Exercise 5.8

$\frac{f(x)}{g(x)}$ $(g(x) \neq 0)$, *sin x, cos x,* e^x, e^{-x}, *log x* $(x > 0)$ are continuous and derivable for all real x.

Note 2: Sum, difference, product of two continuous (derivable) functions is continuous (derivable).

1. Verify Rolle's theorem for $f(x) = x^2 + 2x - 8$, $x \in [-4, 2]$.

Sol. Given: $f(x) = x^2 + 2x - 8$; $x \in [-4, 2]$ ...(*i*)

Here $f(x)$ is a polynomial function of x (of degree 2).

$\therefore$ $f(x)$ is continuous and derivable everywhere *i.e.,* on $(-\infty, \infty)$.

Hence $f(x)$ is continuous in the closed interval $[-4, 2]$ and derivable in open interval $(-4, 2)$.

Putting $x = -4$ in (*i*), $f(-4) = 16 - 8 - 8 = 0$

Putting $x = 2$ in (*i*), $f(2) = 4 + 4 - 8 = 0$

$\therefore$ $f(-4) = f(2)$ $(= 0)$

$\therefore$ All three conditions of Rolle's Theorem are satisfied.

From (*i*), $f'(x) = 2x + 2$.

Putting $x = c$, $f'(c) = 2c + 2 = 0 \Rightarrow 2c = -2$

$\Rightarrow$ $c = -\frac{2}{2} = -1 \in$ open interval $(-4, 2)$.

$\therefore$ Conclusion of Rolle's theorem is true.

$\therefore$ Rolle's theorem is verified.

2. Examine if Rolle's theorem is applicable to any of the following functions. Can you say some thing about the converse of Rolle's theorem from these examples?

(*i*) $f(x) = [x]$ for $x \in [5, 9]$ (*ii*) $f(x) = [x]$ for $x \in [-2, 2]$

(*iii*) $f(x) = x^2 - 1$ for $x \in [1, 2]$.

Sol. (*i*) **Given:** $f(x) = [x]$ for $x \in [5, 9]$ ...(*i*)

(of course $[x]$ denotes the greatest integer $\leq x$)

We know that bracket function $[x]$ is discontinuous at all the integers (See Ex. 15, page 155, NCERT, Part I). Hence $f(x) = [x]$ is discontinuous at all integers between 5 and 9 *i.e.,* discontinuous at $x = 6$, $x = 7$ and $x = 8$ and hence discontinuous in the closed interval $[5, 9]$ and hence not derivable in the open interval $(5, 9)$. ...(*ii*) ($\because$ discontinuity $\Rightarrow$ Non-derivability)

Again from (*i*), $f(5) = [5] = 5$ and $f(9) = [9] = 9$

$\therefore$ $f(5) \neq f(9)$

$\therefore$ Conditions of Rolle's Theorem are not satisfied.

$\therefore$ Rolle's Theorem is not applicable to $f(x) = [x]$ in the closed interval $[5, 9]$.

But converse (conclusion) of Rolle's theorem is true for this function $f(x) = [x]$.

i.e., $f'(c) = 0$ for every real c belonging to open interval

(5, 9) other than integers. (*i.e.,* for every real $c \neq 6, 7, 8$) (even though conditions are not satisfied).

Let us prove it.

$$\text{Left Hand derivative} = \mathrm{L}f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}$$

$$= \lim_{x \to c^-} \frac{[x] - [c]}{x - c} \qquad \text{(By } (i))$$

$$\text{Put } x = c - h,\ h \to 0^+, = \lim_{h \to 0^+} \frac{[c - h] - [c]}{c - h - c}$$

$$= \lim_{h \to 0^+} \frac{[c] - [c]}{-h}$$

[$\because$ We know that for $c \in \mathrm{R} - \mathrm{Z}$, as $h \to 0^+$, $[c - h] = [c]$]

$$= \lim_{h \to 0^+} \frac{0}{-h} = \lim_{h \to 0^+} 0$$

($\because$ $h \to 0^+ \Rightarrow h > 0$ and hence $h \neq 0$)

$$= 0 \qquad ...(iii)$$

$$\text{Right Hand derivative} = \mathrm{R}f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

$$= \lim_{x \to c^+} \frac{[x] - [c]}{x - c} \qquad \text{(By } (i))$$

$$\text{Put } x = c + h,\ h \to 0^+, = \lim_{h \to 0^+} \frac{[c + h] - [c]}{c + h - c} = \lim_{h \to 0^+} \frac{[c] - [c]}{h}$$

[$\because$ We know that for $c \in \mathrm{R} - \mathrm{Z}$, as $h \to 0^+$, $[c + h] = [c]$]

$$= \lim_{h \to 0^+} \frac{0}{h} = \lim_{h \to 0^+} 0$$

($\because$ $h \to 0^+ \Rightarrow h > 0$ and hence $h \neq 0$)

$$= 0 \qquad ...(iv)$$

From (*iii*) and (*iv*) $\mathrm{L}f'(c) = \mathrm{R}f'(c) = 0$

$\therefore$ $f'(c) = 0$ $\forall$ real $c \in$ open interval (5, 9) other than integers $c = 6, 7, 8$.

(*ii*) **Given:** $f(x) = [x]$ for $x \in [-2, 2]$.

Reproduce the solution of (i) part replacing closed interval [5, 9] by [– 2, 2] and integers 6, 7, 8 by – 1, 0 and 1 lying between – 2 and 2.

(*iii*) **Given:** $f(x) = x^2 - 1$ for $x \in [1, 2]$ ...(*i*)

Here $f(x)$ is a polynomial function of x (of degree 2).

$\therefore$ $f(x)$ is continuous and derivable everywhere *i.e.,* on $(-\infty, \infty)$.

Hence $f(x)$ is continuous in the closed interval [1, 2] and derivable in the open interval (1, 2).

Again from (*i*), $f(1) = 1 - 1 = 0$

and $\quad f(2) = 2^2 - 1 = 4 - 1 = 3$

$\therefore \quad f(1) \neq f(2).$

$\therefore$ Conditions of Rolle's Theorem are not satisfied.

$\therefore$ Rolle's theorem is not applicable to $f(x) = x^2 - 1$ in [1, 2].

Let us examine if converse (*i.e.,* conclusion) is true for this function given by (*i*).

From (*i*), $f'(x) = 2x$

Put $x = c$, $f'(c) = 2c = 0 \Rightarrow c = 0$ does not belong to open interval (1, 2).

$\therefore$ Converse (conclusion) of Rolle's Theorem is also not true for this function.

3. If $f : [-5, 5] \to R$ is a differentiable function and if $f'(x)$ does not vanish anywhere, then prove that $f(-5) \neq f(5)$.

Sol. Given: $f : [-5, 5] \to R$ is a differentiable function *i.e.,* f is differentiable on its domain closed interval $[-5, 5]$ (and in particular in open interval $(-5, 5)$ also) and hence is continuous also on closed interval $[-5, 5]$ ...(*i*)

To prove: $f(-5) \neq f(5)$.

If possible, let $f(-5) = f(5)$ ...(*ii*)

From (*i*) and (*ii*) all the three conditions of Rolle's Theorem are satisfied.

$\therefore$ There exists at least one point c in the open interval $(-5, 5)$ such that $f'(c) = 0$.

i.e., $f'(x) = 0$ *i.e.,* $f'(x)$ vanishes (vanishes $\Rightarrow$ zero) for at least one value of x in the open interval $(-5, 5)$. But this is contrary to given that $f'(x)$ does not vanish anywhere.

$\therefore$ Our supposition in (*ii*) *i.e.,* $f(-5) = f(5)$ is wrong.

$\therefore \quad f(-5) \neq f(5).$

4. Verify Mean Value Theorem if $f(x) = x^2 - 4x - 3$ in the interval $[a, b]$ where $a = 1$ and $b = 4$.

Sol. Given: $f(x) = x^2 - 4x - 3$ in the interval $[a, b]$ where $a = 1$ and $b = 4$ *i.e.,* in the interval [1, 4] ...(*i*)

Here $f(x)$ is a polynomial function of x and hence is continuous and derivable everywhere.

$\therefore$ $f(x)$ is continuous in the closed interval [1, 4] and derivable in the open interval (1, 4) also.

$\therefore$ Both conditions of L.M.V.T. are satisfied.

From (*i*), $\quad f'(x) = 2x - 4$

Put $\quad x = c,\ f'(c) = 2c - 4$

from (*i*) $\quad f(a) = f(1) = 1 - 4 - 3 = -6$

and $\qquad f(b) = f(4) = 16 - 16 - 3 = -3$

Putting these values in $f'(c) = \dfrac{f(b) - f(a)}{b - a}$, we have

$$2c - 4 = \frac{-3-(-6)}{4-1} \quad \Rightarrow \quad 2c - 4 = \frac{-3+6}{3}$$

$$\Rightarrow \quad 2c - 4 = \frac{3}{3} = 1 \quad \Rightarrow \quad 2c = 5$$

$$\Rightarrow \quad c = \frac{5}{2} \in \text{open interval } (1, 4).$$

$\therefore$ L.M.V.T. is verified.

5. Verify Mean Value Theorem if $f(x) = x^3 - 5x^2 - 3x$ in the interval $[a, b]$ where $a = 1$ and $b = 3$. Find all $c \in (1, 3)$ for which $f'(c) = 0$.

Sol. Given: $\qquad f(x) = x^3 - 5x^2 - 3x \qquad$ ...(*i*)

In the interval $[a, b]$ where $a = 1$ and $b = 3$ *i.e.*, in the interval $[1, 3]$.

Here $f(x)$ is a polynomial function of x (of degree 3). Therefore, $f(x)$ is continuous and derivable everywhere *i.e.*, on the real line $(-\infty, \infty)$.

Hence $f(x)$ is continuous in the closed interval $[1, 3]$ and derivable in open interval $(1, 3)$.

$\therefore$ Both conditions of Mean Value Theorem are satisfied.

From (*i*), $\qquad f'(x) = 3x^2 - 10x - 3$

Put $x = c$, $\qquad f'(c) = 3c^2 - 10c - 3 \qquad$ ...(*ii*)

From (*i*), $\qquad f(a) = f(1) = 1 - 5 - 3 = 1 - 8 = -7$

and $f(b) = f(3) = 3^3 - 5 \,.\, 3^2 - 3.3 = 27 - 45 - 9 = 27 - 54 = -27$

Putting these values in the conclusion of Mean Value Theorem *i.e.*,

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \text{ we have}$$

$$3c^2 - 10c - 3 = \frac{-27-(-7)}{3-1} = \frac{-27+7}{2} = -\frac{20}{2} = -10$$

$\Rightarrow \quad 3c^2 - 10c - 3 + 10 = 0 \quad \Rightarrow \quad 3c^2 - 10c + 7 = 0$

$\Rightarrow \quad 3c^2 - 3c - 7c + 7 = 0 \quad \Rightarrow \quad 3c(c - 1) - 7(c - 1) = 0$

$\Rightarrow \quad (c - 1)(3c - 7) = 0$

$\therefore$ Either $c - 1 = 0$ or $3c - 7 = 0$

i.e., $c = 1 \notin$ open interval $(1, 3)$ or $3c = 7$ *i.e.*, $c = \dfrac{7}{3}$

which belongs to open interval $(1, 3)$.

Hence mean value theorem is verified.

Now we are to find all $c \in (1, 3)$ for which $f'(c) = 0$.

$\therefore$ From (*ii*), $\quad 3c^2 - 10c - 3 = 0$

Solving for c, $\quad c = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \dfrac{10 \pm \sqrt{100 + 36}}{6}$

$= \dfrac{10 \pm \sqrt{136}}{6} = \dfrac{10 \pm \sqrt{4 \times 34}}{6} = \dfrac{10 \pm 2\sqrt{34}}{6} = 2\left(\dfrac{5 \pm \sqrt{34}}{6}\right) = \dfrac{5 \pm \sqrt{34}}{3}$

Taking positive sign, $c = \dfrac{5 + \sqrt{34}}{3} > 3$ and hence $\notin (1, 3)$

Taking negative sign, $c = \dfrac{5 - \sqrt{34}}{3}$ is negative and hence $\notin (1, 3)$.

6. Examine the applicability of Mean Value Theorem for all the three functions being given below:

(*i*) $f(x) = [x]$ for $x \in [5, 9]$ (*ii*) $f(x) = [x]$ for $x \in [-2, 2]$

(*iii*) $f(x) = x^2 - 1$ for $x \in [1, 2]$.

Sol. (*i*) *Reproduce solution of Q. No. 2(i) upto eqn. (ii)*

$\therefore$ Both conditions of L.M.V.T. are not satisfied.

$\therefore$ L.M.V.T. is not applicable to $f(x) = [x]$ for $x \in [5, 9]$.

(*ii*) *Reproduce solution of Q. No. 2(i) upto eqn. (ii)* replacing [5, 9] by [– 2, 2] and integers 6, 7, 8 by – 1, 0 and 1 lying between – 2 and 2.

$\therefore$ Both conditions of L.M.V.T. are not satisfied.

$\therefore$ L.M.V.T. is not applicable to $f(x) = [x]$ for $x \in [-2, 2]$.

(*iii*) **Given:** $f(x) = x^2 - 1$ for $x \in [1, 2]$ ...(*i*)

Here $f(x)$ is a polynomial function (of degree 2).

Therefore $f(x)$ is continuous and derivable everywhere *i.e.*, on the real line $(-\infty, \infty)$.

Hence $f(x)$ is continuous in the closed interval [1, 2] and derivable in open interval (1, 2).

$\therefore$ Both conditions of Mean Value Theorem are satisfied.

From (*i*), $f'(x) = 2x$

Put $x = c$, $f'(c) = 2c$

From (*i*), $f(a) = f(1) = 1^2 - 1 = 1 - 1 = 0$

$f(b) = f(2) = 2^2 - 1 = 4 - 1 = 3$

Putting these values in the conclusion of Mean Value Theorem *i.e.*, in $f'(c) = \dfrac{f(b) - f(a)}{b - a}$, we have

$$2c = \frac{3 - 0}{2 - 1} \Rightarrow 2c = 3$$

$\Rightarrow \quad c = \dfrac{3}{2} \in (1, 2)$

$\therefore$ Mean Value Theorem is verified.