Exercise 3.3

1. Find the transpose of each of the following matrices:

(i) $\begin{bmatrix} \mathbf{5} \\ \mathbf{\frac{1}{2}} \\ \mathbf{-1} \end{bmatrix}$ ***(ii)*** $\begin{bmatrix} \mathbf{1} & \mathbf{-1} \\ \mathbf{2} & \mathbf{3} \end{bmatrix}$ ***(iii)*** $\begin{bmatrix} \mathbf{-1} & \mathbf{5} & \mathbf{6} \\ \mathbf{\sqrt{3}} & \mathbf{5} & \mathbf{6} \\ \mathbf{2} & \mathbf{3} & \mathbf{-1} \end{bmatrix}$.

Sol. (*i*) Let A = $\begin{bmatrix} 5 \\ \frac{1}{2} \\ -1 \end{bmatrix}$ (is a column matrix 3 × 1)

Changing column of A into a row, (row will automatically become column)

Transpose of A (*i.e.*, A′ or A^T) = $\begin{bmatrix} 5 & \frac{1}{2} & -1 \end{bmatrix}$

(which is a row matrix 1 × 3)

(*ii*) Let A = $\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

Changing rows of A to columns of A,

(columns will automatically beocme rows),

A′ or A^T = $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$.

(*iii*) Let A = $\begin{bmatrix} -1 & 5 & 6 \\ \sqrt{3} & 5 & 6 \\ 2 & 3 & -1 \end{bmatrix}$

(Making) changing rows of A as columns of the new matrix,

we have A′ or A^T = $\begin{bmatrix} -1 & \sqrt{3} & 2 \\ 5 & 5 & 3 \\ 6 & 6 & -1 \end{bmatrix}$.

2. If A = $\begin{bmatrix} -1 & 2 & 3 \\ 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix}$ and B = $\begin{bmatrix} -4 & 1 & -5 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$, then verify that

(*i*) (A + B)′ = A′ + B′   (*ii*) (A – B)′ = A′ – B′.

Sol. (*i*) To verify (A + B)′ = A′ + B′

$$A + B = \begin{bmatrix} -1 & 2 & 3 \\ 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix} + \begin{bmatrix} -4 & 1 & -5 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1-4 & 2+1 & 3-5 \\ 5+1 & 7+2 & 9+0 \\ -2+1 & 1+3 & 1+1 \end{bmatrix} = \begin{bmatrix} -5 & 3 & -2 \\ 6 & 9 & 9 \\ -1 & 4 & 2 \end{bmatrix}$$

(Making) changing rows of A + B as columns of the new matrix, we have

$$\text{L.H.S.} = (A + B)' = \begin{bmatrix} -5 & 6 & -1 \\ 3 & 9 & 4 \\ -2 & 9 & 2 \end{bmatrix} \quad ...(i)$$

$$\text{R.H.S.} = A' + B' = \begin{bmatrix} -1 & 2 & 3 \\ 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix}' + \begin{bmatrix} -4 & 1 & -5 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}'$$

$$= \begin{bmatrix} -1 & 5 & -2 \\ 2 & 7 & 1 \\ 3 & 9 & 1 \end{bmatrix} + \begin{bmatrix} -4 & 1 & 1 \\ 1 & 2 & 3 \\ -5 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1-4 & 5+1 & -2+1 \\ 2+1 & 7+2 & 1+3 \\ 3-5 & 9+0 & 1+1 \end{bmatrix} = \begin{bmatrix} -5 & 6 & -1 \\ 3 & 9 & 4 \\ -2 & 9 & 2 \end{bmatrix} \quad ...(ii)$$

From (*i*) and (*ii*), we have L.H.S. = R.H.S.

i.e., (A + B)′ = A′ + B′

(*ii*) To verify (A – B)′ = A′ – B′

$$A - B = \begin{bmatrix} -1 & 2 & 3 \\ 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix} - \begin{bmatrix} -4 & 1 & -5 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1+4 & 2-1 & 3+5 \\ 5-1 & 7-2 & 9-0 \\ -2-1 & 1-3 & 1-1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 8 \\ 4 & 5 & 9 \\ -3 & -2 & 0 \end{bmatrix}$$

(Making) changing rows of A – B as columns of the new matrix, we have

$$\text{L.H.S.} = (A - B)' = \begin{bmatrix} 3 & 4 & -3 \\ 1 & 5 & -2 \\ 8 & 9 & 0 \end{bmatrix} \quad ...(i)$$

$$\text{R.H.S.} = A' - B' = \begin{bmatrix} -1 & 2 & 3 \\ 5 & 7 & 9 \\ -2 & 1 & 1 \end{bmatrix}' - \begin{bmatrix} -4 & 1 & -5 \\ 1 & 2 & 0 \\ 1 & 3 & 1 \end{bmatrix}'$$

$$= \begin{bmatrix} -1 & 5 & -2 \\ 2 & 7 & 1 \\ 3 & 9 & 1 \end{bmatrix} - \begin{bmatrix} -4 & 1 & 1 \\ 1 & 2 & 3 \\ -5 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1+4 & 5-1 & -2-1 \\ 2-1 & 7-2 & 1-3 \\ 3+5 & 9-0 & 1-1 \end{bmatrix} = \begin{bmatrix} 3 & 4 & -3 \\ 1 & 5 & -2 \\ 8 & 9 & 0 \end{bmatrix} \quad ...(ii)$$

From (*i*) and (*ii*), we have L.H.S. = R.H.S.

Note (A′)′ = A.

3. If $A' = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$**, then verify that**

(*i*) (A + B)′ = A′ + B′ (*ii*) (A – B)′ = A′ – B′.

Sol. Given: $A' = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

Making rows of A′ as columns of the new matrix (transpose of A′ *i.e.,* (A′)′) *i.e.,* $A = \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix}$

(*i*) $A + B = \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 3-1 & -1+2 & 0+1 \\ 4+1 & 2+2 & 1+3 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 5 & 4 & 4 \end{bmatrix}$$

$$\therefore \quad \text{L.H.S.} = (A + B)' = \begin{bmatrix} 2 & 1 & 1 \\ 5 & 4 & 4 \end{bmatrix}' = \begin{bmatrix} 2 & 5 \\ 1 & 4 \\ 1 & 4 \end{bmatrix} \quad ...(i)$$

$$\text{R.H.S.} = A' + B' = \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}' \quad \text{(given)}$$

$$= \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3-1 & 4+1 \\ -1+2 & 2+2 \\ 0+1 & 1+3 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 1 & 4 \\ 1 & 4 \end{bmatrix} \quad ...(ii)$$

From (i) and (ii), we have L.H.S. = R.H.S.

(ii) $A - B = \begin{bmatrix} 3 & -1 & 0 \\ 4 & 2 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 3+1 & -1-2 & 0-1 \\ 4-1 & 2-2 & 1-3 \end{bmatrix} = \begin{bmatrix} 4 & -3 & -1 \\ 3 & 0 & -2 \end{bmatrix}$$

$$\therefore \text{ L.H.S.} = (A - B)' = \begin{bmatrix} 4 & -3 & -1 \\ 3 & 0 & -2 \end{bmatrix}' = \begin{bmatrix} 4 & 3 \\ -3 & 0 \\ -1 & -2 \end{bmatrix} \quad ...(i)$$

$$\text{R.H.S.} = A' - B' = \underset{\text{(given)}}{\begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix}} - \begin{bmatrix} -1 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}'$$

$$= \begin{bmatrix} 3 & 4 \\ -1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3+1 & 4-1 \\ -1-2 & 2-2 \\ 0-1 & 1-3 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -3 & 0 \\ -1 & -2 \end{bmatrix} \quad ...(ii)$$

From (i) and (ii), we have L.H.S. = R.H.S.

4. If $A' = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$, then find $(A + 2B)'$.

Sol. **Given:** $A' = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$

Making rows of A′ as columns of the new matrix (transpose of A′ *i.e.,* (A′)′) *i.e.,* $A = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix}$

$$\therefore \quad A + 2B = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix} + 2\begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -2-2 & 1+0 \\ 3+2 & 2+4 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 5 & 6 \end{bmatrix}$$

Making rows of this matrix as columns of new matrix, we have

$$(A + 2B)' = \begin{bmatrix} -4 & 5 \\ 1 & 6 \end{bmatrix}.$$

5. For the matrices A and B, verify that (AB)′ = B′A′, where

(i) $A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$, $B = [-1 \ \ 2 \ \ 1]$ $\qquad$ (ii) $A = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, $B = [1 \ \ 5 \ \ 7]$.

Sol. (i) **Given:** $A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}$ and $B = [-1 \ \ 2 \ \ 1]$

$$\therefore \quad AB = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}_{3\times 1} [-1 \quad 2 \quad 1]_{1 \times 3} \text{ is a matrix of order}$$

$$3 \times 3 \text{ and} = \begin{bmatrix} -1 & 2 & 1 \\ 4 & -8 & -4 \\ -3 & 6 & 3 \end{bmatrix}$$

(Using row by column multiplication rule)

$$\text{L.H.S.} = (AB)' = \begin{bmatrix} -1 & 2 & 1 \\ 4 & -8 & -4 \\ -3 & 6 & 3 \end{bmatrix}' = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix} \quad ...(i)$$

$$\text{R.H.S.} = B'A' = [-1 \quad 2 \quad 1]' \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}'$$

$$= \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} [1 \quad -4 \quad 3] = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix} \quad ...(ii)$$

From (*i*) and (*ii*), we have L.H.S. = R.H.S. *i.e.,* (AB)′ = B′A′.

(*ii*) **Given:** $A = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ and $B = [1 \quad 5 \quad 7]$

$$\therefore \quad AB = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}_{3\times 1} [1 \quad 5 \quad 7]_{1 \times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 5 & 7 \\ 2 & 10 & 14 \end{bmatrix}$$

$$\text{L.H.S.} = (AB)' = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 5 & 7 \\ 2 & 10 & 14 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 5 & 10 \\ 0 & 7 & 14 \end{bmatrix} \quad ...(i)$$

$$\text{R.H.S.} = B'A' = [1 \quad 5 \quad 7]' \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}' = \begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix}_{3\times 1} [0 \quad 1 \quad 2]_{1 \times 3}$$

$$= \begin{bmatrix} 0 & 1 & 2 \\ 0 & 5 & 10 \\ 0 & 7 & 14 \end{bmatrix} \quad ...(ii)$$

From (*i*) and (*ii*) we have L.H.S. = R.H.S.

i.e., (AB)′ = B′A′.

Remark. Result to remember form this Q.No. 5:

(AB)′ = B′A′ | Reversal Law

6. **(*i*) If A = $\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then verify that A′A = I**

(*ii*) If A = $\begin{bmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{bmatrix}$, then verify that A′A = I.

Sol. (*i*) **Given:** A = $\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$

$$\therefore \quad \text{L.H.S.} = A'A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}' \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & \cos\alpha\sin\alpha - \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha - \cos\alpha\sin\alpha & \sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

(Row by Column Multiplication)

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \ (= I) = \text{R.H.S.}$$

(*ii*) **Given:** A = $\begin{bmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{bmatrix}$

$$\therefore \quad \text{L.H.S.} = A'A = \begin{bmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{bmatrix}' A$$

$$= \begin{bmatrix} \sin\alpha & -\cos\alpha \\ \cos\alpha & \sin\alpha \end{bmatrix} \begin{bmatrix} \sin\alpha & \cos\alpha \\ -\cos\alpha & \sin\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \sin^2\alpha + \cos^2\alpha & \sin\alpha\cos\alpha - \cos\alpha\sin\alpha \\ \cos\alpha\sin\alpha - \sin\alpha\cos\alpha & \cos^2\alpha + \sin^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 \ (= I) = \text{R.H.S.}$$

7. **(*i*) Show that the matrix A = $\begin{bmatrix} 1 & -1 & 5 \\ -1 & 2 & 1 \\ 5 & 1 & 3 \end{bmatrix}$ is a symmetric matrix.**

(*ii*) Show that the matrix A = $\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ is a skew-symmetric matrix.

Sol. (*i*) **Given:** A = $\begin{bmatrix} 1 & -1 & 5 \\ -1 & 2 & 1 \\ 5 & 1 & 3 \end{bmatrix}$ ...(*i*)

(Making) changing rows of matrix A as the columns of the new matrix $A' = \begin{bmatrix} 1 & -1 & 5 \\ -1 & 2 & 1 \\ 5 & 1 & 3 \end{bmatrix} = A$ [By (*i*)]

$\therefore$ $A' = A$

$\therefore$ By definition of symmetric matrix, A is a symmetric matrix.

(*ii*) **Given:** Matrix $A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ ...(*i*)

$$\therefore \quad A' = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}' = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

Taking (– 1) common from R.H.S. of A′, we have

$$A' = -\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} = -A \qquad \text{[By } (i)]$$

$\therefore$ By definition, matrix A is a skew-symmetric matrix.

8. For the matrix $A = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}$, verify that

(*i*) (A + A′) is a symmetric matrix.

(*ii*) (A – A′) is a skew symmetric matrix.

Sol. (*i*) **Given:** $A = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}$

Let $B = A + A' = A + \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}' = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 6 \\ 5 & 7 \end{bmatrix}$

$$= \begin{bmatrix} 1+1 & 5+6 \\ 6+5 & 7+7 \end{bmatrix} = \begin{bmatrix} 2 & 11 \\ 11 & 14 \end{bmatrix} \qquad ...(i)$$

$$\therefore \quad B' = \begin{bmatrix} 2 & 11 \\ 11 & 14 \end{bmatrix}' = \begin{bmatrix} 2 & 11 \\ 11 & 14 \end{bmatrix} = B \qquad \text{[By } (i)]$$

$\therefore$ B *i.e.,* (A + A′) is a symmetric matrix.

(*ii*) **Given:** $A = \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}$

Let $B = A - A' = A - \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix}'$

$$= \begin{bmatrix} 1 & 5 \\ 6 & 7 \end{bmatrix} - \begin{bmatrix} 1 & 6 \\ 5 & 7 \end{bmatrix} = \begin{bmatrix} 1-1 & 5-6 \\ 6-5 & 7-7 \end{bmatrix}$$

or $\qquad B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ ...(*i*)

$\therefore \qquad B' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Taking (– 1) common from R.H.S. of B′,

$$B' = -\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -B \qquad [\text{By } (i)]$$

$\therefore$ Matrix B *i.e.,* A – A′ is a skew symmetric matrix.

9. Find $\frac{1}{2}$ (A + A′) and $\frac{1}{2}$ (A – A′) when A = $\begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$.

Sol. Given: $\qquad A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$

$$\therefore \qquad A' = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}' = \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix}$$

$$\therefore \qquad A + A' = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} + \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0 & a-a & b-b \\ -a+a & 0+0 & c-c \\ -b+b & -c+c & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\therefore \quad \frac{1}{2}(A + A') = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Again $\quad A - A' = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} - \begin{bmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{bmatrix}$

$$= \begin{bmatrix} 0-0 & a+a & b+b \\ -a-a & 0-0 & c+c \\ -b-b & -c-c & 0-0 \end{bmatrix} = \begin{bmatrix} 0 & 2a & 2b \\ -2a & 0 & 2c \\ -2b & -2c & 0 \end{bmatrix}$$

$$\therefore \qquad \frac{1}{2}(A - A') = \frac{1}{2}\begin{bmatrix} 0 & 2a & 2b \\ -2a & 0 & 2c \\ -2b & -2c & 0 \end{bmatrix}$$

Multiplying each entry by $\frac{1}{2}$, $= \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$.

10. Express the following matrices as the sum of a symmetric and skew symmetric matrix:

(*i*) $\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ **(*ii*)** $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

(*iii*) $\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$ **(*iv*)** $\begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix}$

Note Formula. Every square matrix A can be expressed as the sum of a symmetric matrix $\frac{1}{2}$ (A + A′) and skew symmetric matrix $\frac{1}{2}$ (A – A′).

Sol. (*i*) **Given:** Matrix (say) $A = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$

therefore, $A' = \begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}' = \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}$

By Formula above, symmetric matrix part of A

$$= \frac{1}{2}(A + A') = \frac{1}{2}\left(\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{pmatrix} 6 & 6 \\ 6 & -2 \end{pmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix} \quad ...(i)$$

and skew symmetric matrix part of A.

$$= \frac{1}{2}(A - A') = \frac{1}{2}\left(\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 1 \\ 5 & -1 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 3-3 & 5-1 \\ 1-5 & -1+1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 4 \\ -4 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(0) & \frac{1}{2}(4) \\ \frac{1}{2}(-4) & \frac{1}{2}(0) \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \quad ...(ii)$$

∴ Given matrix A is sum of matrices (*i*) and (*ii*)

= symmetric matrix $\begin{bmatrix} 3 & 3 \\ 3 & -1 \end{bmatrix}$ + skew symmetric matrix $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$.

(*ii*) **Given:** matrix say $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

$$\therefore \quad A' = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}' = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

$\therefore$ Symmetric part of A = $\mathbf{\frac{1}{2}\,(A + A')}$

$$= \frac{1}{2}\left(\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 12 & -4 & 4 \\ -4 & 6 & -2 \\ 4 & -2 & 6 \end{bmatrix} = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \qquad ...(i)$$

and skew symmetric part of A = $\mathbf{\frac{1}{2}\,(A - A')}$

$$= \frac{1}{2}\left(\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} - \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 6-6 & -2+2 & 2-2 \\ -2+2 & 3-3 & -1+1 \\ 2-2 & -1+1 & 3-3 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad ...(ii)$$

$\therefore$ Given matrix A = sum of matrices (*i*) and (*ii*)

$$= \text{symmetric matrix} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

$$+ \text{ skew symmetric matrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(*iii*) **Given:** matrix say A = $\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}$

$$\therefore \quad A' = \begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix}' = \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}$$

$\therefore$ Symmetric part of A = $\mathbf{\frac{1}{2}\,(A + A')}$

$$= \frac{1}{2}\left(\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 6 & 1 & -5 \\ 1 & -4 & -4 \\ -5 & -4 & 4 \end{bmatrix} = \begin{bmatrix} 3 & \frac{1}{2} & -\frac{5}{2} \\ \frac{1}{2} & -2 & -2 \\ -\frac{5}{2} & -2 & 2 \end{bmatrix} \qquad ...(i)$$

and skew symmetric part of A = $\mathbf{\frac{1}{2}\,(A - A')}$

$$= \frac{1}{2}\left(\begin{bmatrix} 3 & 3 & -1 \\ -2 & -2 & 1 \\ -4 & -5 & 2 \end{bmatrix} - \begin{bmatrix} 3 & -2 & -4 \\ 3 & -2 & -5 \\ -1 & 1 & 2 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 3-3 & 3+2 & -1+4 \\ -2-3 & -2+2 & 1+5 \\ -4+1 & -5-1 & 2-2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 3-3 & 3+2 & -1+4 \\ -2-3 & -2+2 & 1+5 \\ -4+1 & -5-1 & 2-2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{5}{2} & \frac{3}{2} \\ -\frac{5}{2} & 0 & 3 \\ -\frac{3}{2} & -3 & 0 \end{bmatrix} \qquad ...(ii)$$

$\therefore$ Given matrix A = sum of matrices (*i*) and (*ii*)

$$= \text{symmetric matrix} \begin{bmatrix} 3 & \frac{1}{2} & -\frac{5}{2} \\ \frac{1}{2} & -2 & -2 \\ -\frac{5}{2} & -2 & 2 \end{bmatrix}$$

$$+ \text{ skew symmetric matrix} \begin{bmatrix} 0 & \frac{5}{2} & \frac{3}{2} \\ -\frac{5}{2} & 0 & 3 \\ -\frac{3}{2} & -3 & 0 \end{bmatrix}$$

(*iv*) **Given:** matrix say $A = \begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix}$ $\therefore A' = \begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix}' = \begin{bmatrix} 1 & -1 \\ 5 & 2 \end{bmatrix}$

$\therefore$ Symmetric part of A = $\frac{1}{2}$ **(A + A′)**

$$= \frac{1}{2}\left(\begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 5 & 2 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 2 & 4 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} \quad ...(i)$$

and skew symmetric part of A = $\frac{1}{2}$ **(A – A′)**

$$= \frac{1}{2}\left(\begin{bmatrix} 1 & 5 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & -1 \\ 5 & 2 \end{bmatrix}\right) = \frac{1}{2}\left(\begin{bmatrix} 1-1 & 5+1 \\ -1-5 & 2-2 \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 6 \\ -6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} \quad ...(ii)$$

$\therefore$ Given matrix = Sum of matrices (*i*) and (*ii*)

= Symmetric matrix $\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$

+ skew-symmetric matrix $\begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$.

Choose the correct answer in Exercises 11 and 12

11. If A and B are symmetric matrices of same order, AB – BA is a

(A) Skew-symmetric matrix **(B) Symmetric Matrix**

(C) Zero matrix **(D) Identity matrix.**

Sol. Given: A and B are symmetric matrices

$\Rightarrow$ A′ = A and B′ = B ...(*i*)

Now (AB – BA)′ = (AB)′ – (BA)′ [$\because$ (P – Q)′ = P′ – Q′]

= B′A′ – A′B′ [Reversal Law]

= BA – AB [Using (*i*)]

= – (AB – BA)

$\therefore$ (AB – BA) is a skew symmetric.

Thus, option (A) is the correct answer.

12. If $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$, **then A + A′ = I, if the value of α is**

(A) $\frac{\pi}{6}$ **(B)** $\frac{\pi}{3}$ **(C)** π **(D)** $\frac{3\pi}{2}$.

Sol. Given: $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

Also given A + A′ = I

$$\Rightarrow \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} + \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}' = I = I_2$$

$$\Rightarrow \quad \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} + \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 2\cos\alpha & 0 \\ 0 & 2\cos\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equating corresponding entries, we have

$$2\cos\alpha = 1$$

$$\Rightarrow \quad \cos\alpha = \frac{1}{2} = \cos\frac{\pi}{3} \qquad \therefore \qquad \alpha = \frac{\pi}{3}.$$

Thus, option (B) is the correct answer.