cdot y = x \quad \Rightarrow \quad y\, dy = x\, dx$$

Now variables are separated.

Integrating both sides $\int y\ dy = \int x\ dx$ $\quad \therefore \quad \frac{y^2}{2} = \frac{x^2}{2} + c$

Multiplying by L.C.M. = 2, $y^2 = x^2 + 2c$

or $\quad y^2 = x^2 + \text{A}$ $\quad$ ...(i)

where A = 2c.

Given: Curve (i) passes through the point (0, – 2).

Putting x = 0 and y = – 2 in (i), 4 = A.

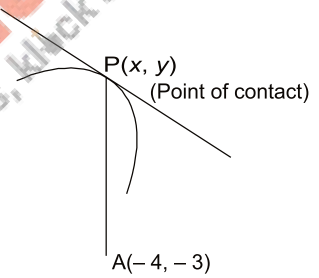
Putting A = 4 in (i), equation of required curve is

$$y^2 = x^2 + 4 \quad \text{or} \quad y^2 - x^2 = 4.$$

18. At any point (x, y) of a curve the slope of the tangent is twice the slope of the line segment joining the point of contact to the point (– 4, – 3). Find the equation of the curve given that it passes through (– 2, 1).

Sol. According to question, slope of the tangent at any point P(x, y) of the required curve.

= 2 . (Slope of the line joining the point of contact P(x, y) to the given point A(– 4, – 3)).



$$\Rightarrow \quad \frac{dy}{dx} = 2\left(\frac{y-(-3)}{x-(-4)}\right) \quad \Bigg| \ \frac{y_2 - y_1}{x_2 - x_1}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2(y+3)}{(x+4)}$$

Cross-multiplying, $(x + 4)\, dy = 2(y + 3)\, dx$

Separating variables, $\frac{dy}{y+3} = \frac{2}{x+4}\, dx$

Integrating both sides, $\int \frac{1}{y+3}\, dy = 2 \int \frac{1}{x+4}\, dx$

$\Rightarrow \quad \log |\, y + 3\, | = 2 \log |\, x + 4\, | + \log |\, c\, |$

(For log | c |, see Foot Note page 612)

$\Rightarrow \quad \log |\, y + 3\, | = \log |\, x + 4\, |^2 + \log |\, c\, | = \log |\, c\, |\, (x + 4)^2$

$\Rightarrow \quad |\, y + 3\, | = |\, c\, |\, (x + 4)^2$

$\Rightarrow \quad y + 3 = \pm\, |\, c\, |\, (x + 4)^2$

$\Rightarrow \quad y + 3 = \text{C}(x + 4)^2$ $\quad$ ...(i) where C = ± | c |

To find C. Given that curve (i) passes through the point (– 2, 1).

Putting $x = -2$ and $y = 1$ in (i),

$$1 + 3 = C(-2 + 4)^2 \quad \text{or} \quad 4 = 4C \quad \Rightarrow \quad C = \frac{4}{4} = 1.$$

Putting C = 1 in (i), equation of required curve is

$$y + 3 = (x + 4)^2 \quad \text{or} \quad (x + 4)^2 = y + 3.$$

19. The volume of a spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of balloon after *t* seconds.

Sol. Let x be the radius of the spherical balloon at time t.

Given: Rate of change of volume of spherical balloon is constant $= k$ (say)

$$\Rightarrow \frac{d}{dt}\left(\frac{4\pi}{3}x^3\right) = k \Rightarrow \frac{4\pi}{3}3x^2\frac{dx}{dt} = k \Rightarrow 4\pi x^2\frac{dx}{dt} = k$$

Separating variables, $\quad 4\pi x^2\, dx = k\, dt$

Integrating both sides, $\quad 4\pi\int x^2\, dx = k\int 1\, dt$

$$\Rightarrow 4\pi\frac{x^3}{3} = kt + c \qquad \text{...}(i)$$

To find c: Given: Initially radius is 3 units.

$\Rightarrow$ When $t = 0$, $x = 3$

Putting $t = 0$ and $x = 3$ in (i), we have

$$\frac{4\pi}{3}(27) = c \quad \text{or} \quad c = 36\pi \qquad \text{...}(ii)$$

To find k: Given: When $t = 3$ sec, $x = 6$ units

Putting $t = 3$ and $x = 6$ in (i), $\quad \frac{4\pi}{3}(6)^3 = 3k + c.$

Putting $c = 36\pi$ from (ii), $\quad \frac{4\pi}{3}(216) = 3k + 36\pi$

or $\quad 4\pi\,(72) - 36\pi = 3k \quad \Rightarrow \quad 288\pi - 36\pi = 3k$

or $\quad 3k = 252\pi \quad \Rightarrow \quad k = 84\pi \qquad \text{...}(iii)$

Putting values of c and k from (ii) and (iii) in (i), we have

$$\frac{4\pi}{3}x^3 = 84\pi t + 36\pi$$

Dividing both sides by 4π, $\quad \frac{x^3}{3} = 21t + 9$

$\Rightarrow \quad x^3 = 63t + 27 \quad \Rightarrow \quad x = (63t + 27)^{1/3}.$

20. In a bank principal increases at the rate of *r* % per year. Find the value of *r* if ₹ 100 double itself in 10 years. ($\log_e 2 = 0.6931$)

Sol. Let P be the principal (amount) at the end of t years.

According to given, rate of increase of principal per year

$= r\%$ (of the principal)

$$\Rightarrow \quad \frac{dP}{dt} = \frac{r}{100} \times P$$

Separating variables, $\quad \frac{dP}{P} = \frac{r}{100}\, dt$

Integrating both sides, $\quad \log P = \frac{r}{100}\, t + c \qquad ...(i)$

(Clearly P being principal is > 0, and hence $\log |P| = \log P$)

To find *c*. Initial principal = ₹ 100 (given)

i.e., When $t = 0$, $P = 100$

Putting $t = 0$ and $P = 100$ in (i), $\log 100 = c$.

Putting $c = \log 100$ in (i), $\log P = \frac{r}{100}\, t + \log 100$

$$\Rightarrow \quad \log P - \log 100 = \frac{r}{100}\, t \quad \Rightarrow \quad \log \frac{P}{100} = \frac{r}{100}\, t \qquad ...(ii)$$

Putting P = double of itself = 2×100 = ₹ 200

When $t = 10$ years (given) in (ii),

$$\log \frac{200}{100} = \frac{r}{100} \times 10 \quad \Rightarrow \quad \log 2 = \frac{r}{10}$$

$\Rightarrow \quad r = 10 \log 2 = 10\,(0.6931) = 6.931\%$ (given).

21. In a bank, principal increases at the rate of 5% per year. An amount of ₹ 1000 is deposited with this bank, how much will it worth after 10 years ($e^{0.5} = 1.648$).

Sol. Let P be the principal (amount) at the end of t years.

According to given rate of increase of principal per year

$= 5\%$ (of the principal)

$$\Rightarrow \quad \frac{dP}{dt} = \frac{5}{100} \times P \quad \Rightarrow \quad \frac{dP}{dt} = \frac{P}{20}$$

$\Rightarrow \quad 20\, dP = P\, dt$

Separating variables, $\frac{dP}{P} = \frac{dt}{20}$

Integrating both sides, we have

$$\log P = \frac{1}{20} t + c \qquad ...(i)$$

To find *c*. Given: Initial principal deposited with the bank is ₹ 1000.

$\Rightarrow \quad$ When $t = 0$, $P = 1000$

Putting $t = 0$ and $P = 1000$ in (i), we have $\log 1000 = c$

Putting $c = \log 1000$ in (i), $\log P = \frac{t}{20} + \log 1000$

$$\Rightarrow \quad \log P - \log 1000 = \frac{t}{20} \quad \Rightarrow \quad \log \frac{P}{1000} = \frac{t}{20}$$

Putting $t = 10$ years (given), we have

$$\log \frac{P}{1000} = \frac{10}{20} = \frac{1}{2} = 0.5$$

$\Rightarrow \quad \frac{P}{1000} = e^{0.5}$ $\qquad$ [$\because$ If $\log x = t$, then $x = e^t$]

$\Rightarrow \quad P = 1000\, e^{0.5} = 1000\,(1.648)$ $\qquad$ [$\because$ $e^{0.8} = 1.648$ (given)]

$$= 1000 \left(\frac{1648}{1000}\right) = ₹\ 1648.$$

22. In a culture the bacteria count is 1,00,000. The number is increased by 10% in 2 hours. In how many hours will the count reach 2,00,000, if the rate of growth of bacteria is proportional to the number present.

Sol. Let x be the bacteria present in the culture at time t hours.

According to given,

Rate of **growth** of bacteria is proportional to the number present.

i.e., $\frac{dx}{dt}$ is proportional to x.

$\therefore \quad \frac{dx}{dt} = kx$ where k is the constant of proportionality ($k > 0$ because rate of growth (*i.e.,* increase) of bacteria is given.)

$\Rightarrow \quad dx = kx\, dt \qquad \Rightarrow \quad \frac{dx}{x} = k\, dt$

Integrating both sides, $\int \frac{1}{x}\, dx = k \int 1\, dt$

$\Rightarrow \qquad \log x = kt + c \qquad$ ...(*i*)

To find c. Given: Initially the bacteria count is x_0 (say) = 1,00,000.

$\Rightarrow$ When $t = 0$, $x = x_0$.

Putting these value in (*i*), $\log x_0 = c$.

Putting $c = \log x_0$ in (*i*), $\log x = kt + \log x_0$

$\Rightarrow \quad \log x - \log x_0 = kt \qquad \Rightarrow \quad \log \frac{x}{x_0} = kt \qquad$ ...(*ii*)

To find k: According to given, the number of bacteria is increased by 10% in 2 hours.

$\therefore$ Increase in bacteria in 2 hours $= \frac{10}{100} \times 1,00,000 = 10,000$

$\therefore$ x, the amount of bacteria at $t = 2$

$= 1,00,000 + 10,000 = 1,10,000 = x_1$ (say)

Putting $x = x_1$ and $t = 2$ in (*ii*),

$$\log \frac{x_1}{x_0} = 2k \qquad \Rightarrow \quad k = \frac{1}{2} \log \frac{x_1}{x_0}$$

$$\Rightarrow \quad k = \frac{1}{2} \log \frac{1,10,000}{1,00,000} = \frac{1}{2} \log \frac{11}{10}$$

Putting this value of k in (ii), we have $\log \frac{x}{x_0} = \frac{1}{2}\left(\log \frac{11}{10}\right) t$

When $x = 2{,}00{,}000$ (given);

then $\log \frac{2,00,000}{1,00,000} = \left(\frac{1}{2} \log \frac{11}{10}\right) t \quad \Rightarrow \quad \log 2 = \frac{1}{2} \log \left(\frac{11}{10}\right) t$

Cross-multiplying $2 \log 2 = \left(\log \frac{11}{10}\right) t \Rightarrow t = \frac{2 \log 2}{\left(\log \frac{11}{10}\right)}$ hours.

23. The general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$ is

(A) $e^x + e^{-y} = c$ (B) $e^x + e^y = c$ (C) $e^{-x} + e^y = c$ (D) $e^{-x} + e^{-y} = c$

Sol. The given D.E. is $\frac{dy}{dx} = e^{x+y}$

$\Rightarrow \frac{dy}{dx} = e^x \cdot e^y \qquad \Rightarrow dy = e^x \cdot e^y\, dx$

Separating variables, $\frac{dy}{(e^y)} = e^x\, dx$ or $e^{-y}\, dy = e^x\, dx$

Integrating both sides $\int e^{-y}\, dy = \int e^x\, dx$

$\Rightarrow \frac{e^{-y}}{-1} = e^x + c \Rightarrow -e^{-y} - e^x = c$

Dividing by -1, $e^{-y} + e^x = -c$

or $e^x + e^{-y} = C$ where $C = -c$ which is the required solution.

$\therefore$ Option (A) is the correct answer.