Evaluate the following limits:

1. $\lim_{x\to 0} \frac{5^x - 1}{\sqrt{4+x} - 2}$

Solution:

Given: $\lim_{x\to 0} \frac{5^x - 1}{\sqrt{4+x} - 2}$

The limit $\lim_{x\to 0} \frac{5^x - 1}{\sqrt{4+x} - 2}$

When x = 0, the expression $\lim_{x\to 0} \frac{5^x - 1}{\sqrt{4+x} - 2}$ assumes the form (0/0).

So,

As $Z = \lim_{x\to 0} \frac{5^x - 1}{\sqrt{4+x} - 2}$

Now, multiply both numerator and denominator by √(4+x) +2 so that we can remove the indeterminate form.

$$Z = \lim_{x\to 0} \frac{(5^x - 1)\sqrt{4+x}+2}{(\sqrt{4+x})^2 - 2^2}$$

$$= \lim_{x\to 0} \frac{5^x - 1}{\sqrt{4+x}-2} \times \frac{\sqrt{4+x}+2}{\sqrt{4+x}+2}$$

{By using $a^2 - b^2 = (a + b)(a - b)$}

$$Z = \lim_{x\to 0} \frac{(5^x - 1)\sqrt{4+x}+2}{4+x-4}$$

$$= \lim_{x\to 0} \frac{(5^x - 1)\sqrt{4+x}+2}{x}$$

By using basic algebra of limits, we get

$$Z = \lim_{x\to 0} \frac{(5^x - 1)}{x} \times \lim_{x\to 0} \sqrt{4+x} + 2 = \{\sqrt{4+0} + 2\} \lim_{x\to 0} \frac{(5^x - 1)}{x}$$

$= 4 \lim_{x\to 0} \frac{(5^x - 1)}{x}$ [By using the formula: $\lim_{x\to 0} \frac{(a^x - 1)}{x} = \log a$]

$Z = 4\log 5$

∴ The value of $\lim_{x\to 0} \frac{5^x - 1}{\sqrt{4+x} - 2} = 4\log 5$

2. $\lim_{x\to 0} \frac{\log(1+x)}{3^x - 1}$

Solution:

Given: $\lim_{x\to 0} \frac{\log(1+x)}{3^x - 1}$

The limit

When x = 0, the expression $\lim_{x\to 0} \frac{\log(1+x)}{3^x - 1}$ assumes the form (0/0).

So,

As $Z = \lim_{x\to 0} \frac{\log(1+x)}{3^x-1}$

Let us divide numerator and denominator by x, we get

$$Z = \lim_{x\to 0} \frac{\frac{\log(1+x)}{x}}{\frac{3^x-1}{x}} = \frac{\lim_{x\to 0}\frac{\log(1+x)}{x}}{\lim_{x\to 0}\frac{3^x-1}{x}}$$ {by using basic limit algebra}

[By using the formula: $\lim_{x\to 0} \frac{(a^x-1)}{x} = \log a$]

$$= \frac{1}{\log 3}$$

∴ The value of $\lim_{x\to 0} \frac{\log(1+x)}{3^x - 1} = \frac{1}{\log 3}$

3. $\lim_{x\to 0} \frac{a^x + a^{-x} - 2}{x^2}$

Solution:

Given: $\lim_{x\to 0} \frac{a^x + a^{-x} - 2}{x^2}$

The limit

When x = 0, the expression $\lim_{x\to 0} \frac{a^x + a^{-x} - 2}{x^2}$ assumes the form (0/0).

So,

As $Z = \lim_{x\to 0} \frac{a^x+a^{-x}-2}{x^2}$

$$= \lim_{x\to 0} \frac{a^{-x}(a^{2x}-2a^x+1)}{x^2}$$

$$= \lim_{x\to 0} \frac{(a^{2x}-2a^x+1)}{a^x x^2}$$

$$= \lim_{x\to 0} \frac{(a^x-1)^2}{a^x x^2}$$ {By using $(a+b)^2 = a^2 + b^2 + 2ab$}

Let us use algebra of limit, we get

$$Z = \lim_{x\to 0}\left(\frac{a^x-1}{x}\right)^2 \times \lim_{x\to 0}\frac{1}{a^x}$$

[By using the formula: $\lim_{x\to 0}\frac{(a^x-1)}{x} = \log a$]

$$Z = (\log a)^2 \frac{1}{a^0} = (\log a)^2$$

$\therefore$ The value of $\lim_{x\to 0}\frac{a^x + a^{-x} - 2}{x^2} = (\log a)^2$

4. $\lim_{x\to 0}\frac{a^{mx}-1}{b^{nx}-1}, n \neq 0$

Solution:

Given: $\lim_{x\to 0}\frac{a^{mx}-1}{b^{nx}-1}, n \neq 0$

The limit $\lim_{x\to 0}\frac{a^{mx}-1}{b^{nx}-1}, n \neq 0$

When x = 0, the expression assumes the form (0/0).

So, let us include mx and nx as follows:

$$Z = \lim_{x\to 0}\frac{a^{mx}-1}{b^{nx}-1} = \lim_{x\to 0}\frac{\frac{a^{mx}-1}{mx}\times mx}{\frac{b^{nx}-1}{nx}\times nx}$$

$$= \frac{m}{n}\lim_{x\to 0}\frac{\frac{a^{mx}-1}{mx}}{\frac{b^{nx}-1}{nx}}$$

By using algebra of limits, we get

$$Z = \frac{m}{n}\frac{\lim_{x\to 0}\frac{a^{mx}-1}{mx}}{\lim_{x\to 0}\frac{b^{nx}-1}{nx}}$$

[By using the formula: $\lim_{x\to 0}\frac{(a^x-1)}{x} = \log a$]

$$Z = \frac{m}{n}\frac{\log a}{\log b}, n \neq 0$$

$\therefore$ The value of $\lim_{x\to 0}\frac{a^{mx}-1}{b^{nx}-1} = \frac{m}{n}\frac{\log a}{\log b}, n \neq 0$

5. $\lim_{x\to 0} \frac{a^x + b^x - 2}{x}$

Solution:

Given: $\lim_{x\to 0} \frac{a^x + b^x - 2}{x}$

The limit

When $x = 0$, the expression $\lim_{x\to 0} \frac{a^x + b^x - 2}{x}$ assumes the form (0/0).

So,

As $Z = \lim_{x\to 0} \frac{a^x+b^x-2}{x}$

$= \lim_{x\to 0} \frac{a^x-1+b^x-1}{x}$

By using algebra of limits, we get

$Z = \lim_{x\to 0} \frac{a^x-1}{x} + \lim_{x\to 0} \frac{b^x-1}{x}$

[By using the formula: $\lim_{x\to 0} \frac{(a^x-1)}{x} = \log a$]

$Z = \log a + \log b = \log ab$

$\therefore$ The value of $\lim_{x\to 0} \frac{a^x + b^x - 2}{x} = \log ab$