NCERT Class 12 Maths Solutions

Exercise 10.4

1. Find $|\vec{a} \times \vec{b}|$, if $\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$.

Sol. Given: $\vec{a} = \hat{i} - 7\hat{j} + 7\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 2\hat{k}$.

Therefore, $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -7 & 7 \\ 3 & -2 & 2 \end{vmatrix}$

$\left[\because \text{ If } \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \text{ and } \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k};\right.$

$\left. \text{then } \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \right]$

Expanding along first row,

$$\vec{a} \times \vec{b} = \hat{i}\begin{vmatrix} -7 & 7 \\ -2 & 2 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 7 \\ 3 & 2 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & -7 \\ 3 & -2 \end{vmatrix}$$

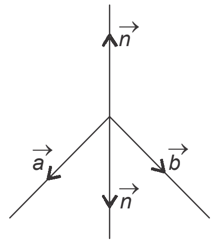
$$\Rightarrow \quad \vec{a} \times \vec{b} = \hat{i}(-14 + 14) - \hat{j}(2 - 21) + \hat{k}(-2 + 21)$$

$$= 0\hat{i} + 19\hat{j} + 19\hat{k}$$

$$\therefore \quad |\vec{a} \times \vec{b}| = \sqrt{0^2 + (19)^2 + (19)^2} = \sqrt{2(19)^2} = \sqrt{2}\,(19) = 19\sqrt{2}.$$

Result: We know that $\vec{n} = \vec{a} \times \vec{b}$ is a vector perpendicular to both the vectors $\vec{a}$ and $\vec{b}$.

Therefore, a unit vector perpendicular to both the vectors $\vec{a}$ and $\vec{b}$ is



$$\hat{n} = \pm \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}. \qquad \left[\because \hat{A} = \frac{\vec{A}}{|\vec{A}|}\right]$$

2. Find a unit vector perpendicular to each of the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ where $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$.

Sol. Given: $\vec{a} = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$

Adding, $\vec{c} = \vec{a} + \vec{b} = 4\hat{i} + 4\hat{j} + 0\hat{k}$

Subtracting $\vec{d} = \vec{a} - \vec{b} = 2\hat{i} + 0\hat{j} + 4\hat{k}$

Therefore, $\vec{n} = \vec{c} \times \vec{d} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix}$

Expanding along first row $= \hat{i}(16 - 0) - \hat{j}(16 - 0) + \hat{k}(0 - 8)$

$$\Rightarrow \quad \vec{n} = 16\hat{i} - 16\hat{j} - 8\hat{k}$$

$$\therefore |\vec{n}| = \sqrt{(16)^2 + (-16)^2 + (-8)^2} = \sqrt{256 + 256 + 64} = \sqrt{576} = 24.$$

Therefore, a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ is

$$\hat{n} = \pm \frac{\vec{n}}{|\vec{n}|} = \pm \frac{(16\hat{i} - 16\hat{j} - 8\hat{k})}{24}$$

$$= \pm \left(\frac{16}{24}\hat{i} - \frac{16}{24}\hat{j} - \frac{8}{24}\hat{k}\right) = \pm \left(\frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}\right).$$

3. If a unit vector $\hat{a}$ makes an angle $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$ and an acute angle θ with $\hat{k}$, then find θ and hence, the components of $\hat{a}$.

Sol. Let $\hat{a} = x\hat{i} + y\hat{j} + z\hat{k}$ be a unit vector ...(i)

$\Rightarrow \quad |\hat{a}| = 1 \Rightarrow \sqrt{x^2 + y^2 + z^2} = 1$

Squaring both sides, $x^2 + y^2 + z^2 = 1$ ...(ii)

Given: Angle between vectors $\hat{a}$ and $\hat{i} = \hat{i} + 0\hat{j} + 0\hat{k}$ is $\frac{\pi}{3}$.

$\therefore \quad \cos\frac{\pi}{3} = \frac{\hat{a}.\hat{i}}{|\hat{a}||\hat{i}|}$ $\left[\because \cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}\right]$

$\Rightarrow \quad \frac{1}{2} = \frac{x(1) + y(0) + z(0)}{(1)(1)}$ or $\frac{1}{2} = x$ ...(iii)

Again, **Given:** Angle between vectors $\hat{a}$ and $\hat{j} = 0\hat{i} + \hat{j} + 0\hat{k}$ is $\frac{\pi}{4}$.

$\therefore \quad \cos\frac{\pi}{4} = \frac{\hat{a}.\hat{j}}{|\hat{a}||\hat{j}|} \Rightarrow \frac{1}{\sqrt{2}} = \frac{x(0) + y(1) + z(0)}{(1)(1)}$

$\Rightarrow \quad \frac{1}{\sqrt{2}} = y$ ...(iv)

Again, **Given:** Angle between vectors $\hat{a}$ and $\hat{k} = 0\hat{i} + 0\hat{j} + \hat{k}$ is θ where θ is acute.

$\therefore \quad \cos\theta = \frac{\hat{a}.\hat{k}}{|\hat{a}||\hat{k}|} = \frac{x(0) + y(0) + z(1)}{(1)(1)} = z$ ...(v)

Putting values of x, y and z from (iii), (iv) and (v) in (ii),

$$\frac{1}{4} + \frac{1}{2} + \cos^2\theta = 1$$

$\Rightarrow \quad \cos^2\theta = 1 - \frac{1}{4} - \frac{1}{2} = \frac{4 - 1 - 2}{4} = \frac{1}{4} \Rightarrow \cos\theta = \pm\frac{1}{2}$

But θ is acute angle (given)

$\Rightarrow \quad \cos\theta$ is positive and hence $= \frac{1}{2} = \cos\frac{\pi}{3} \Rightarrow \theta = \frac{\pi}{3}$

From (v), $z = \cos\theta = \frac{1}{2}$

Putting values of x, y, z in (i), $\hat{a} = \frac{1}{2}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{2}\hat{k}$

$\therefore$ Components of $\hat{a}$ are coefficients of $\hat{i}, \hat{j}, \hat{k}$ in $\hat{a}$

i.e., $\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}$ and acute angle $\theta = \frac{\pi}{3}$.

4. Show that $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2\vec{a} \times \vec{b}$.

Sol. L.H.S. $= (\vec{a} - \vec{b}) \times (\vec{a} + \vec{b})$

$= \vec{a} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{a} - \vec{b} \times \vec{b}$

$= \vec{0} + \vec{a} \times \vec{b} + \vec{a} \times \vec{b} - \vec{0}$

$[\because \vec{a} \times \vec{a} = \vec{0},\ \vec{b} \times \vec{b} = \vec{0}$ and $\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}]$

$= 2\vec{a} \times \vec{b}$ = R.H.S.

5. Find λ and μ if $(2\hat{i} + 6\hat{j} + 27\hat{k}) \times (\hat{i} + \lambda\hat{j} + \mu\hat{k}) = \vec{0}$.

Sol. Given: $(2\hat{i} + 6\hat{j} + 27\hat{k}) \times (\hat{i} + \lambda\hat{j} + \mu\hat{k}) = \vec{0}$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 6 & 27 \\ 1 & \lambda & \mu \end{vmatrix} = \vec{0}$$

Expanding along first row,

$\hat{i}(6\mu - 27\lambda) - \hat{j}(2\mu - 27) + \hat{k}(2\lambda - 6) = \vec{0} = 0\hat{i} + 0\hat{j} + 0\hat{k}$

Comparing coefficients of $\hat{i}, \hat{j}, \hat{k}$ on both sides, we have

$6\mu - 27\lambda = 0$ ...(*i*)

$2\mu - 27 = 0$ ...(*ii*)

and $2\lambda - 6 = 0$ ...(*iii*)

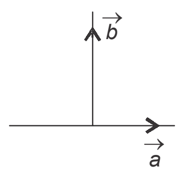
From (*ii*), $2\mu = 27 \Rightarrow \mu = \frac{27}{2}$

From (*iii*), $2\lambda = 6 \Rightarrow \lambda = \frac{6}{2} = 3$

Putting $\lambda = 3$ and $\mu = \frac{27}{2}$ in (*i*), $6\left(\frac{27}{2}\right) - 27(3) = 0$

or $81 - 81 = 0$ or $0 = 0$ which is true. $\therefore$ $\lambda = 3$ and $\mu = \frac{27}{2}$.

6. Given that $\vec{a} \cdot \vec{b} = 0$ and $\vec{a} \times \vec{b} = \vec{0}$. What can you conclude about the vectors $\vec{a}$ and $\vec{b}$?



Sol. Given: $\vec{a} \cdot \vec{b} = 0 \Rightarrow |\vec{a}|\,|\vec{b}| \cos\theta = 0$

$\Rightarrow$ Either $|\vec{a}| = 0$

or $|\vec{b}| = 0$ or $\cos\theta = 0$ ($\Rightarrow \theta = 90°$)

$\Rightarrow$ Either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$

or vector $\vec{a}$ is perpendicular to $\vec{b}$. ...(*i*)

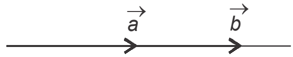
($\because$ By definition, vector $\vec{a}$ is zero vector if and only if $|\vec{a}| = 0$)

Again given $\vec{a} \times \vec{b} = \vec{0} \quad \Rightarrow \quad |\vec{a} \times \vec{b}| = 0$

$\Rightarrow \quad |\vec{a}|\,|\vec{b}| \sin\theta = 0$

$[\because \quad |\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}| \sin\theta]$

$\Rightarrow$ Either $|\vec{a}| = 0$ or $|\vec{b}| = 0$ or $\sin\theta = 0$ ($\Rightarrow \theta = 0$)



$\Rightarrow$ Either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or vectors $\vec{a}$ and $\vec{b}$ are collinear (or parallel) vectors. ...(*ii*)

We know from common sense that vectors $\vec{a}$ and $\vec{b}$ are perpendicular to each other as well as are parallel (or collinear) is impossible. ...(*iii*)

$\therefore$ From (*i*), (*ii*) and (*iii*), either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$

$\therefore \quad \vec{a} \cdot \vec{b} = 0$ and $\vec{a} \times \vec{b} = \vec{0}$

$\Rightarrow$ Either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$.

7. Let the vectors $\vec{a}, \vec{b}, \vec{c}$ be given as $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, $c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$.

Then show that $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.

Sol. Given: Vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$,

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$\therefore \quad \vec{b} + \vec{c} = (b_1 + c_1)\hat{i} + (b_2 + c_2)\hat{j} + (b_3 + c_3)\hat{k}$

$$\text{L.H.S.} = \vec{a} \times (\vec{b} + \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 + c_1 & b_2 + c_2 & b_3 + c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

[By Property of Determinants]

$$= \vec{a} \times \vec{b} + \vec{a} \times \vec{c} = \text{R.H.S.}$$

8. If either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$, then $\vec{a} \times \vec{b} = \vec{0}$. Is the converse true? Justify your answer with an example.

Sol. Given: Either $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$.

$\therefore \quad |\vec{a}| = |\vec{0}| = 0$ or $|\vec{b}| = |\vec{0}| = 0$ ...(*i*)

$\therefore \quad |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta = 0 (\sin\theta) = 0$ [By (*i*)]

$\therefore \quad \vec{a} \times \vec{b} = \vec{0}$ (By definition of zero vector)

But the converse is not true.

Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ $\therefore$ $|\vec{a}| = \sqrt{1+1+1} = \sqrt{3} \neq 0.$

$\therefore \quad \vec{a}$ is a non-zero vector.

Let $|\vec{b}| = 2(\hat{i} + \hat{j} + \hat{k}) = 2\hat{i} + 2\hat{j} + 2\hat{k}$

$\therefore |\vec{b}| = \sqrt{4+4+4}$ or $|\vec{b}| = \sqrt{12} = \sqrt{4 \times 3} = 2\sqrt{3} \neq 0.$

$\therefore \quad \vec{b}$ is a non-zero vector.

But $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{vmatrix}$

Taking 2 common from R_3, $= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \vec{0}$

($\because$ R_2 and R_3 are identical)

9. Find the area of the triangle with vertices A(1, 1, 2), B(2, 3, 5) and C(1, 5, 5).

Sol. Vertices of ΔABC are A(1, 1, 2), B(2, 3, 5) and C(1, 5, 5).

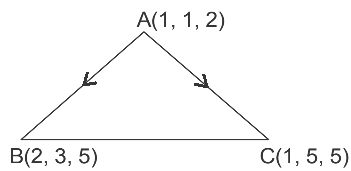
$\therefore$ Position Vector (P.V.) of point A is (1, 1, 2) $= \hat{i} + \hat{j} + 2\hat{k}$

P.V. of point B is (2, 3, 5)

$= 2\hat{i} + 3\hat{j} + 5\hat{k}$

P.V. of point C is (1, 5, 5)

$= \hat{i} + 5\hat{j} + 5\hat{k}$



$\therefore \quad \vec{AB}$ = P.V. of point B – P.V. of point A

$= 2\hat{i} + 3\hat{j} + 5\hat{k} - (\hat{i} + \hat{j} + 2\hat{k})$

$= 2\hat{i} + 3\hat{j} + 5\hat{k} - \hat{i} - \hat{j} - 2\hat{k} = \hat{i} + 2\hat{j} + 3\hat{k}$

and $\vec{AC}$ = P.V. of point C – P.V. of point A

$= \hat{i} + 5\hat{j} + 5\hat{k} - (\hat{i} + \hat{j} + 2\hat{k}) = \hat{i} + 5\hat{j} + 5\hat{k} - \hat{i} - \hat{j} - 2\hat{k}$

$= 0\hat{i} + 4\hat{j} + 3\hat{k}$

$$\therefore \quad \vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 0 & 4 & 3 \end{vmatrix}$$

$= \hat{i}(6 - 12) - \hat{j}(3 - 0) + \hat{k}(4 - 0) = -6\hat{i} - 3\hat{j} + 4\hat{k}$

We know that **area of triangle ABC**

$$= \frac{1}{2} |\vec{AB} \times \vec{AC}| = \frac{1}{2}\sqrt{36 + 9 + 16} \quad | \sqrt{x^2 + y^2 + z^2}$$

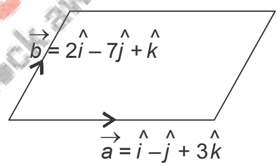
$$= \frac{1}{2}\sqrt{61} \text{ sq. units.}$$

10. Find the area of the parallelogram whose adjacent sides are determined by the vectors $\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$ and $\vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$.

Sol. Given: Vectors representing two adjacent sides of a parallelogram are

$$\vec{a} = \hat{i} - \hat{j} + 3\hat{k}$$

and $\quad \vec{b} = 2\hat{i} - 7\hat{j} + \hat{k}$.



$$\therefore \quad \vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ 2 & -7 & 1 \end{vmatrix}$$

$= \hat{i}(-1 + 21) - \hat{j}(1 - 6) + \hat{k}(-7 + 2) = 20\hat{i} + 5\hat{j} - 5\hat{k}$

We know that **area of parallelogram** $= |\vec{a} \times \vec{b}|$

$$= \sqrt{400 + 25 + 25} = \sqrt{450} = \sqrt{25 \times 9 \times 2}$$

$$= 5(3)\sqrt{2} = 15\sqrt{2} \text{ square units.}$$

Note. Area of parallelogram whose **diagonal vectors** are $\vec{\alpha}$ and $\vec{\beta}$ is $\frac{1}{2}|\vec{\alpha} \times \vec{\beta}|$.

11. Let the vectors $\vec{a}$ and $\vec{b}$ be such that $|\vec{a}| = 3$, $|\vec{b}| = \frac{\sqrt{2}}{3}$, then $\vec{a} \times \vec{b}$ is a unit vector, if the angle between $\vec{a}$ and $\vec{b}$ is

(A) $\frac{\pi}{6}$ **(B)** $\frac{\pi}{4}$ **(C)** $\frac{\pi}{3}$ **(D)** $\frac{\pi}{2}$.

Sol. Given: $|\vec{a}| = 3$, $|\vec{b}| = \frac{\sqrt{2}}{3}$ and $\vec{a} \times \vec{b}$ is a unit vector.

$\Rightarrow \quad |\vec{a} \times \vec{b}| = 1 \qquad \Rightarrow \quad |\vec{a}|\,|\vec{b}| \sin\theta = 1$

where θ is the angle between vectors $\vec{a}$ and $\vec{b}$.

Putting values of $|\vec{a}|$ and $|\vec{b}|$, $3\left(\frac{\sqrt{2}}{3}\right)\sin\theta = 1$

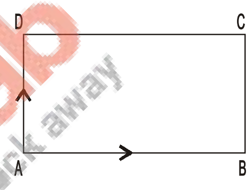
$\Rightarrow \quad \sqrt{2}\sin\theta = 1 \quad \Rightarrow \quad \sin\theta = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4} \quad \Rightarrow \quad \theta = \frac{\pi}{4}$

∴ Option (B) is the correct answer.

12. Area of a rectangle having vertices A, B, C and D with position vectors $-\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}$, $\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}$, $\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$ and $-\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k}$, respectively, is

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) 4

Sol. Given: ABCD is a rectangle.



We know that $\vec{AB}$ = P.V. of point B – P.V. of point A

$= \hat{i} + \frac{1}{2}\hat{j} + 4\hat{k} - \left(-\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}\right)$

$= \hat{i} + \frac{1}{2}\hat{j} + 4\hat{k} + \hat{i} - \frac{1}{2}\hat{j} - 4\hat{k} = 2\hat{i} + 0\hat{j} + 0\hat{k}$

∴ $AB = |\vec{AB}| = \sqrt{4+0+0} = \sqrt{4} = 2$

and $\vec{AD}$ = P.V. of point D – P.V. of point A

$= -\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k} - \left(-\hat{i} + \frac{1}{2}\hat{j} + 4\hat{k}\right)$

$= -\hat{i} - \frac{1}{2}\hat{j} + 4\hat{k} + \hat{i} - \frac{1}{2}\hat{j} - 4\hat{k} = -\hat{j} = 0\hat{i} - \hat{j} + 0\hat{k}$

∴ $AD = |\vec{AD}| = \sqrt{0+1+0} = \sqrt{1} = 1$

∴ Area of rectangle ABCD = (AB)(AD) (= Length × Breadth)
= 2(1) = 2 sq. units

∴ Option (C) is the correct answer.

or Area of rectangle ABCD = $\left|\vec{AB} \times \vec{AD}\right|$.