Exercise 7.4

Integrate the following functions in Exercises 1 to 9:

1. $\dfrac{3x^2}{x^6+1}$

Sol. Let $\text{I} = \displaystyle\int \frac{3x^2}{x^6+1}\,dx = \int \frac{3x^2}{(x^3)^2+1^2}\,dx$ ...(*i*)

Put $x^3 = t$

$\therefore \quad 3x^2 = \dfrac{dt}{dx} \quad \Rightarrow \quad 3x^2\,dx = dt$

$\therefore \quad$ From (*i*), $\text{I} = \displaystyle\int \frac{dt}{t^2+1^2} = \frac{1}{1}\tan^{-1}\frac{t}{1} + \text{C}$

$$\left[\because \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

Putting $t = x^3$; $= \tan^{-1}(x^3) + \text{C}$.

Note. $ax^2 + b$ $(a \neq 0)$ is called a **pure quadratic.**

2. $\dfrac{1}{\sqrt{1+4x^2}}$

Sol. Let $\text{I} = \displaystyle\int \frac{1}{\sqrt{1+4x^2}}\,dx = \int \frac{1}{\sqrt{(2x)^2+1^2}}\,dx$

Using $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \log\left|x+\sqrt{x^2+a^2}\right|$,

$$\text{I} = \frac{\log\left|(2x)+\sqrt{(2x)^2+1^2}\right|}{2 \to \text{Coeff. of } x} + \text{C} \quad = \frac{1}{2}\log\left|2x+\sqrt{4x^2+1}\right| + \text{C}.$$

3. $\dfrac{1}{\sqrt{(2-x)^2+1}}$

Sol. Let $\text{I} = \displaystyle\int \frac{1}{\sqrt{(2-x)^2+1}}\,dx = \int \frac{1}{\sqrt{(2-x)^2+1}}\,dx$

Using $\displaystyle\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \log\left|x+\sqrt{x^2+a^2}\right|$,

$$= \frac{\log\left|(2-x)+\sqrt{(2-x)^2+1^2}\right|}{-1 \to \text{Coeff. of } x} + \text{C}$$

$$= -\log\left|2-x+\sqrt{4+x^2-4x+1}\right| + \text{C}$$

$$= \log\left|\frac{1}{2-x+\sqrt{x^2-4x+5}}\right| + \text{C}.$$

$$\left[\because -\log\frac{m}{n} = -(\log m - \log n) = \log n - \log m = \log\frac{n}{m}\right]$$

4. $\dfrac{1}{\sqrt{9-25x^2}}$

Sol. Let $\text{I} = \displaystyle\int \frac{1}{\sqrt{9-25x^2}}\,dx = \int \frac{1}{\sqrt{3^2-(5x)^2}}\,dx$

$$= \frac{\sin^{-1}\frac{5x}{3}}{5 \to \text{Coeff. of } x} + \text{C} \qquad \left[\because \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}\right]$$

$$= \frac{1}{5}\sin^{-1}\left(\frac{5x}{3}\right) + \text{C}.$$

5. $\dfrac{3x}{1+2x^4}$

Sol. Let $\text{I} = \displaystyle\int \frac{3x}{1+2x^4}\,dx = \frac{3}{2}\int \frac{2x}{1+2(x^2)^2}\,dx$ ...(*i*)

Put $x^2 = t$. $\therefore$ $2x = \dfrac{dt}{dx}$ $\Rightarrow$ $2x\,dx = dt$

$\therefore$ From (*i*), $\text{I} = \dfrac{3}{2}\displaystyle\int \frac{dt}{1+2t^2} = \frac{3}{2}\int \frac{1}{(\sqrt{2}t)^2+1^2}\,dt$

$$= \frac{3}{2}\,\frac{\frac{1}{1}\tan^{-1}\frac{\sqrt{2}t}{1}}{\sqrt{2} \to \text{Coeff. of } t} + \text{C} = \frac{3}{2\sqrt{2}}\tan^{-1}(\sqrt{2}\,t) + \text{C}$$

Putting $t = x^2$, $= \dfrac{3}{2\sqrt{2}}\tan^{-1}(\sqrt{2}\,x^2) + \text{C}.$

6. $\dfrac{x^2}{1-x^6}$

Sol. Let $\text{I} = \displaystyle\int \frac{x^2}{1-x^6}\,dx = \int \frac{x^2}{1-(x^3)^2}\,dx = \frac{1}{3}\int \frac{3x^2}{1-(x^3)^2}\,dx$

Put $x^3 = t$. Therefore $3x^2 = \dfrac{dt}{dx}$ $\Rightarrow$ $3x^2\,dx = dt.$

$$\therefore \quad \text{I} = \frac{1}{3}\int \frac{dt}{1-t^2} = \frac{1}{3}\int \frac{1}{1^2-t^2}\,dt = \frac{1}{3}\,\frac{1}{2\times 1}\log\left|\frac{1+t}{1-t}\right| + \text{C}$$

$$\left[\because \int \frac{1}{a^2-x^2}\,dx = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|\right]$$

Putting $t = x^3$, $= \dfrac{1}{6}\log\left|\dfrac{1+x^3}{1-x^3}\right| + \text{C}.$

7. $\dfrac{x-1}{\sqrt{x^2-1}}$

Sol. Let $\text{I} = \displaystyle\int \frac{x-1}{\sqrt{x^2-1}}\,dx = \int \left(\frac{x}{\sqrt{x^2-1}} - \frac{1}{\sqrt{x^2-1}}\right)dx$

$$= \int \frac{x}{\sqrt{x^2 - 1}}\, dx - \int \frac{1}{\sqrt{x^2 - 1^2}}\, dx$$

$$= \frac{1}{2} \int \frac{2x}{\sqrt{x^2 - 1}}\, dx - \log \left| x + \sqrt{x^2 - 1^2} \right| \quad ...(i)$$

$$\left(\because \int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \log | x + \sqrt{x^2 - a^2} | \right)$$

Let $I_1 = \int \frac{2x}{\sqrt{x^2 - 1}}\, dx$

Put $x^2 - 1 = t$. Therefore $2x = \frac{dt}{dx} \Rightarrow 2x\, dx = dt$

$\therefore \quad I_1 = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2}\, dt = \frac{t^{1/2}}{\frac{1}{2}} = 2\sqrt{t} \qquad = 2\sqrt{x^2 - 1} + C$

Putting this value of $I_1 = \int \frac{2x}{\sqrt{x^2 - 1}}\, dx$ in (i),

$$I = \frac{1}{2} (2\sqrt{x^2 - 1} + C) - \log | x + \sqrt{x^2 - 1} |$$

$$= \sqrt{x^2 - 1} + \frac{C}{2} - \log | x + \sqrt{x^2 - 1} |$$

$$= \sqrt{x^2 - 1} - \log | x + \sqrt{x^2 - 1} | + C_1 \text{ where } C_1 = \frac{C}{2}.$$

8. $\boldsymbol{\frac{x^2}{\sqrt{x^6 + a^6}}}$

Sol. Let $I = \int \frac{x^2}{\sqrt{x^6 + a^6}}\, dx = \frac{1}{3} \int \frac{3x^2}{\sqrt{(x^3)^2 + a^6}}\, dx \quad ...(i)$

Put $\boldsymbol{x^3 = t}$. Therefore $3x^2 = \frac{dt}{dx} \Rightarrow 3x^2\, dx = dt$.

$\therefore$ From (i), $I = \frac{1}{3} \int \frac{dt}{\sqrt{t^2 + a^6}} = \frac{1}{3} \int \frac{1}{\sqrt{t^2 + (a^3)^2}}\, dt$

$$= \frac{1}{3} \log \left| t + \sqrt{t^2 + (a^3)^2} \right| + C \left[\because \int \frac{1}{\sqrt{x^2 + a^2}}\, dx = \log \left| x + \sqrt{x^2 + a^2} \right| \right]$$

Putting $t = x^3$, $= \frac{1}{3} \log \left| x^3 + \sqrt{x^6 + a^6} \right| + C.$

9. $\boldsymbol{\frac{\sec^2 x}{\sqrt{\tan^2 x + 4}}}$

Sol. Let $I = \int \frac{\sec^2 x}{\sqrt{\tan^2 x + 4}}\, dx \quad ...(i)$

Put $\tan x = t$. $\therefore \quad \sec^2 x = \dfrac{dt}{dx} \quad \Rightarrow \quad \sec^2 x \, dx = dt$

$\therefore$ From (*i*), $\text{I} = \displaystyle\int \frac{dt}{\sqrt{t^2 + 4}} = \int \frac{1}{\sqrt{t^2 + 2^2}} \, dt$

$$= \log \left| t + \sqrt{t^2 + 2^2} \right| + \text{C} \left[\because \int \frac{1}{\sqrt{x^2 + a^2}} dx = \log \left| x + \sqrt{x^2 + a^2} \right| \right]$$

Putting $t = \tan x$, $\text{I} = \log \left| \tan x + \sqrt{\tan^2 x + 4} \right| + \text{C}$.

Integrate the following functions in Exercises 10 to 18:

Note. Rule to evaluate

$$\int \frac{1}{\textbf{Quadratic}} dx \textbf{ or } \int \frac{1}{\sqrt{\textbf{Quadratic}}} dx \textbf{ or } \int \sqrt{\textbf{Quadratic}} \, dx$$

Write Quadratic. Take coefficient of x^2 common to make it unity. Then complete squares by adding and subtracting $\left(\frac{1}{2} \text{ coefficient of } x \right)^2$

10. $\dfrac{1}{\sqrt{x^2 + 2x + 2}}$

Sol. $\displaystyle\int \frac{1}{\sqrt{x^2 + 2x + 2}} \, dx = \int \frac{1}{\sqrt{x^2 + 2x + 1 + 1}} \, dx = \int \frac{1}{\sqrt{(x+1)^2 + 1^2}} \, dx$

Using $\displaystyle\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \log | x + \sqrt{x^2 + a^2} |$

$= \log | x + 1 + \sqrt{(x+1)^2 + 1^2} | + c = \log | x + 1 + \sqrt{x^2 + 2x + 2} | + c.$

11. $\dfrac{1}{9x^2 + 6x + 5}$

Sol. Let $\text{I} = \displaystyle\int \frac{1}{9x^2 + 6x + 5} \, dx$ ...(*i*)

$\displaystyle\int \frac{1}{\text{Quadratic}} \, dx$

Here Quadratic expression $= 9x^2 + 6x + 5$

Making coefficient of x^2 unity, $= 9 \left(x^2 + \frac{6x}{9} + \frac{5}{9} \right)$

$= 9 \left(x^2 + \frac{2x}{3} + \frac{5}{9} \right)$

To complete squares, adding and subtracting $\left(\frac{1}{2} \text{ Coefficient of } x \right)^2$

$= \left(\left(\frac{1}{2} \times \frac{2}{3} \right)^2 = \left(\frac{1}{3} \right)^2 = \frac{1}{9} \right) = 9 \left(x^2 + \frac{2x}{3} + \left(\frac{1}{3} \right)^2 - \frac{1}{9} + \frac{5}{9} \right)$

$$= 9\left(\left(x + \frac{1}{3}\right)^2 + \frac{4}{9}\right) \quad \Rightarrow \quad 9x^2 + 6x + 5 = 9\left[\left(x + \frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2\right]$$

Putting this value in (*i*), $\quad I = \int \frac{1}{9\left[\left(x + \frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2\right]} dx$

$$= \frac{1}{9} \int \frac{1}{\left(x + \frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2} dx$$

$$= \frac{1}{9} \frac{1}{\left(\frac{2}{3}\right)} \tan^{-1} \frac{x + \frac{1}{3}}{\frac{2}{3}} + c \qquad \left(\because \int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}\right)$$

$$= \frac{1}{9} \cdot \frac{3}{2} \tan^{-1} \left(\frac{\frac{3x+1}{3}}{\frac{2}{3}}\right) + c \;=\; \frac{1}{6} \tan^{-1} \left(\frac{3x+1}{2}\right) + c.$$

12. $\dfrac{\mathbf{1}}{\sqrt{\mathbf{7 - 6}\boldsymbol{x} \mathbf{-} \boldsymbol{x}^{\mathbf{2}}}}$

Sol. Let $I = \int \frac{1}{\sqrt{7 - 6x - x^2}} dx$ ...(*i*) $\Big|$ Type $\int \frac{1}{\sqrt{\text{Quadratic}}} dx$

Here Quadratic expression is $7 - 6x - x^2 = -x^2 - 6x + 7$.
Making coefficient of x^2 unity, $= -(x^2 + 6x - 7)$.

To complete squares, adding and subtracting $\left(\frac{1}{2} \text{ coefficient of } x\right)^2$

$$= \left(\frac{1}{2} \times 6\right)^2 = 9$$

$$= -[x^2 + 6x + 9 - 9 - 7] = -[(x + 3)^2 - 16] \qquad ...(ii)$$
$$= -(x + 3)^2 + 16 = 4^2 - (x + 3)^2 \qquad ...(iii)$$

(**Note.** Must adjust negative sign outside Eqn. (*ii*) in the bracket as shown above because otherwise we shall get $\sqrt{-1} = i$ on taking square roots.]

Putting the value of quadratic expression from (*iii*) in (*i*),

$$I = \int \frac{1}{\sqrt{4^2 - (x + 3)^2}} dx = \sin^{-1}\left(\frac{x + 3}{4}\right) + c$$

$$\left[\because \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}\right]$$

13. $\dfrac{\mathbf{1}}{\sqrt{(\boldsymbol{x} \mathbf{- 1})(\boldsymbol{x} \mathbf{- 2})}}$

Sol. Let $I = \int \frac{1}{\sqrt{(x - 1)(x - 2)}} dx = \int \frac{1}{\sqrt{x^2 - 2x - x + 2}} dx$

$$= \int \frac{1}{\sqrt{x^2 - 3x + 2}} \qquad ...(i)$$

Here quadratic expression is $x^2 - 3x + 2$. Coefficient of x^2 is already unity. To complete squares, adding and subtracting

$$\left(\frac{1}{2} \text{ coefficient of } x\right)^2, \text{ i.e., } \left(-\frac{3}{2}\right)^2 = \left(\frac{3}{2}\right)^2$$

$$x^2 - 3x + 2 = x^2 - 3x + \left(\frac{3}{2}\right)^2 - \frac{9}{4} + 2$$

$$= \left(x - \frac{3}{2}\right)^2 - \frac{1}{4} \qquad \left[\because \frac{-9}{4} + 2 = \frac{-9+8}{4} = \frac{-1}{4}\right]$$

$$= \left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2 \qquad ...(ii)$$

Putting this value in (i), $\text{I} = \int \frac{1}{\sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}\, dx$

$$= \log \left| x - \frac{3}{2} + \sqrt{\left(x - \frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2} \right| + c$$

$$\left[\because \int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \log \left| x + \sqrt{x^2 - a^2} \right|\right]$$

$$= \log \left| x - \frac{3}{2} + \sqrt{x^2 - 3x + 2} \right| + c. \qquad \text{[By } (ii)\text{]}$$

14. $\dfrac{\mathbf{1}}{\sqrt{\mathbf{8 + 3}\boldsymbol{x} - \boldsymbol{x}^{\mathbf{2}}}}$

Sol. Let $\text{I} = \int \frac{1}{\sqrt{8 + 3x - x^2}}\, dx \qquad ...(i)$

Here quadratic expression is $8 + 3x - x^2 = -x^2 + 3x + 8$.
Making coefficient of x^2 unity, $= -(x^2 - 3x - 8)$.
To complete squares, adding and subtracting

$$\left(\frac{1}{2} \text{ coefficient of } x\right)^2 = \left(\frac{3}{2}\right)^2$$

$$8 + 3x - x^2 = -\left(x^2 - 3x + \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2 - 8\right)$$

$$= -\left[\left(x - \frac{3}{2}\right)^2 - \frac{9}{4} - 8\right] = -\left[\left(x - \frac{3}{2}\right)^2 - \frac{41}{4}\right] = \frac{41}{4} - \left(x - \frac{3}{2}\right)^2$$

(See **Note** given in the solution of Q.N. 12)

$$= \left(\frac{\sqrt{41}}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2 \qquad ...(ii)$$

Putting this value in (i), $\text{I} = \int \frac{1}{\sqrt{\left(\frac{\sqrt{41}}{2}\right)^2 - \left(x - \frac{3}{2}\right)^2}}\, dx$

$$= \sin^{-1} \frac{x - \frac{3}{2}}{\frac{\sqrt{41}}{2}} + c \qquad \left[\because \int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1} \frac{x}{a}\right]$$

$$= \sin^{-1} \left(\frac{2x - 3}{\sqrt{41}}\right) + c.$$

15. $\dfrac{\mathbf{1}}{\sqrt{(\boldsymbol{x} - \boldsymbol{a})(\boldsymbol{x} - \boldsymbol{b})}}$

Sol. Let $\text{I} = \int \frac{1}{\sqrt{(x-a)(x-b)}}\, dx = \int \frac{1}{\sqrt{x^2 - bx - ax + ab}}\, dx$

$$= \int \frac{1}{\sqrt{x^2 - x(a+b) + ab}} \qquad ...(i)$$

Here Quadratic expression $= x^2 - x(a + b) + ab$

Adding and subtracting $\left(\frac{1}{2}\text{ coefficient of } x\right)^2 = \left(\frac{a+b}{2}\right)^2$

$$= x^2 - x(a + b) + \left(\frac{a+b}{2}\right)^2 - \left(\frac{a+b}{2}\right)^2 + ab$$

$$= \left(x - \left(\frac{a+b}{2}\right)\right)^2 - \left(\frac{(a+b)^2}{4} - ab\right)$$

$$= \left(x - \left(\frac{a+b}{2}\right)\right)^2 - \left(\frac{(a+b)^2 - 4ab}{4}\right) = \left(x - \left(\frac{a+b}{2}\right)\right)^2 - \frac{(a-b)^2}{4}$$

$$= \left(x - \left(\frac{a+b}{2}\right)\right)^2 - \left(\frac{a-b}{2}\right)^2 \qquad ...(ii)$$

$(\because\ (a + b)^2 - 4ab = a^2 + b^2 + 2ab - 4ab$
$= a^2 + b^2 - 2ab = (a - b)^2)$

Putting this value in (i),

$$\text{I} = \int \frac{1}{\sqrt{\left(x - \left(\frac{a+b}{2}\right)\right)^2 - \left(\frac{a-b}{2}\right)^2}}\, dx$$

$$= \log \left| x - \left(\frac{a+b}{2}\right) + \sqrt{\left(x - \left(\frac{a+b}{2}\right)\right)^2 - \left(\frac{a-b}{2}\right)^2} \right| + c$$

$$\left[\because \int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \log | x + \sqrt{x^2 - a^2} |\right]$$

$$= \log \left| x - \left(\frac{a+b}{2}\right) + \sqrt{x^2 - x(a+b) + ab} \right| + c \quad \text{[By } (ii)]$$

Note. Method to evaluate $\int \dfrac{\textbf{Linear}}{\textbf{Quadratic}} dx$ **or** $\int \dfrac{\textbf{Linear}}{\sqrt{\textbf{Quadratic}}} dx$

or $\int \textbf{Linear} \sqrt{\textbf{Quadratic}}\ dx$**.**

Write linear = A $\dfrac{d}{dx}$ (Quadratic) + B.

Find values of A and B by comparing coefficients of x and constant terms on both sides.

16. $\dfrac{4x+1}{\sqrt{2x^2+x-3}}$

Sol. Let I = $\displaystyle\int \frac{4x+1}{\sqrt{2x^2+x-3}}\, dx$ ...(i)

Here $\dfrac{d}{dx}$ (Quadratic $2x^2 + x - 3$) is $(4x + 1)$, the numerator.

So put $2x^2 + x - 3 = t$.

$\therefore \quad (4x + 1) = \dfrac{dt}{dx} \quad \Rightarrow \quad (4x + 1)\, dx = dt$

$\therefore$ From (i), $\text{I} = \displaystyle\int \frac{dt}{\sqrt{t}} = \int t^{-1/2}\, dt = \frac{t^{1/2}}{\frac{1}{2}} + c$

$$= 2\sqrt{t} + c = 2\sqrt{2x^2 + x - 3} + c.$$

17. $\dfrac{x+2}{\sqrt{x^2-1}}$

Sol. Let I = $\displaystyle\int \frac{x+2}{\sqrt{x^2-1}}\, dx = \int \left(\frac{x}{\sqrt{x^2-1}} + \frac{2}{\sqrt{x^2-1}}\right) dx$

$$= \int \frac{x}{\sqrt{x^2-1}}\, dx + 2\int \frac{1}{\sqrt{x^2-1^2}}\, dx$$

$$= \int \frac{x}{\sqrt{x^2-1}}\, dx + 2 \log | x + \sqrt{x^2 - 1^2} | + c \quad ...(i)$$

Let $\text{I}_1 = \displaystyle\int \frac{x}{\sqrt{x^2-1}}\, dx = \frac{1}{2} \int \frac{2x}{\sqrt{x^2-1}}\, dx$

Put $x^2 - 1 = t$. Therefore $2x = \dfrac{dt}{dx}$ or $2x\ dx = dt$

$\therefore \quad \text{I}_1 = \displaystyle\frac{1}{2} \int \frac{dt}{\sqrt{t}} = \frac{1}{2} \int t^{-1/2}\, dt = \frac{1}{2} \frac{t^{1/2}}{\frac{1}{2}} = \sqrt{t} = \sqrt{x^2 - 1}$

Putting this value of ($I_1 =$) $\int \frac{x}{\sqrt{x^2-1}}\,dx = \sqrt{x^2-1}$ in (i)

$$I = \sqrt{x^2-1} + 2\log|x + \sqrt{x^2-1}| + c.$$

18. $\mathbf{\dfrac{5x-2}{1+2x+3x^2}}$

Sol. Let $I = \int \frac{5x-2}{1+2x+3x^2}\,dx$ ...(i) $\quad\Big|\ \int \frac{\text{Linear}}{\text{Quadratic}}\,dx$

Let Linear = A $\frac{d}{dx}$ (Quadratic) + B

i.e., $5x - 2 = A\frac{d}{dx}(1 + 2x + 3x^2) + B$

or $5x - 2 = A(2 + 6x) + B$ ...(ii)

i.e., $5x - 2 = 2A + 6Ax + B$

Comparing coefficients of x, $6A = 5 \Rightarrow A = \frac{5}{6}$

Comparing constants, $2A + B = -2$

Putting $A = \frac{5}{6}$, $\frac{10}{6} + B = -2$

$\Rightarrow B = -2 - \frac{10}{6} = \frac{-22}{6}$ or $B = \frac{-11}{3}$

Putting values of A and B in (ii), $5x - 2 = \frac{5}{6}(2 + 6x) - \frac{11}{3}$

Putting this value of $5x - 2$ in (i),

$$I = \int \frac{\frac{5}{6}(2+6x) - \frac{11}{3}}{1+2x+3x^2}\,dx$$

$$\Rightarrow I = \frac{5}{6}\int \frac{2+6x}{1+2x+3x^2}\,dx - \frac{11}{3}\int \frac{1}{1+2x+3x^2}\,dx$$

$$= \frac{5}{6} I_1 - \frac{11}{3} I_2 \quad ...(iii)$$

Here $I_1 = \int \frac{2+6x}{1+2x+3x^2}\,dx$

Put Denominator $1 + 2x + 3x^2 = t$.

$\therefore 2 + 6x = \frac{dt}{dx} \Rightarrow (2 + 6x)\,dx = dt$

$\therefore I_1 = \int \frac{dt}{t} = \int \frac{1}{t}\,dt = \log|t| = \log|1 + 2x + 3x^2|$ ...(iv)

Again $I_2 = \int \frac{1}{1+2x+3x^2}\,dx = \int \frac{1}{3x^2+2x+1}\,dx \quad\Big|\ \int \frac{1}{\text{Quadratic}}\,dx$

Now Quadratic Expression $= 3x^2 + 2x + 1$.

Making coefficient of x^2 unity $= 3\left(x^2 + \frac{2}{3}x + \frac{1}{3}\right)$

Completing squares $= 3\left[x^2 + \frac{2}{3}x + \left(\frac{1}{3}\right)^2 + \frac{1}{3} - \frac{1}{9}\right]$

$$= 3\left[\left(x + \frac{1}{3}\right)^2 + \frac{2}{9}\right] \qquad \left|\because \frac{1}{3} - \frac{1}{9} = \frac{3-1}{9} = \frac{2}{9}\right.$$

$$\Rightarrow \quad I_2 = \int \frac{1}{3\left[\left(x+\frac{1}{3}\right)^2 + \frac{2}{9}\right]} dx = \frac{1}{3}\int \frac{1}{\left(x+\frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2} dx$$

$$= \frac{1}{3} \frac{1}{\left(\frac{\sqrt{2}}{3}\right)} \tan^{-1} \frac{x+\frac{1}{3}}{\frac{\sqrt{2}}{3}} \qquad \left[\because \int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}\right]$$

$$\Rightarrow \quad I_2 = \frac{1}{3} \cdot \frac{3}{\sqrt{2}} \tan^{-1} \frac{3x+1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) \qquad ...(v)$$

Putting values of I_1 and I_2 from (*iv*) and (*v*) in (*iii*), we have

$$I = \frac{5}{6} \log |1 + 2x + 3x^2| - \frac{11}{3} \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{3x+1}{\sqrt{2}}\right) + c.$$

Integrate the functions in Exercises 19 to 23:

19. $\dfrac{\mathbf{6x+7}}{\sqrt{\mathbf{(x-5)(x-4)}}}$

Sol. Let $I = \int \frac{6x+7}{\sqrt{(x-5)(x-4)}} dx = \int \frac{6x+7}{\sqrt{x^2 - 4x - 5x + 20}} dx$

i.e., $\quad I = \int \frac{6x+7}{\sqrt{x^2 - 9x + 20}} dx \qquad ...(i) \qquad \left| \int \frac{\text{Linear}}{\sqrt{\text{Quadratic}}} dx \right.$

Let Linear = A $\frac{d}{dx}$ (Quadratic) + B

i.e., $\quad 6x + 7 = A(2x - 9) + B \qquad ...(ii)$

$\qquad = 2Ax - 9A + B$

Comparing coefficients of x, $2A = 6 \Rightarrow A = 3$

Comparing constants, $-9A + B = 7$.

Putting $A = 3$, $-27 + B = 7 \Rightarrow B = 34$

Putting values of A and B in (*ii*),

$\qquad 6x + 7 = 3(2x - 9) + 34$

Putting this value of $6x + 7$ in (*i*),

$$I = \int \frac{3(2x-9)+34}{\sqrt{x^2-9x+20}} dx$$

$$= 3\int \frac{2x-9}{\sqrt{x^2-9x+20}} dx + 34 \int \frac{1}{\sqrt{x^2-9x+20}} dx$$

$$= 3 I_1 + 34 I_2 \qquad ...(iii)$$

$$I_1 = \int \frac{2x-9}{\sqrt{x^2-9x+20}} dx$$

Put $x^2 - 9x + 20 = t$. $\therefore \quad 2x - 9 = \frac{dt}{dx}$

$\Rightarrow \quad (2x - 9)\, dx = dt$

$$\therefore \quad I_1 = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2}\, dt = \frac{t^{1/2}}{1/2} = 2\sqrt{t}$$

$$= 2\sqrt{x^2 - 9x + 20} \qquad \text{...}(iv)$$

$$I_2 = \int \frac{1}{\sqrt{x^2 - 9x + 20}}\, dx = \int \frac{1}{\sqrt{x^2 - 9x + \left(\frac{9}{2}\right)^2 + 20 - \frac{81}{4}}}\, dx$$

$$= \int \frac{1}{\sqrt{\left(x - \frac{9}{2}\right)^2 - \frac{1}{4}}}\, dx = \int \frac{1}{\sqrt{\left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2}}\, dx$$

$$= \log \left| x - \frac{9}{2} + \sqrt{\left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2} \right|$$

$$\left[\because \int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \log | x + \sqrt{x^2 - a^2} |\right]$$

$$I_2 = \log \left| x - \frac{9}{2} + \sqrt{x^2 - 9x + 20} \right| \qquad \text{...}(v)$$

$$\left(\because \left(x - \frac{9}{2}\right)^2 - \left(\frac{1}{2}\right)^2 = x^2 + \frac{81}{4} - 9x - \frac{1}{4} = x^2 - 9x + 20\right)$$

Putting values of I_1 and I_2 from (iv) and (v) in (iii),

$$I = 6\sqrt{x^2 - 9x + 20} + 34 \log \left| x - \frac{9}{2} + \sqrt{x^2 - 9x + 20} \right| + c.$$

20. $\dfrac{\boldsymbol{x + 2}}{\sqrt{\boldsymbol{4x - x^2}}}$

Sol. Let $I = \int \frac{x + 2}{\sqrt{4x - x^2}}\, dx$ ...(i) $\quad \left| \int \frac{\text{Linear}}{\sqrt{\text{Quadratic}}}\, dx \right.$

Let Linear $= A \frac{d}{dx}$ (Quadratic) + B

i.e., $x + 2 = A(4 - 2x) + B$ ...(ii)

$= 4A - 2Ax + B$

Comparing coefficients of x: $-2A = 1 \Rightarrow A = \frac{-1}{2}$

Comparing constants: $4A + B = 2$

Putting $A = \frac{-1}{2}$, $-2 + B = 2 \Rightarrow B = 4$

Putting values of A and B in (ii), $x + 2 = \frac{-1}{2}(4 - 2x) + 4$

Putting this value of $x + 2$ in (i),

$$\text{I} = \int \frac{\frac{-1}{2}(4-2x)+4}{\sqrt{4x-x^2}}\,dx = \frac{-1}{2}\int \frac{4-2x}{\sqrt{4x-x^2}}\,dx + 4\int \frac{1}{\sqrt{4x-x^2}}\,dx$$

$$= \frac{-1}{2}\,\text{I}_1 + 4\,\text{I}_2 \quad ...(iii) \qquad \text{I}_1 = \int \frac{4-2x}{\sqrt{4x-x^2}}\,dx$$

Put $4x - x^2 = t \quad \therefore \quad 4 - 2x = \frac{dt}{dx} \quad \Rightarrow \quad (4-2x)\,dx = dt$

$$\therefore \quad \text{I}_1 = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2}\,dt = \frac{t^{1/2}}{1/2} = 2\sqrt{t} = 2\sqrt{4x-x^2} \qquad ...(iv)$$

$$\text{I}_2 = \int \frac{1}{\sqrt{4x-x^2}}\,dx$$

Quadratic Expression is $4x - x^2 = -x^2 + 4x$
$= -(x^2 - 4x) = -(x^2 - 4x + 4 - 4) = -((x-2)^2 - 2^2) = 2^2 - (x-2)^2$

$$\therefore \quad \text{I}_2 = \int \frac{1}{\sqrt{2^2-(x-2)^2}}\,dx = \sin^{-1}\frac{x-2}{2} \qquad ...(v)$$

$$\left(\because \int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}\right)$$

Putting values of I_1 and I_2 from (*iv*) and (*v*) in (*iii*),

$$\text{I} = -\sqrt{4x-x^2} + 4\sin^{-1}\frac{x-2}{2} + c.$$

21. $\dfrac{x+2}{\sqrt{x^2+2x+3}}$

Sol. Let $\text{I} = \int \frac{x+2}{\sqrt{x^2+2x+3}}\,dx$ ...(*i*)

Let Linear = A $\frac{d}{dx}$ (Quadratic) + B

i.e., $x + 2 = \text{A}(2x+2) + \text{B}$ ...(*ii*)
$= 2\text{A}x + 2\text{A} + \text{B}$

Comparing coefficients of x, $2\text{A} = 1 \quad \Rightarrow \quad \text{A} = \frac{1}{2}$

Comparing constants, $2\text{A} + \text{B} = 2$

Putting $\text{A} = \frac{1}{2}$, $1 + \text{B} = 2 \quad \Rightarrow \quad \text{B} = 1$

Putting values of A and B in (*ii*), $x + 2 = \frac{1}{2}(2x+2) + 1$

Putting this value of $(x + 2)$ in (*i*),

$$\text{I} = \int \frac{\frac{1}{2}(2x+2)+1}{\sqrt{x^2+2x+3}}\,dx = \frac{1}{2}\int \frac{2x+2}{\sqrt{x^2+2x+3}}\,dx + \int \frac{dx}{\sqrt{x^2+2x+3}}$$

$\Rightarrow \quad I = \frac{1}{2} I_1 + I_2 \quad ...(iii) \qquad I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}}\, dx$

Put $x^2 + 2x + 3 = t \quad \therefore \quad (2x + 2) = \frac{dt}{dx} \Rightarrow (2x + 2)\, dx = dt$

$$I_1 = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2}\, dt = \frac{t^{1/2}}{\frac{1}{2}} = 2\sqrt{t} = 2\sqrt{x^2+2x+3} \qquad ...(iv)$$

$$I_2 = \int \frac{1}{\sqrt{x^2+2x+3}}\, dx = \int \frac{1}{\sqrt{x^2+2x+1+2}}\, dx$$

$$= \int \frac{1}{\sqrt{(x+1)^2+(\sqrt{2})^2}}\, dx = \log | x + 1 + \sqrt{(x+1)^2+(\sqrt{2})^2} |$$

$$\left[\because \int \frac{1}{\sqrt{x^2+a^2}}\, dx = \log | x + \sqrt{x^2+a^2} |\right]$$

$$= \log | x + 1 + \sqrt{x^2+2x+3} | \qquad ...(v)$$

Putting values from (iv) and (v) in (iii),

$$I = \sqrt{x^2+2x+3} + \log | x + 1 + \sqrt{x^2+2x+3} | + c.$$

22. $\dfrac{\boldsymbol{x+3}}{\boldsymbol{x^2-2x-5}}$

Sol. Let $I = \int \frac{x+3}{x^2-2x-5}\, dx \qquad ...(i)$

Let $x + 3 = A \frac{d}{dx}(x^2 - 2x - 5) + B$

or $\quad x + 3 = A(2x - 2) + B \qquad ...(ii)$

$\qquad\qquad = 2Ax - 2A + B$

Comparing coefficients of x on both sides, $2A = 1 \Rightarrow A = \frac{1}{2}$

Comparing constants, $-2A + B = 3$

Putting $A = \frac{1}{2}$, $-1 + B = 3 \qquad \Rightarrow \quad B = 4$

Putting values of A and B in (ii), $x + 3 = \frac{1}{2}(2x - 2) + 4$

Putting this value in (i),

$$I = \int \frac{\frac{1}{2}(2x-2)+4}{x^2-2x-5}\, dx = \frac{1}{2}\int \frac{2x-2}{x^2-2x-5}\, dx + 4\int \frac{1}{x^2-2x-5}\, dx$$

$$= \frac{1}{2} I_1 + 4 I_2 \qquad ...(iii)$$

$$I_1 = \int \frac{2x-2}{x^2-2x-5}\, dx$$

Put $x^2 - 2x - 5 = t$. Therefore $(2x - 2) = \frac{dt}{dx} \Rightarrow (2x - 2)\, dx = dt$

$\therefore \quad I_1 = \int \frac{dt}{t} = \log |t| = \log |x^2 - 2x - 5|$ ...(*iv*)

Again $I_2 = \int \frac{1}{x^2 - 2x - 5}\, dx$

$$= \int \frac{1}{x^2 - 2x + 1 - 1 - 5}\, dx = \int \frac{1}{(x-1)^2 - 6}\, dx$$

$$= \int \frac{1}{(x-1)^2 - 6}\, dx = \frac{1}{2\sqrt{6}} \log \left| \frac{x - 1 - \sqrt{6}}{x - 1 + \sqrt{6}} \right| \quad ...(v)$$

$$\left[\because \int \frac{1}{x^2 - a^2}\, dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| \right]$$

Putting values of I_1 and I_2 from (*iv*) and (*v*) in (*iii*),

$$I = \frac{1}{2} \log |x^2 - 2x - 5| + \frac{2}{\sqrt{6}} \log \left| \frac{x - 1 - \sqrt{6}}{x - 1 + \sqrt{6}} \right| + c.$$

23. $\mathbf{\dfrac{5x + 3}{\sqrt{x^2 + 4x + 10}}}$

Sol. Let $I = \int \frac{5x + 3}{\sqrt{x^2 + 4x + 10}}\, dx$ ...(*i*)

Let Linear = A $\frac{d}{dx}$ (Quadratic) + B

i.e., $5x + 3 = A(2x + 4) + B$ ...(*ii*)

$= 2Ax + 4A + B$

Comparing coefficients of x on both sides, $2A = 5 \Rightarrow A = \frac{5}{2}$

Comparing constants, $4A + B = 3$

Putting $A = \frac{5}{2}$, $10 + B = 3 \quad \Rightarrow \quad B = -7$

Putting values of A and B in (*ii*), $5x + 3 = \frac{5}{2}(2x + 4) - 7$

Putting this value in (*i*), $I = \int \frac{\frac{5}{2}(2x+4) - 7}{\sqrt{x^2 + 4x + 10}}\, dx$

$$= \frac{5}{2} \int \frac{2x + 4}{\sqrt{x^2 + 4x + 10}}\, dx - 7 \int \frac{1}{\sqrt{x^2 + 4x + 10}}\, dx$$

or $\quad I = \frac{5}{2} I_1 - 7 I_2$ ...(*iii*)

$$I_1 = \int \frac{2x + 4}{\sqrt{x^2 + 4x + 10}}\, dx$$

Put $x^2 + 4x + 10 = t$. Therefore $2x + 4 = \frac{dt}{dx} \Rightarrow (2x + 4)\, dx = dt$

$$\therefore \quad I_1 = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2}\ dt = \frac{t^{1/2}}{\frac{1}{2}} = 2\sqrt{t}$$

$$= 2\sqrt{x^2 + 4x + 10} \qquad \ldots(iv)$$

$$I_2 = \int \frac{1}{\sqrt{x^2 + 4x + 10}}\ dx = \int \frac{1}{\sqrt{x^2 + 4x + 4 + 6}}\ dx$$

$$= \int \frac{1}{\sqrt{(x+2)^2 + (\sqrt{6})^2}}\ dx = \log \left| x + 2 + \sqrt{(x+2)^2 + (\sqrt{6})^2} \right|$$

$$\left[\because \int \frac{1}{\sqrt{x^2 + a^2}}\, dx = \log \left| x + \sqrt{x^2 + a^2} \right| \right]$$

$$= \log \left| x + 2 + \sqrt{x^2 + 4x + 10} \right| \qquad \ldots(v)$$

Putting values of I_1 and I_2 from (iv) and (v) in (iii),

$I = 5\sqrt{x^2 + 4x + 10} - 7 \log \left| x + 2 + \sqrt{x^2 + 4x + 10} \right| + c.$

Choose the correct answer in Exercises 24 and 25.

24. $\displaystyle\int \frac{dx}{x^2 + 2x + 2}$ **equals**

(A) $x \tan^{-1} (x + 1) + C$ **(B)** $\tan^{-1} (x + 1) + C$

(C) $(x + 1) \tan^{-1} x + C$ **(D)** $\tan^{-1} x + C.$

Sol. $\displaystyle\int \frac{dx}{x^2 + 2x + 2} = \int \frac{1}{x^2 + 2x + 1 + 1}\ dx = \int \frac{1}{(x+1)^2 + 1^2}\ dx$

$$= \frac{1}{1} \tan^{-1} \frac{(x+1)}{1} + C \quad \left[\because \int \frac{1}{x^2 + a^2}\, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}\right]$$

$$= \tan^{-1} (x + 1) + C$$

$\therefore$ Option (B) is the correct answer.

25. $\displaystyle\int \frac{dx}{\sqrt{9x - 4x^2}}$ **equals**

(A) $\frac{1}{9} \sin^{-1} \left(\frac{9x - 8}{8}\right) + C$ **(B)** $\frac{1}{2} \sin^{-1} \left(\frac{8x - 9}{9}\right) + C$

(C) $\frac{1}{3} \sin^{-1} \left(\frac{9x - 8}{8}\right) + C$ **(D)** $\frac{1}{2} \sin^{-1} \left(\frac{9x - 8}{8}\right) + C.$

Sol. Let $\displaystyle I = \int \frac{dx}{\sqrt{9x - 4x^2}} = \int \frac{dx}{\sqrt{-4x^2 + 9x}} \qquad \ldots(i)$

Here Quadratic expression is $-4x^2 + 9x = -4\left(x^2 - \frac{9}{4}x\right)$

$$= -4\left[x^2 - \frac{9}{4}x + \left(\frac{9}{8}\right)^2 - \left(\frac{9}{8}\right)^2\right] = -4\left[\left(x - \frac{9}{8}\right)^2 - \left(\frac{9}{8}\right)^2\right]$$

$$= 4\left[-\left(x - \frac{9}{8}\right)^2 + \left(\frac{9}{8}\right)^2\right] = 4\left[\left(\frac{9}{8}\right)^2 - \left(x - \frac{9}{8}\right)^2\right]$$

Putting this value in (*i*),

$$I = \int \frac{1}{\sqrt{4\left[\left(\frac{9}{8}\right)^2 - \left(x - \frac{9}{8}\right)^2\right]}}\, dx = \frac{1}{2}\int \frac{1}{\sqrt{\left[\left(\frac{9}{8}\right)^2 - \left(x - \frac{9}{8}\right)^2\right]}}\, dx$$

$$= \frac{1}{2}\sin^{-1}\frac{x - \frac{9}{8}}{\frac{9}{8}} + C$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{8x-9}{9}\right) + C \qquad \left[\because \int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\frac{x}{a}\right]$$

$\therefore$ Option (B) is the correct answer.