Exercise 10.1

Q1

Show that $f(x) = |x-3|$ is continuous but not differentiable at $x = 3$

Solution

$$f(x) = |x-3|$$
$$= \begin{cases} -(x-3), & \text{if } x < 3 \\ x-3, & \text{if } x \geq 3 \end{cases}$$
$$f(3) = 3-3 = 0$$

LHL $= \lim_{x \to 3^-} f(x)$
$= \lim_{h \to 0} f(3-h)$
$= \lim_{h \to 0} 3-(3-h)$
$= \lim_{h \to 0} 0$

RHL $= \lim_{x \to 3^+} f(x)$
$= \lim_{h \to 0} f(3+h)$
$= \lim_{h \to 0} 3+h-3$
$= 0$

LHL $= f(3) =$ RHL

$\therefore$ $f(x)$ is continuous at $x = 3$

(LHD at $x = 3$) $= \lim_{x \to 3^-} \frac{f(x) - f(3)}{x-3}$
$= \lim_{h \to 0} \frac{f(3-h) - f(3)}{3-h-3}$
$= \lim_{h \to 0} \frac{3-(3-h)-0}{-h}$
$= \lim_{h \to 0} \frac{h}{-h}$
$= -1$

(RHD at $x = 3$) $= \lim_{x \to 3^+} \frac{f(x) - f(3)}{x-3}$
$= \lim_{h \to 0} \frac{f(3+h) - f(3)}{3+h-3}$
$= \lim_{h \to 0} \frac{3+h-3-0}{h}$
$= \lim_{h \to 0} \frac{h}{h}$
$= 1$

(LHD at $x = 3$) $\neq$ (RHD at $x = 3$)

$\therefore$ $f(x)$ is continuous but not differentiable at $x = 3$.

Q2

Show that $f(x) = x^{\frac{1}{3}}$ is not differentiable at $x = 0$.

Solution

$$f(x) = x^{\frac{1}{3}}$$

$$(\text{LHD at } x = 0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}$$
$$= \lim_{h \to 0} \frac{f(0 - h) - f(0)}{0 - h - 0}$$
$$= \lim_{h \to 0} \frac{(-h)^{\frac{1}{3}} - 0}{-h}$$
$$= \lim_{h \to 0} \frac{(-h)^{\frac{1}{3}}}{-h}$$
$$= \lim_{h \to 0} \frac{(-1)^{\frac{1}{3}} h^{\frac{1}{3}}}{(-1) h}$$
$$= \lim_{h \to 0} (-1)^{\frac{-2}{3}} h^{\frac{-2}{3}}$$
$$= \text{Not defined}$$

$$(\text{RHD at } x = 0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0}$$
$$= \lim_{h \to 0} \frac{f(0 + h) - f(0)}{0 + h - 0}$$
$$= \lim_{h \to 0} \frac{h^{\frac{1}{3}} - 0}{h}$$
$$= \lim_{h \to 0} h^{\frac{-2}{3}}$$
$$= \text{Not defined}$$

Since,

LHD and RHD does not exists at $x = 0$

$\therefore$ $f(x)$ is not differentiable at $x = 0$

Q3

Show that $f(x) = \begin{cases} 12x - 13, & \text{if } x \le 3 \\ 2x^2 + 5, & \text{if } x > 3 \end{cases}$ is differentiable at $x = 3$. Also, find $f'(3)$.

Solution

$$f(x) = \begin{cases} 12x - 13, & \text{if } x \le 3 \\ 2x^2 + 5, & \text{if } x > 3 \end{cases}$$

$$\begin{aligned}
(\text{LHD at } x = 3) &= \lim_{x \to 3^-} \frac{f(x) - f(3)}{x - 3} \\
&= \lim_{h \to 0} \frac{f(3-h) - f(3)}{3 - h - 3} \\
&= \lim_{h \to 0} \frac{[12(3-h) - 13] - [12(3) - 13]}{-h} \\
&= \lim_{h \to 0} \frac{36 - 12h - 13 - 36 + 13}{-h} \\
&= \lim_{h \to 0} \frac{-12h}{-h} \\
&= 12
\end{aligned}$$

$$\begin{aligned}
(\text{RHD at } x = 3) &= \lim_{x \to 3^+} \frac{f(x) - f(3)}{x - 3} \\
&= \lim_{h \to 0} \frac{f(3+h) - f(3)}{x - 3} \\
&= \lim_{h \to 0} \frac{\left[2\left(3 + h^2\right) + 5\right] - [12(3) - 13]}{3 + h - 3} \\
&= \lim_{h \to 0} \frac{18 + 12h + 2h^2 + 5 - 36 + 13}{h} \\
&= \lim_{h \to 0} \frac{2h^2 + 12h}{h} \\
&= \lim_{h \to 0} \frac{h(2h + 12)}{h} \\
&= 12
\end{aligned}$$

Now,

(LHD at $x = 3$) = (RHD at $x = 3$)

$\therefore$ $f(x)$ is differentiable at $x = 3$

$f'(x) = 12$

Q4

Show that the function defined as follows, is continuous at $x = 2$, but not differentiable thereat: $f(x) = \begin{cases} 3x - 2, & 0 < x \le 1 \\ 2x^2 - x, & 1 < x \le 2 \\ 5x - 4, & x > 2 \end{cases}$

Solution

$$f(x)=\begin{cases}3x-2\ , & 0<x\le 1\\ 2x^2-x\ , & 1<x\le 2\\ 5x-4\ , & x>2\end{cases}$$

$$f(2)=2(2)^2-2$$
$$=8-2=6$$

LHL $= \lim_{x\to 2^-} f(x)$
$= \lim_{h\to 0} f(2-h)$
$= \lim_{h\to 0}\left[2(2-h)^2-(2-h)\right]$
$= 8-2$
$= 6$

RHL $= \lim_{x\to 2^+} f(x)$
$= \lim_{h\to 0} f(2+h)$
$= \lim_{h\to 0} 5(2+h)-4$
$= 6$

LHL $= f(2) =$ RHL

$f(x)$ is continuous at $x=2$

(LHD at $x=2$) $= \lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2}$
$= \lim_{h\to 0}\frac{f(2-h)-f(2)}{2-h-2}$
$= \lim_{h\to 0}\frac{\left[2(2-h)^2-(2-h)\right]-[8-2]}{-h}$
$= \lim_{h\to 0}\frac{8-8h+2h^2-h-6}{-h}$
$= \lim_{h\to 0}\frac{2h^2-6h}{-h}$
$= \lim_{h\to 0}\frac{h(2h-6)}{-h}$
$= \lim_{h\to 0}(6-2h)$
$= 6$

(RHD at $x=2$) $= \lim_{x\to 2^+}\frac{f(x)-f(2)}{x-2}$
$= \lim_{h\to 0}\frac{f(2+h)-f(2)}{2+h-2}$
$= \lim_{h\to 0}\frac{\left[5(2+h)-4\right]-[8-2]}{h}$
$= \lim_{h\to 0}\frac{10+5h-4-6}{h}$
$= 5$

Since,

(LHD at $x=2$) $\ne$ (RHD at $x=2$)

So,

$f(x)$ is continuous at $x=2$ but not differentiable at $x=2$.

Q5

Discuss the continuity and differentiability of the function $f(x)=|x|+|x-1|$ in the interval $(-1, 2)$.

Solution

f(x) = |x|+|x-1| in theinterval (-1, 2).

$$f(x)=\begin{cases}x+x+1 & -1<x<0\\1 & 0\le x\le 1\\-x-x+1 & 1<x<2\end{cases}$$

$$f(x)=\begin{cases}2x+1 & -1<x<0\\1 & 0\le x\le 1\\-2x+1 & 1<x<2\end{cases}$$

We know that a polynomial and a constant function is continuous and differentiable everywhere.
So, f(x) is continuous and differentiable for $x \in (-1, 0)$, $x \in (0, 1)$ and $(1, 2)$.

We need to check continuity and differentiability at x = 0 and x = 1.

Continuity at x = 0

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 2x+1 = 1$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} 1 = 1$$

$$f(0) = 1$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$
$\therefore$ f(x) is continuous at x = 0.

Continuity at x = 1

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 1 = 1$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 1 = 1$$

$$f(1) = 1$$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(1)$$
$\therefore$ f(x) is continuous at x = 1.

Differentiability at x = 0

$$(\text{LHD at } x = 0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{2x + 1 - 1}{x - 0} = \lim_{x \to 0^-} \frac{2x}{x} = 2$$

$$(\text{RHD at } x = 0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{1 - 1}{x} = \lim_{x \to 0^+} \frac{0}{x} = 0$$

$\therefore$ (LHD at x = 0) $\neq$ (RHD at x = 0)

So, f(x) is differentiable at x = 0.

Differentiability at x = 1

$$(\text{LHD at } x = 1) = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{1 - 1}{x - 1} = 0$$

$$(\text{RHD at } x = 1) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{-2x + 1 - 1}{x - 1} \to \infty$$

$\therefore$ (LHD at x = 1) $\neq$ (RHD at x = 1)

So, f(x) is not differentiable at x = 1.

So, f(x) is continuous on (-1, 2) but not differentiable at x = 0, 1.

Q6

Find whether the following functions is differentiable at x = 1 and x = 2 or not:

$$f(x) = \begin{cases} x, & x \leq 1 \\ 2 - x, & 1 \leq x \leq 2 \\ -2 + 3x - x^2, & x > 2 \end{cases}$$

Solution

$$f(x) = \begin{cases} x, & x \le 1 \\ 2 - x, & 1 \le x \le 2 \\ -2 + 3x - x^2, & x > 2 \end{cases}$$

Differentiability at x = 1

$$(\text{LHD at } x = 1) = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^-} \frac{x - 1}{x - 1} = 1$$

$$(\text{RHD at } x = 1) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1^+} \frac{2 - x - 1}{x - 1} = \lim_{x \to 1^+} \frac{1 - x}{x - 1} = -1$$

$\therefore$ (LHD at x = 1) $\ne$ (RHD at x = 1)

So, f(x) is not differentiable at x = 1.

Differentiability at x = 2

$$(\text{LHD at } x = 2) = \lim_{x \to 2^-} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^-} \frac{2 - x - 0}{x - 2} = \lim_{x \to 2^-} \frac{2 - x}{x - 2} = -1$$

$$(\text{RHD at } x = 2) = \lim_{x \to 2^+} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2^+} \frac{-2 + 3x - x^2 - 0}{x - 2} = \lim_{x \to 2^+} \frac{(1 - x)(x - 2)}{x - 2} = -1$$

$\therefore$ (LHD at x = 2) = (RHD at x = 2)

So, f(x) is differentiable at x = 2.

Q7

Show that $f(x) = \begin{cases} x^m \sin\left(\frac{1}{x}\right), & x \ne 0 \\ 0, & x = 0 \end{cases}$ is differentiable at $x = 0$, if $m > 1$.

Solution

$$f(x) = \begin{cases} x^m \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$(\text{LHD at } x = 0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{h \to 0} \frac{f(0-h) - f(0)}{(0-h) - 0}$$

$$= \lim_{h \to 0} \frac{(0-h)^m \sin\left(\frac{1}{-h}\right) - 0}{-h}$$

$$= \lim_{h \to 0} \frac{(-h)^m \sin\left(-\frac{1}{h}\right)}{-h}$$

$$= \lim_{h \to 0} (-h)^{m-1} \sin\left(-\frac{1}{h}\right)$$

$$= \lim_{h \to 0} -(-h)^{m-1} \sin\left(\frac{1}{h}\right)$$

$$= 0 \times k \qquad [\text{When } -1 \le k \le 1]$$

$$= 0$$

$$(\text{RHD at } x = 0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{h \to 0} \frac{f(0+h) - f(0)}{(0+h) - 0}$$

$$= \lim_{h \to 0} \frac{h^m \sin\left(\frac{1}{h}\right) - 0}{h}$$

$$= \lim_{h \to 0} \left(h^{m-1}\right) \sin\left(\frac{1}{h}\right)$$

$$= 0 \times k' \qquad [\text{Since } -1 \le k' \le 1]$$

$$= 0$$

$(\text{LHD at } x = 0) = (\text{RHD at } x = 0)$

$\therefore$ $f(x)$ is differentiable at $x = 0$

Q8

Show that $f(x) = \begin{cases} x^m \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$ is continuous but not differentiable at $x = 0$, if $0 < m < 1$.

Solution

LHL $= \lim_{x \to 0^-} f(x)$

$= \lim_{h \to 0} f(0-h)$

$= \lim_{h \to 0} (-h)^m \sin\left(-\frac{1}{h}\right)$

$= -\lim_{h \to 0} (-h)^m \sin\left(\frac{1}{h}\right)$

$= 0 \times k$ [When $-1 \le k \le 1$]

$= 0$

RHL $= \lim_{x \to 0^+} f(x)$

$= \lim_{h \to 0} f(0+h)$

$= \lim_{h \to 0} (0+h)^m \sin \frac{1}{(0+h)}$

$= \lim_{h \to 0} h^m \sin\left(\frac{1}{h}\right)$

$= 0 \times k'$ [Where $-1 \le k' \le 1$]

$= 0$

LHL $= f(0) =$ RHL

$\therefore$ $f(x)$ is continuons at $x = 0$

For differentiablility at $x = 0$

(LHD at $x = 0$) $= \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}$

$= \lim_{h \to 0} \frac{(0-h) - f(0)}{(0-h) - 0}$

$= \lim_{h \to 0} \frac{(-h)^m \sin\left(-\frac{1}{h}\right)}{-h}$

$= \lim_{h \to 0} - (-h)^{m-1} \sin\left(\frac{1}{h}\right)$

= Not definded [Since $0 < m < 1$]

(RHD at $x = 0$) $= \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0}$

$= \lim_{h \to 0} \frac{f(0+h) - f(0)}{0 + h - 0}$

$= \lim_{h \to 0} \frac{h^m \sin\left(\frac{1}{h}\right)}{h}$

$= \lim_{h \to 0} \left(h^{m-1}\right) \sin\left(\frac{1}{h}\right)$

= Not defined [as $0, m < 1$]

$\therefore$ (LHD at $x = 0$) and (RHD at $x = 0$) are not defined, so

$f(x)$ is continuous but not differentiable at $x = 0$, when $0 < m < 1$.

Q9

Show that $f(x) = \begin{cases} x^m \sin\left(\frac{1}{x}\right), & x \ne 0 \\ 0, & x = 0 \end{cases}$ is neither continuous nor differentiable, if $m \le 0$.

Solution

$$\text{LHL} = \lim_{x\to 0^-} f(x)$$

$$= \lim_{h\to 0} f(0-h)$$

$$= \lim_{h\to 0} (-h)^m \sin\left(-\frac{1}{h}\right)$$

$$= \text{Not defined as } m \le 0$$

$$\text{RHL} = \lim_{x\to 0^+} f(x)$$

$$= \lim_{h\to 0} f(0+h)$$

$$= \lim_{h\to 0} h^m \sin\left(\frac{1}{h}\right)$$

$$= \text{Not defined, as } m \le 0$$

Since RHL and LHL are not difined, so $f(x)$ is not continuous

Let $x = 0$ for $m \le 0$.

Now,

$$(\text{LHD at } x = 0) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0}$$

$$= \lim_{h\to 0} \frac{f(0-h)-0}{0-h-0}$$

$$= \lim_{h\to 0} \frac{(-h)^m \sin\left(-\frac{1}{h}\right)}{-h}$$

$$= \lim_{h\to 0} -(-h)^{m-1} \sin\left(\frac{1}{h}\right)$$

$$= \text{Not defined, as } m \le 0$$

$$\text{RHD} = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0}$$

$$= \lim_{h\to 0} \frac{f(0+h)-f(0)}{0+h-0}$$

$$= \lim_{h\to 0} \frac{h^m \sin\left(\frac{1}{h}\right)}{h}$$

$$= \lim_{h\to 0} \left(h^{m-1}\right) \sin\left(\frac{1}{h}\right)$$

$$= \text{Not defined, as } m \le 0$$

Thus,

$f(x)$ is neither continuous not differentiable at $x = 0$ for $m \le 0$.

Q10

Find the value of a and b so that the function $f(x) = \begin{cases} x^2 + 3x + a, & \text{if } x \le 1 \\ bx + 2 \quad , & \text{if } x > 1 \end{cases}$ is differentiable at each $x \in R$.

Solution

$$f(x) = \begin{cases} x^2 + 3x + a, & \text{if } x \le 1 \\ bx + 2 \quad , & \text{if } x > 1 \end{cases}$$

$$(\text{LHD at } x = 1) = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{h \to 0} \frac{f(1-h) - f(1)}{1 - h - 1}$$

$$= \lim_{h \to 0} \frac{\left[(1-h)^2 + 3(1-h) + a\right] - [1 + 3 + a]}{-h}$$

$$= \lim_{h \to 0} \frac{h^2 - 5h}{-h}$$

$$= -5$$

$$(\text{RHD at } x = 1) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{h \to 0} \frac{f(1+h) - f(1)}{1 + h - 1}$$

$$= \lim_{h \to 0} \frac{[b(1+h) + 2] - (b + 2)}{h}$$

$$= \lim_{h \to 0} \frac{b + bh + 2 - b - 2}{h}$$

$$= b$$

Since $f(x)$ is differentiable, so

$(\text{LHD at } x = 1) = (\text{RHD at } x = 1)$

$5 = b$

$f(1) = 1 + 3 + a$

$= 4 + a$

LHL $= \lim_{x \to 1^{-}} f(x)$

$= \lim_{h \to 0} f(1-h)$

$= \lim_{h \to 0} (1-h)^2 + 3(1-h) + a$

$= 4 + a$

RHL $= \lim_{x \to 1^+} f(x)$

$= \lim_{h \to 0} f(1+h)$

$= \lim_{h \to 0} b(1+b) + 2$

$= b + 2$

Since $f(x)$ is differentiable

$\Rightarrow$ $f(x)$ is continuous

$\Rightarrow$ LHL = RHL

$4 + a = b + 2$

$4 + a = 5 + 2$

$a = 7 - 4$

$a = 3, \quad b = 5$

Q11

Show that the function $f(x) = \begin{cases} |2x - 3|\,[x], & x \ge 1 \\ \sin\left(\frac{\pi x}{2}\right) \quad , & x < 1 \end{cases}$ is continuous but not differentiable at $x = 1$.

Solution

$$f(x) = \begin{cases} |2x-3|\,[x], & x \geq 1 \\ \sin\left(\frac{\pi x}{2}\right), & x < 1 \end{cases}$$

$$f(x) = \begin{cases} (2x-3)[x], & x \geq \frac{3}{2} \\ -(2x-3), & 1 \leq x \leq \frac{3}{2} \\ \sin\left(\frac{\pi x}{2}\right), & x < 1 \end{cases}$$

For continuity at $x = 1$

$f(1) = -(2.1-3) = 1$

LHL $= \lim_{x \to 1^-} f(x)$

$= \lim_{h \to 0} f(1-h)$

$= \lim_{h \to 0} \sin\left(\frac{\pi(1-h)}{2}\right)$

$= \sin\frac{\pi}{2}$

$= 1$

RHL $= \lim_{x \to 0^+} f(x)$

$= \lim_{h \to 0} f(1+h)$

$= \lim_{h \to 0} -(2(1+h)-3)$

$= -1(-1)$

$= 1$

Since,

LHL = $f(1)$ = RHL

So, $f(x)$ is continuous at $x = 1$

For differentiablility at $x = 1$

$$(\text{LHD at } x = 1) = \lim_{x \to 0^-} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{h \to 0} \frac{f(1-h) - 1}{1 - h - 1}$$

$$= \lim_{h \to 0} \frac{\sin\left(\frac{\pi(1-h)}{2}\right) - 1}{-h}$$

$$= \lim_{h \to 0} \frac{\sin\left(\frac{\pi}{2} - \frac{\pi}{2}h\right) - 1}{-h}$$

$$= \lim_{h \to 0} \frac{\cos\left(\frac{\pi}{2}h\right) - 1}{\frac{-h}{2}}$$

$$= \lim_{h \to 0} \frac{2\sin^2\left(\frac{\pi}{4}h\right)}{h} \times \frac{\left(\frac{\pi}{4}h\right)^2}{\left(\frac{\pi}{4}h\right)^2}$$

$$= 0$$

$$(\text{RHD at } x = 1) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{h \to 0} \frac{f(1+h) - f(1)}{1 + h - 1}$$

$$= \lim_{h \to 0} \frac{-[2(1+h) - 3] - 1}{h}$$

$$= \lim_{h \to 0} \frac{-2 - 2h + 3 - 1}{h}$$

$$= \lim_{h \to 0} \frac{-2h}{h}$$

$$= -2$$

$(\text{LHD at } x = 1) \neq (\text{RHD at } x = 1)$

$\therefore$ $f(x)$ is continuous but differentiable at $x = 1$.

Q12

If $f(x) = \begin{cases} ax^2 - b & , \text{ if } |x| < 1 \\ \frac{1}{|x|} & , \text{ if } |x| \geq 1 \end{cases}$ is differentiable at $x = 1$, find a, b.

Solution

Her,e $f(x) = \begin{cases} ax^2 - b & , \text{ if } |x| < 1 \\ \frac{1}{|x|} & , \text{ if } |x| \ge 1 \end{cases}$

$= \begin{cases} -\frac{1}{x} & , \text{ if } x \le -1 \\ ax^2 - b & , \text{ if } -1 < x < 1 \\ \frac{1}{x} & , \text{ if } x \ge 1 \end{cases}$

LHL $= \lim_{x \to 1^-} f(x)$

$= \lim_{h \to 0} f(1-h)$

$= \lim_{h \to 0} a(1-h)^2 - b$

$= a - b$

RHL $= \lim_{x \to 1^+} f(x)$

$= \lim_{h \to 0} f(1+h)$

$= \lim_{h \to 0} \frac{1}{1+h}$

$= 1$

Since, $f(x)$ is continuous, so

LHS = RHS

$a - b = 1$ ---(i)

$(\text{LHD at } x = 1) = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1}$

$= \lim_{h \to 0} \frac{f(1-h) - 1}{1 - h - 1}$

$= \lim_{h \to 0} \frac{a(1-h)^2 - b - 1}{-h}$

$= \lim_{h \to 0} \frac{a(1-h)^2 - (a-1) - 1}{-h}$

Using equation (i),

$= \lim_{h \to 0} \frac{a + ah^2 - 2ah - a + 1 - 1}{-h}$

$= \lim_{h \to 0} \frac{ah^2 - 2ah}{-h}$

$= \lim_{h \to 0} (2a - ah)$

$= 2a$

$$(\text{RHD at } x = 1) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1}$$
$$= \lim_{h \to 0} \frac{f(1+h) - f(1)}{1 + h - 1}$$
$$= \lim_{h \to 0} \frac{\frac{1}{1+h} - 1}{h}$$
$$= \lim_{h \to 0} \frac{1 - 1 - h}{(1+h)h}$$
$$= \lim_{h \to 0} \frac{-1}{1+h}$$
$$= -1$$

Since $f(x)$ is differentiable at $x = 1$,

$$(\text{LHD at } x = 1) = (\text{RHD at } x = 1)$$
$$2a = -1$$
$$a = \frac{-1}{2}$$

Put $a = \frac{-1}{2}$ in equation (i),

$$a - b = 1$$
$$\left(\frac{-1}{2}\right) - b = 1$$
$$b = \frac{-1}{2} - 1$$
$$b = \frac{-3}{2}$$
$$a = \frac{-1}{2}$$

Exercise 10.2

Q1

If f is defined by $f(x) = x^2$, find $f'(2)$.

Solution

Here, $f(x) = x^2$ is a polynomial function so, it is differentiable at $x = 2$.

$$f'(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$$
$$= \lim_{h \to 0} \frac{(2+h)^2 - (2)^2}{h}$$
$$= \lim_{h \to 0} \frac{4 + h^2 + 4h - 4}{h}$$
$$= \lim_{h \to 0} \frac{4h + h^2}{h}$$
$$= \lim_{h \to 0} (4 + h)$$
$$= 4$$
$$\therefore \quad f'(2) = 4$$

Q2

If f is defined by $f(x) = x^2 - 4x + 7$, show that $f'(5) = 2f'\left(\frac{7}{2}\right)$

Solution

$f(x) = x^2 - 4x + 7$ is a polynomial function, So it is differentiable everywhere.

$$f'(5) = \lim_{h\to 0}\frac{f(5+h)-f(5)}{h}$$
$$= \lim_{h\to 0}\frac{\left\{(5+h)^2-4(5+h)+7\right\}-[25-20+7]}{h}$$
$$= \lim_{h\to 0}\frac{h^2+25+10h-20-4h+7-12}{h}$$
$$= \lim_{h\to 0}\frac{h^2+6h}{h}$$
$$= \lim_{h\to 0}(h+6)$$
$$= 6$$

$$f'\left(\frac{7}{2}\right) = \lim_{h\to 0}\frac{f\left(\frac{7}{2}+h\right)-f\left(\frac{7}{2}\right)}{h}$$
$$= \lim_{h\to 0}\frac{\left[\left(\frac{7}{2}+h\right)^2-4\left(\frac{7}{2}+h\right)+7\right]-\left[\left(\frac{7}{2}\right)^2-4\left(\frac{7}{2}\right)+7\right]}{h}$$
$$= \lim_{h\to 0}\frac{\left[\frac{49}{2}+h^2+7h-14-4h+7\right]-\left[\frac{49}{4}-14+7\right]}{h}$$
$$= \lim_{h\to 0}\frac{\frac{49}{4}+h^2+7h-14-4h+7-\frac{49}{4}+14-7}{h}$$
$$= \lim_{h\to 0}\frac{h^2+3h}{h}$$
$$= \lim_{h\to 0}(h+3)$$
$$= 3$$

Now,

$$f'(5) = 6$$
$$= 2(3)$$
$$f'(5) = 2f'\left(\frac{7}{2}\right)$$

Q3

Show that the derivative of the function f given by

$f(x) = 2x^3 - 9x^2 + 12x + 9$, at $x = 1$ and $x = 2$ are equal.

Solution

We know that, $f(x) = 2x^3 - 9x^2 + 12x + 9$ is a polynomial function. So, it is differentiable every where. For $x = 1$

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$
$$= \lim_{h \to 0} \frac{\left[2(1+h)^3 - 9(1+h)^2 + 12(1+h) + 9\right] - \left[2 - 9 + 12 + 9\right]}{h}$$
$$= \lim_{h \to 0} \frac{\left[2\left(1 + h^3 + 3h^2 + 3h\right) - 9\left(1 + h^2 + 2h\right) + 12 + 12h + 9\right] - [14]}{h}$$
$$= \lim_{h \to 0} \frac{\left[2 + 2h^3 + 6h^2 + 6h - 9 - 9h^2 - 18h + 12 + 12h + 9 - 14\right]}{h}$$
$$= \lim_{h \to 0} \frac{2h^3 - 3h^2}{h}$$
$$= \lim_{h \to 0} \frac{h^2(2h - 3)}{h}$$
$$= \lim_{h \to 0} h(2h - 3)$$
$$f'(1) = 0 \qquad \text{---(i)}$$

For $x = 2$

$$f'(2) = \lim_{h \to 0} \frac{\left[2(2+h)^3 - 9(2+h)^2 + 12(12+h) + 9\right] - \left[16 - 36 + 24 + 9\right]}{h}$$
$$= \lim_{h \to 0} \frac{\left[2\left(8 + h^3 + 12h + 6h^2\right) - 9\left(4 + h^2 + 4h\right) + 24 + 12h + 9\right] - [13]}{h}$$
$$= \lim_{h \to 0} \frac{\left[16 + 2h^3 + 24h + 12h^2 - 36 - 9h^2 - 36h + 33 + 12h - 13\right]}{h}$$
$$= \lim_{h \to 0} \frac{2h^3 + 3h^2}{h}$$
$$= \lim_{h \to 0} \frac{h^2(2h + 3)}{h}$$
$$= \lim_{h \to 0} h(2h + 3)$$
$$f'(2) = 0 \qquad \text{---(ii)}$$

From equation (i) and (ii),

$$f'(1) = f'(2)$$

Q4

If for the function $\Phi(x) = \lambda x^2 + 7x - 4, \Phi'(5) = 97$, find λ.

Solution

$$\Phi(x) = \lambda x^2 + 7x - 4 \text{ and } \Phi'(5) = 97$$

$$\Phi'(5) = \lim_{h\to 0} \frac{\left[\lambda(5+h)^2 + 7(5+h) - 4\right] - [25\lambda + 35 - 4]}{h}$$

$$97 = \lim_{h\to 0} \frac{\lambda\left(25 + h^2 + 10h\right) + 35 + 7h - 4 - 25\lambda - 35 + 4}{h}$$

$$= \lim_{h\to 0} \frac{25\lambda + \lambda h^2 + 10\lambda h - 25\lambda + 7h}{h}$$

$$= \lim_{h\to 0} \frac{\lambda h^2 + h(10\lambda + 7)}{h}$$

$$= \lim_{h\to 0} \frac{f(\lambda h + 10\lambda + 7)}{h}$$

$$97 = 10\lambda + 7$$

$$10\lambda = 97 + 7$$

$$\lambda = \frac{90}{10}$$

$$\lambda = 9$$

Q5

If $f(x) = x^3 + 7x^2 + 8x - 9$, find $f'(4)$.

Solution

$f(x) = x^3 + 7x^2 + 8x - 9$ is a polynomial function. So, it is differentiable every where.

$$f'(4) = \lim_{h\to 0} \frac{f(4+h) - h(4)}{h}$$

$$= \lim_{h\to 0} \frac{\left[(4+h)^3 + 7(4+h)^2 + 8(4+h) - 9\right] - [64 + 112 + 32 - 9]}{h}$$

$$= \lim_{h\to 0} \frac{\left[64 + h^3 + 48h + 12h^2 + 112 + 7h^2 + 56h + 32 + 8h - 9\right] - [210 - 9]}{h}$$

$$= \lim_{h\to 0} \frac{h^3 + 19h^2 + 112h + 210 - 9 - 210 + 9}{h}$$

$$= \lim_{h\to 0} \frac{h^3 + 19h^2 + 112h}{h}$$

$$= \lim_{h\to 0} \frac{h\left(h^2 + 19h + 112\right)}{h}$$

$$f'(4) = 112$$

Q6

Find the derivative of the function f defined by $f(x) = mx + c$ at $x = 0$.

Solution

$$f(x) = mx + c$$
$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$
$$= \lim_{h \to 0} \frac{f(h) - h(0)}{h}$$
$$= \lim_{h \to 0} \frac{(mh + c) - (m \times 0 + c)}{h}$$
$$= \lim_{h \to 0} \frac{mh + c - c}{h}$$
$$= \lim_{h \to 0} \frac{mh}{h}$$
$$= m$$
$$f'(0) = m$$

Q7

Examine the differentiability of the function f defined by

$$f(x) = \begin{cases} 2x + 3, \text{ if } -3 \le x \le -2 \\ x + 1, \text{ if } -2 \le x < 0 \\ x + 2, \text{ if } 0 \le x \le 1 \end{cases}$$

Solution

$$f(x) = \begin{cases} 2x + 3, \text{ if } -3 \le x < -2 \\ x + 1, \text{ if } -2 \le x < 0 \\ x + 2, \text{ if } 0 \le x \le 1 \end{cases}$$

We know that polynomial funtiona are continuous and differentiable everywhere.
So f(x) is differentiable on $x \in [-3,2)$, $x \in (-2,0)$ and $x \in [0,1]$
We need to check the differentiability at x = -2 and x = 0

Differentiability at x = -2

$$(\text{LHD at } x = -2) = \lim_{x \to -2^-} \frac{f(x) - f(-2)}{x - (-2)} = \lim_{x \to -2^-} \frac{2x + 3 + 1}{x + 2} = \lim_{x \to -2^-} \frac{2(x+2)}{x + 2} = 2$$

$$(\text{RHD at } x = -2) = \lim_{x \to -2^+} \frac{f(x) - f(-2)}{x - (-2)} = \lim_{x \to -2^+} \frac{x + 1 + 1}{x + 2} = \lim_{x \to -2^+} \frac{x + 2}{x + 2} = 1$$

$\therefore$ (LHD at x = -2) $\neq$ (RHD at x = -2)
So, f(x) is not differentiable at x = -2.

Differentiability at x = 0

$$(\text{LHD at } x = 0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{x + 1 - 2}{x} = \lim_{x \to 0^-} \frac{x - 1}{x} \to \infty$$

$$(\text{RHD at } x = 0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x + 2 - 2}{x} = \lim_{x \to 0^+} \frac{x}{x} = 1$$

$\therefore$ (LHD at x = 0) $\neq$ (RHD at x = 0)
So, f(x) is not differentiable at x = 0.

Q8

Write an example of a function which is everywhere continuous but fails to be differentiable exactly at five points.

Solution

We know that, modulus function

$f(x) = |x|$ is continuous but bot differentiable at $x = 0$,

So,

$f(x) = |x| + |x-1| + |x-2| + |x-3| + |x-4|$ is continuous but not differentiable $x = 0, 1, 2, 3, 4.$

Q9

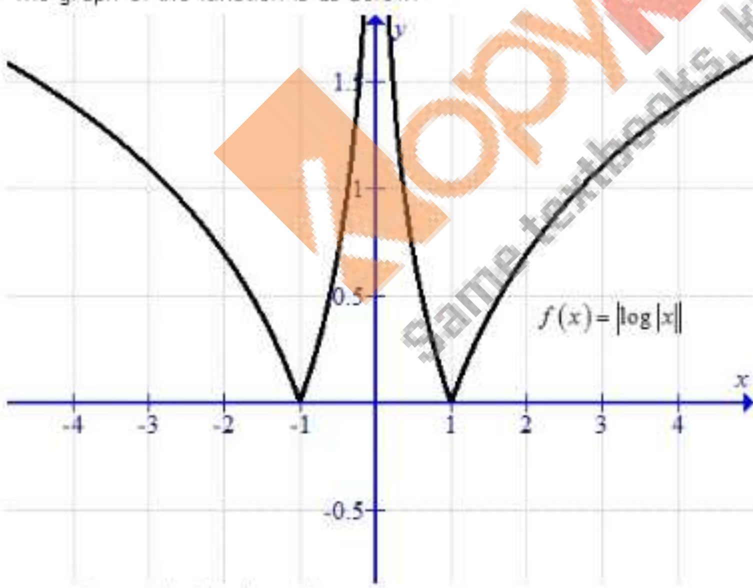
Discuss the continuity and differentiability of $f(x) = |\log|x||$.

Solution

$f(x) = |\log|x||$

Since, it is an absolute function. So, it is continuous function.

The graph of the function is as below:-



From the graph, it is clear that

$f(x)$ is not differentiable at $x = -1, 1$ but continuous for all x

Q10

Discuss the continuity and differentiability of $f(x) = e^{|x|}$

Solution

$f(x) = e^{|x|}$

$f(x) = \begin{cases} e^{-x} & , x < 0 \\ e^{x} & , x \geq 0 \end{cases}$

For continuity at $x = 0$

RHL $= \lim_{x \to 0^+} f(x)$

$= \lim_{h \to 0} f(0 + h)$

$= \lim_{h \to 0} e^{(0+h)}$

$= \lim_{h \to 0} e^{h}$

$= e^{0}$

RHL $= 1$

LHL $= \lim_{x \to 0^-} f(x)$

$= \lim_{h \to 0} f(0 - h)$

$= \lim_{h \to 0} e^{-(0-h)}$

$= \lim_{h \to 0} e^{h}$

LHL $= 1$

$f(0) = e^{0}$

$= 1$

Now,

LHL $= f(0) =$ RHL

So, $f(x)$ is continuous at $x = 0$

For differentiablility at $x = 0$

LHD $= \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}$

$= \lim_{h \to 0} \frac{f(0 - h) - e^{0}}{(0 - h) - 0}$

$= \lim_{h \to 0} \frac{e^{-(0-h)} - 1}{-h}$

$= \lim_{h \to 0} \frac{e^{h} - 1}{-h}$

$= 1$ $\left[\text{Since } \lim_{x \to 0} \frac{e^{x} - 1}{x} = 1\right]$

RHD $= \lim_{hx \to 0^+} \frac{f(x) - f(0)}{x - 0}$

$= \lim_{h \to 0} \frac{f(0 + h) - f(0)}{(0 + h) - 0}$

$= \lim_{h \to 0} \frac{e^{x} - e^{0}}{h}$

$= \lim_{h \to 0} \frac{e^{h} - 1}{h}$

$= 1$ $\left[\text{Since } \lim_{x \to 0} \frac{e^{x} - 1}{x} = 1\right]$

Clearly,

LHD $\neq$ RHD

So,

$f(x)$ is not differentiable at $x = 0$.

Q11

Discuss the continuity and differentiability of

$$f(x) = \begin{cases} (x-c)\cos\frac{1}{(x-c)} & , x \neq c \\ 0 & , x = c \end{cases}$$

Solution

$$f(x) = \begin{cases} (x-c)\cos\frac{1}{(x-c)} & , x \neq c \\ 0 & , x = c \end{cases}$$

$$\begin{aligned}
(\text{LHL at } x = c) &= \lim_{x \to c^-} f(x) \\
&= \lim_{h \to 0} f(-h) \\
&= \lim_{h \to 0} (c - h - c)\cos\left(\frac{1}{c-h-c}\right) \\
&= \lim_{h \to 0} -h\cos\left(-\frac{1}{h}\right) \\
&= \lim_{h \to 0} -h\cos\left(\frac{1}{h}\right) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
(\text{RHL at } x = c) &= \lim_{x \to c^+} f(x) \\
&= \lim_{h \to 0} f(c+h) \\
&= \lim_{h \to 0} (c + h - c)\cos\left(\frac{1}{c+h-c}\right) \\
&= \lim_{h \to 0} h\cos\left(\frac{1}{h}\right) \\
&= 0
\end{aligned}$$

$f(e) = 0$

Since, LHL = $f(x)$ = RHL at $x = c$

$\Rightarrow$ $f(x)$ is continuous at $x = c$

$$\begin{aligned}
(\text{LHD at } x = c) &= \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h} \\
&= \lim_{h \to 0} \frac{(c-h-c)\cos\left(\frac{1}{c-h-c}\right) - 0}{-h} \\
&= \lim_{h \to 0} \cos\left(-\frac{1}{h}\right) \\
&= \lim_{h \to 0} \cos\left(\frac{1}{h}\right) \\
&= k
\end{aligned}$$

$$\begin{aligned}
(\text{RHD at } x = c) &= \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \\
&= \lim_{h \to 0} \frac{(c+h-c)\cos\left(\frac{1}{c+h-c}\right) - 0}{h} \\
&= \lim_{h \to 0} \frac{h\cos\left(\frac{1}{h}\right)}{h} \\
&= \lim_{h \to 0} \cos\left(\frac{1}{h}\right) \\
&= k
\end{aligned}$$

$(\text{LHD at } x = c) = (\text{RHD at } x = c)$

So,

$f(x)$ is differentiable and continuous at $x = c$.

Q12

Is $|\sin x|$ differentiable ? What about $\cos|x|$?

Solution

$$f(x) = |\sin x| = \begin{cases} -\sin x & , x < n\pi \\ \sin x & , x \geq n\pi \end{cases}$$

For $x = n\pi$ (n even)

$$(\text{LHD at } x = n\pi) = \lim_{x \to n\pi^-} \frac{f(x) - f(n\pi)}{x - n\pi}$$
$$= \lim_{h \to 0} \frac{-\sin(n\pi - h) - \sin n\pi}{n\pi - h - n\pi}$$
$$= \lim_{h \to 0} \frac{\sin h - 0}{-h}$$
$$= -1$$

$$(\text{RHD at } x = n\pi) = \lim_{h \to 0} \frac{\sin(n\pi + h) - \sin n\pi}{h}$$
$$= \lim_{h \to 0} \frac{\sin h}{h}$$
$$= 1$$

$(\text{LHD at } x = n\pi) \neq (\text{RHD at } x = n\pi)$

For $x = n\pi$ (n is odd)

$$(\text{LHD at } x = n\pi) = \lim_{h \to 0} \frac{-\sin(n\pi - h) - \sin n\pi}{-h}$$
$$= \lim_{h \to 0} \frac{-\sin h}{-h}$$
$$= 1$$

$$(\text{RHD at } x = n\pi) = \lim_{h \to 0} \frac{\sin(n\pi + h) - \sin n\pi}{h}$$
$$= \lim_{h \to 0} \frac{-\sin h - 0}{h}$$
$$= -1$$

$(\text{LHD at } x = n\pi) \neq (\text{RHD at } x = n\pi)$

Thus,

$f(x) = |\sin x|$ is not differentiable at $x = n\pi$

$f(x) = \cos|x|$

Since, $\cos(-x) = \cos x$

$\Rightarrow \quad f(x) = \cos x$

$\Rightarrow \quad f(x) = \cos|x|$ is differnetiable everywhere.

Exercise MCQ

Q1

Let f(x) = |x| and g(x) = $|x^3|$, then
f(x) and g(x) both are continuous at x = 0
f(x) and g(x) both are differentiable at x = 0
f(x) is differentiable but g(x) is not differentiable at x = 0
f(x) and g(x) both are not differentiable at x = 0

Solution

Correct option: (a)
Absolute value function is continuous on R.

Q2

The function f(x) = $\sin^{-1}$ (cos x) is
discontinuous at x = 0
continuous at x = 0
differentiable at x = 0
none of these

Solution

Correct option: (b)

$$f(x) = \sin^{-1}(\cos x)$$
$$f(x) = \sin^{-1}\left[\sin\left(\frac{\pi}{2} - x\right)\right]$$
$$f(x) = \frac{\pi}{2} - x$$

Function is continuous at x = 0.

Q3

The set of points where the function f(x) = x|x| is differentiable is
(-∞, ∞)
(-∞, 0) ∪ (0, ∞)
(0, ∞)
[0, ∞]

Solution

Correct option: (a)

$$f(x) = x|x|$$
$$f(x) = x^2 \quad x > 0$$
$$= -x^2 \quad x < 0$$
$$= 0 \quad x = 0$$

Which is polynomial function. Hence, it is differentiable on $(-\infty, \infty)$

Q4

If $f(x)=\begin{cases}\dfrac{|x+2|}{\tan^{-1}(x+2)}, & x \neq -2 \\ 2, & x=-2\end{cases}$, then f(x) is

continuous at x = -2
not continuous at x = -2
differentiable at x = -2
continuous but not derivable at x = -2

Solution

Correct option: (b)

$\lim_{x\to -2}\dfrac{|x+2|}{\tan^{-1}(x+2)}$

Let, $x = -2+h$

$x \to -2 \Rightarrow h \to 0$

$\lim_{h\to 0}\dfrac{|-2+h+2|}{\tan^{-1}(-2+h+2)}$

$\lim_{h\to 0}\dfrac{h}{\tan^{-1}h} = 1$

$\lim_{x\to -2}\dfrac{|x+2|}{\tan^{-1}(x+2)} \neq f(-2)$

Function is not continuous at $x = -2$

Q5

Let f(x) = (x + |x|) |x|. Then, for all x
f is continuous
f is differentiable for some x
f ' is continuous
f'' is continuous

Solution

Correct option: (a), (c)

$f(x) = (x + |x|)|x|$

$\Rightarrow$

$f(x) = 2x^2 \quad x > 0$

$\quad = 0 \quad x < 0$

$\lim_{x\to 0} 2x^2 = 0$

Function is continuous at $x = 0$.

Also, differentiable at $x = 0$ as it is polynomial function.

Q6

The function $f(x) = e^{-|x|}$ is
continuous everywhere but not differentiable at x =0
continuous and differentiable everywhere
not continuous at x =0
none of these

Solution

Correct option: (a)

$f(x) = e^{-|x|}$

$\Rightarrow$

$f(x) = e^{x} \quad x > 0$

$f(x) = e^{-x} \quad x < 0$

Function is continuous at $x = 0$ but not differentiable at $x = 0$

Q7

The function f (x) =|cos x| is

everywhere continuous and differentiable

everywhere continuous but not differentiable at (2n+1)π /2, n ∊ Z

neither continuous nor differentiable at (2n +1) π/2,n∊ Z

none of these

Solution

Correct option : (b)

As cos x is even function it is continuous everywhere

but not differentiable at $(2n+1)\frac{\pi}{2},\ n \in Z$

$\cos\left[(2n+1)\frac{\pi}{2}\right] = \cos\left(n\pi + \frac{\pi}{2}\right) = -\sin n\pi$

For n as an integer $\Rightarrow \sin n\pi = 0$

For n as a rational $\Rightarrow \sin n\pi = -1$

Q8

If $f(x) = \sqrt{1 - \sqrt{1 - x^2}}$, Then f(x) is

continuous on [-1,1] and differentiable on (-1,1)

continuous on [-1,1] and differentiable (-1,0)∪ ϕ(0,1)

continuous and differentiable on [-1,1]

none of these

Solution

Correct option: (b)

$f(x)= \sqrt{1-\sqrt{1-x^2}}$

$1-x^2 > 0$

$-1 \le x \le 1$

$\Rightarrow$ function is continuous on $[-1,1]$.

To check differentiability,

$$\lim_{x\to 0^-}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{\sqrt{1-\sqrt{1-x^2}}}{x} = -\infty$$

$$\lim_{x\to 0^+}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{\sqrt{1-\sqrt{1-x^2}}}{x} = \infty$$

Function is not differentiable at $x = 0$.

Hence, differentiable on $(-1,0)\cup\phi(0,1)$

Q9

If $f(x) = a|\sin x| + b e^{|x|} + c|x|^3$ and if $f(x)$ is differentiable at $x = 0$, then

$a = b = c = 0$

$a = 0,\ b = 0;\ c \in R$

$b = c = 0;\ a \in R$

$c = 0,\ a = 0,\ b \in R$

Solution

Correct option: (b)

Given function can be written as

$$f(x) = -a\sin x + be^{-x} - cx^3 \quad x < 0$$
$$= a\sin x + be^{x} + cx^3 \quad x > 0$$

Function is differentiable at $x = 0$.

$$\Rightarrow \lim_{h\to 0^+}\frac{f(h)-f(0)}{h} = \lim_{h\to 0^-}\frac{f(-h)-f(0)}{h}$$

$$\Rightarrow \lim_{h\to 0}\frac{a\sin h + be^h + ch^3 - b}{h} = \lim_{h\to 0^-}\frac{-a\sin(-h) + be^h + ch^3 - b}{-h}$$

$$\Rightarrow \lim_{h\to 0}\frac{a\cos h + be^h + 3ch^2}{1} = \lim_{h\to 0^-}\frac{a\cos h + be^h + 3ch^2}{-1}$$

$\Rightarrow a + b = -a - b$

$\Rightarrow a + b = 0$

This is true for all value of c

Q10

If $f(x) = x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \dots + \frac{x^2}{(1+x^2)^n} + \dots,$

then at $x = 0$, $f(x)$

has no limit

is discontinuous

is continuous but not differentiable

is differentiable

Solution

Correct option: (b)

$$\lim_{x\to0} f(x) = \lim_{x\to0}\left(x^2 + \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \ldots + \frac{x^2}{(1+x^2)^n} + \ldots\right)$$

$$\lim_{x\to0} f(x) = \lim_{x\to0} x^2\left(1 + \frac{1}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \ldots + \frac{1}{(1+x^2)^n} + \ldots\right)$$

$$\lim_{x\to0} f(x) = \lim_{x\to0} x^2\left(\frac{1}{1-\frac{1}{1+x^2}}\right)$$

$$\lim_{x\to0} f(x) = \lim_{x\to0}(1+x^2)$$

$$\lim_{x\to0} f(x) = 1$$

But, $f(0) = 0$

$f(0) \neq \lim_{x\to0} f(x)$

Function is discontinuous.

Q11

If f (x)= $|\log_e x|$, then

f' (1^+) =1
f' (1^-) =1
f' (1) =1
f' (1) = -1

Solution

Correct option: (a), (b)

$$\lim_{x\to1^+}\frac{f(x)-f(1)}{x-1} = \lim_{x\to1}\frac{|\log_e x| - 0}{x-1}$$

$x = 1 + h$ say

$x \to 1 \Rightarrow h \to 0$

$$\lim_{x\to1^+}\frac{f(x)-f(1)}{x-1} = \lim_{h\to0}\frac{\log(1+h)}{1+h-1}$$

$$\lim_{x\to1^+}\frac{f(x)-f(1)}{x-1} = \lim_{h\to0}\frac{\log(1+h)}{h} = 1$$

similarly, $\lim_{x\to1^-}\frac{f(x)-f(1)}{x-1} = -1$

Q12

If f (x) = $\log_e |x|$, then

f(x) is continuous and differentiable for all x in its domain
f (x) is continuous for all for all x in its domain but not differentiable at x = ± 1
f (x) is neither continuous nor differentiable at x = ± 1
none of these

Solution

Correct option: (b)

$f(x) = \log_e |x|$

Logarithmic function is always continuous on R - {0}

To check differentiability,

$\lim_{x \to 1^+} \frac{f(x) - f(0)}{x - 1}$

$\lim_{x \to 1^+} \frac{\log_e x - 0}{x - 1}$

$x = 1 + h$

$x \to 1 \Rightarrow h \to 0$

$\lim_{h \to 0} \frac{\log_e (1 + h)}{1 + h - 1} = 1$

similarly, $\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = -1$

Also, you can check it for $x = -1$.

Function iscontinuous but not differentiable at $x = \pm 1$.

Q13

Let $f(x) = \begin{cases} \frac{1}{|x|} & \text{for } |x| \geq 1 \\ ax^2 + b & \text{for } |x| < 1 \end{cases}$. If $f(x)$ is continuous and differentiable at any point, then

a. $a = \frac{1}{2}, b = -\frac{3}{2}$

b. $a = -\frac{1}{2}, b = \frac{3}{2}$

c. a = 1 , b = -1

d. none of these

Solution

Correct option: (b)

Given function is continuous at x = 1.

$\Rightarrow \lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x)$

$\Rightarrow \lim\limits_{x\to 1} \frac{1}{x} = \lim\limits_{x\to 1^+} ax^2 + b$

$\Rightarrow 1 = a + b \qquad \ldots(i)$

Function is derivable at x = 1.

$\Rightarrow \lim\limits_{h\to 0^+} \frac{f(1+h) - f(1)}{h} = \lim\limits_{h\to 0^-} \frac{f(0-h) - f(1)}{h}$

$\Rightarrow \lim\limits_{h\to 0^+} \frac{\frac{1}{1+h} - 1}{h} = \lim\limits_{h\to 0^-} \frac{a(1+h)^2 - a}{h}$

$\Rightarrow -1 = \lim\limits_{h\to 0^-} \frac{h(2a+h)}{h}$

$\Rightarrow 2a = -1$

$\Rightarrow a = \frac{-1}{2}$

$a + b = 1 \quad$ (From (i))

$\frac{-1}{2} + b = 1$

$\Rightarrow b = \frac{3}{2}$

Q14

The function f(x) =x -[x], where [.] denotes the greatest integer function is
continuous everywhere
continuous at integer point only
continuous at non-integer points only
differentiable everywhere

Solution

Correct option: (c)

$f(x) = x - [x]$

$\lim\limits_{x\to n^-} f(x) = [x - (n-1)] = x - n + 1$

$\lim\limits_{x\to n^+} f(x) = (x - n) = x - n$

Hence, given function is continuous on non – integers only.

Q15

Let $f(x) = \begin{cases} ax^2 + 1, & x > 1 \\ x + 1/2, & x \le 1. \end{cases}$ Then, f (x)is derivable at x = 1, if

a = 2
a = 1
a =0
a =1/2

Solution

Correct option: (d)

Given function is derivable.

$$\Rightarrow \lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h}=\lim_{h\to 0^-}\frac{f(1-h)-f(1)}{h}$$

$$\Rightarrow \lim_{h\to 0}\frac{a(1+h)^2-(a+1)}{h}=\lim_{h\to 0}\frac{1+h+\frac{1}{2}-\frac{3}{2}}{h}$$

$$\Rightarrow \lim_{h\to 0}\frac{ah(h+2)}{h}=\lim_{h\to 0}\frac{h}{h}$$

$$\Rightarrow \lim_{h\to 0}a(h+2)=1$$

$$\Rightarrow 2a=1$$

$$\Rightarrow a=\frac{1}{2}$$

Q16

Let f (x) =|sin x|. Then,

f (x) is everywhere differentiable.

f (x) everywhere continuous but not differentiable at x = n π, n ∈ Z

c. $f(x)$ is everywhere continuous but not differentiable at $x=(2n+1)\frac{\pi}{2}, n\in Z$.

None of these

Solution

Correct option: (b)

$f(x)=|\sin x|$

Given function is continuous and differentiable on $(2n\pi,(2n+1)\pi)$

But not differentiable at $x=n\pi,\ n\in Z$.

As $\sin n\pi=0$ for $n\in Z$.

Q17

Let f (x) =|cos x|. Then,

f (x) is everywhere differentiable

f (x) everywhere continuous but not differentiable at x = n π, n ∈ Z

c. $f(x)$ is everywhere continuous but not differentiable at $x=(2n+1)\frac{\pi}{2}, n\in Z$.

Solution

Correct option: (c)

$f(x)=|\cos x|$

Given function is trigonometric function.

$\Rightarrow$ Hence, it is continuous.

Function is not differentiable at odd multiples of $\frac{\pi}{2}$

$\Rightarrow f(x)$ is not differentiable at $x=(2n+1)\frac{\pi}{2}$

Q18

The function f (x) =1+|cos x| is
Continuous no where
Continuous everywhere
Not differentiable at x =0
Not differentiable at x =n π, n ∈ Z

Solution

Correct option: (b)

$f(x) = 1 + |\cos x|$

cosx is even function hence,

$\lim_{x\to a} f(x) = \lim_{x\to a^+} f(x)$

Function is continuous on R.

But it is not differentiable at $x = (2n+1)\frac{\pi}{2}, n \in Z$

Q19

The function f (x) = |cos x| is
Differentiable at x = (2n+1) π/2, n∈ Z
Continuous but not differentiable at x =(2n+1) π/2, n ∈ Z
Neither differentiable nor continuous at x = n π, n ∈ Z
None of these

Solution

Correct option: (b)

$f(x) = |\cos x|$

Given function is trigonometric function.

⇒ Hence, it is continuous.

Function is not differentiable at odd multiples of $\frac{\pi}{2}$

⇒ $f(x)$ is not differentiable at $x = (2n+1)\frac{\pi}{2}$

Q20

The function $f(x) = \frac{\sin(\pi[x-\pi])}{4+[x]^2}$, where [.] denotes the greatest integer function,is
Continuous as well differentiable for all x ∈ R
Continuous for all x but not continuous at some x.
Differentiable for all but not continuous at same x'.
None of these

Solution

Correct option: (a)

$$f(x) = \frac{\sin(\pi[x-\pi])}{4+[x]^2}$$

As $4+[x^2] \neq 0$

$\Rightarrow f(x) = 0$ for all x

$f(x)$ is continuous and differentiable on R.

Q21

Let f (x) = $\alpha + b|x| + c|x|^4$, where α, b and c are real constants. Then f (x) is differentiable at x =0 , if

a =0

b =0

c =0

none of these

Solution

Correct option: (b)

$$f(x) = a + b|x| + c|x|^4$$

$$f(x) = a - bx + cx^4 \quad x < 0$$

$$= a + bx + cx^4 \quad x \leq 0$$

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x} = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x}$$

$$\lim_{x\to 0^-} \frac{a-bx+cx^4-a}{x} = \lim_{x\to 0^+} \frac{a+bx+cx^4-a}{x}$$

$$\lim_{x\to 0^-} \frac{-bx+cx^4}{x} = \lim_{x\to 0^+} \frac{bx+cx^4}{x}$$

$$\lim_{x\to 0^-} \frac{x(-b+cx^3)}{x} = \lim_{x\to 0^+} \frac{x(b+cx^3)}{x}$$

$$\lim_{x\to 0^-} -b+cx^3 = \lim_{x\to 0^+} b+cx^3$$

$$-b = b$$

$$2b = 0$$

$$b = 0$$

Q22

If f (x) = |3-x|+(3+x) , where (x) denotes the least integer greater than or equal to x, then f (x) is

continuous and differentiable at x =3

continuous but not differentiable at x = 3

differentiable but not continuous at x =3

neither differentiable nor continuous at x =3

Solution

Correct option: (d)

Given function can be written as

$f(x) = -x + 9 \quad x < 3$

$= x + 4 \quad x > 3$

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3} -x + 9 = 6$

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3} x + 4 = 7$

Function is not continuous at $x = 3$.

$\Rightarrow$ Function is not differentiable at $x = 3$

Q23

If $f(x) = \begin{cases} \frac{1}{1+e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then $f(x)$ is

continuous as well as differentiable at x =0
continuous but not differentiable at x = 0
differentiable but not continuous at x =0
none of these

Solution

Correct option: (d)

$\lim_{x \to 0^-} \frac{1}{1+e^{\frac{1}{x}}} = \lim_{x \to 0} \frac{1}{1+e^{\frac{-1}{x}}} = 1 \quad \left(\because \lim_{x \to 0} e^{\frac{-1}{x}} = 0\right)$

$\lim_{x \to 0^-} f(x) \neq f(0)$

Function is not continuous.

$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x} = \lim_{h \to 0} \frac{f(-h) - f(0)}{-h}$

$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x} = \lim_{h \to 0} \frac{\frac{1}{1+e^{\frac{-1}{h}}} - 0}{-h}$

$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x} = \lim_{h \to 0} \frac{1}{-\left(1+e^{\frac{-1}{h}}\right)h} = -\infty$

Similarly,

$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x} = \infty$

Q24

If $f(x) = \begin{cases} \frac{1-\cos x}{x \sin x}, & x \neq 0 \\ \frac{1}{2}, & x = 0 \end{cases}$ then at $x = 0$, $f(x)$ is

continuous and differentiable
differentiable but not continuous
continuous but not differentiable
neither continuous nor differentiable

Solution

Correct option: (a)

Given function is continuous at $x = 0$.

You can check it by definition,

$\lim_{x\to 0} f(x) = f(0)$

Also,

$$\lim_{h\to 0^-} \frac{f(0-h)-f(0)}{h} = \lim_{h\to 0} \frac{\frac{1-\cos(-h)}{(-h)\sin(-h)} - 0}{h}$$

$$\lim_{h\to 0^-} \frac{f(-h)-f(0)}{h} = \lim_{h\to 0} \frac{1-\cosh}{h\sinh} = \frac{1}{2}$$

Also, $\lim_{h\to 0^+} \frac{f(h)-f(0)}{h} = \frac{1}{2}$

Hence, function is continuous and differentiable at $x = 0$

Q25

The set of point where the function f (x) = |x-3| cos x is differentiable

R

R - {3}

(0,∞)

None of these

Solution

Correct option: (b)

$$\lim_{x\to 3^+} \frac{f(x)-f(3)}{x-3} = \lim_{h\to 0} \frac{f(3+h)-f(3)}{h}$$

$$\lim_{x\to 3^+} \frac{f(x)-f(3)}{x-3} = \lim_{h\to 0} \frac{|3+h-3|\cos 3 - 0}{h}$$

$$\lim_{x\to 3^+} \frac{f(x)-f(3)}{x-3} = \lim_{h\to 0} \frac{h\cos 3}{h} = \cos 3$$

cos3 is not differentiable.

Function is differentiable on $R - \{3\}$.

Q26

$$\text{Let } f(x) = \begin{cases} 1, & x \le -1 \\ |x|, & -1 < x < 1 \\ 0, & x \ge 1 \end{cases} \text{. Then, f is}$$

Continuous at x =1

Differentiable at x = -1

Everywhere continuous

Everywhere differentiable

Solution

Correct option: (b)

$$f(x) = \begin{cases} 1, & x \le -1 \\ |x|, & -1 < x < 1 \\ 0, & x \ge 1 \end{cases}$$

$$\lim_{x \to -1^-} \frac{f(x) - f(-1)}{x + 1} = \lim_{x \to -1^-} \frac{-x + 1}{x + 1} = 0$$

Similarly,

$$\lim_{x \to -1^+} \frac{f(x) - f(-1)}{x + 1} = \lim_{x \to -1^+} \frac{x - 1}{x + 1} = 0$$

Function is differentiable at $x = -1$.

Exercise 10VSAQ

Q1

Define diferentiability of a function at a point.

Solution

Differentiability of a function at a point:

A real valued function $f(x)$ defined on (a,b) is said to be differentiable at $x = c \in (a,b)$ if and only if,

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \text{ exists finitely}$$

$$\Rightarrow \quad \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$$

$$\Rightarrow \quad \lim_{h \to 0} \frac{f(c-h) - f(c)}{-h} = \lim_{h \to c^+} \frac{f(c+h) - f(c)}{h}$$

$$\Rightarrow \quad (\text{LHD at } x = c) = (\text{RHD at } x = c)$$

Q2

Is every differentiable function continuous?

Solution

Yes, every differentiable function is continuous.

Q3

Is every continuous function differentiable?

Solution

No, every continuous function is not differentiable

For example, $f(x) = |x|$ is continuous at $x = 0$ but not differentiable at $x = 0$.

Q4

Give an example of a function which is continuous but not differentiable at a point.

Solution

$$f(x) = |x|$$
$$= \begin{cases} -x & , x < 0 \\ x & , x \geq 0 \end{cases}$$

For continuity at $x = 0$

LHL $= \lim_{x \to 0^-} f(x)$
$= \lim_{h \to 0} f(0-h)$
$= \lim_{h \to 0} -(0-h)$
$= 0$

RHL $= \lim_{x \to 0^+} f(x)$
$= \lim_{h \to 0} f(0+h)$
$= \lim_{h \to 0} (0+h)$
$= 0$

$f(0) = 0$

LHL $= f(0) =$ RHL

So,

$f(x)$ is continuous at $x = 0$.

For differentiability at $x = 0$

LHD $= \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}$
$= \lim_{h \to 0} \frac{f(0-h) - f(0)}{(0-h) - 0}$
$= \lim_{h \to 0} \frac{-(0-h) - 0}{-h}$
$= -1$

RHD $= \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0}$
$= \lim_{h \to 0} \frac{f(0+h) - f(0)}{0 + h - 0}$
$= \lim_{h \to 0} \frac{(0+h)}{h}$
$= 1$

(LHL at $x = 0$) $\neq$ (RHD at $x = 0$)

So,

$f(x)$ is not differentiable.

Thus,

$f(x) = |x|$ is continuous but not differentiable at $x = 0$.

Q5

If $f(x)$ is differentiable at $x = c$, then write the value of $\lim_{x \to c} f(x)$.

Solution

Given,

$f(x)$ is differentiable at $x - c$

$\lim_{x \to c} f(x) = f(c)$

Q6

If $f(x) = |x-2|$ write wheteher $f'(2)$ exists or not.

Solution

$$f(x) = |x-2|$$
$$= \begin{cases} -(x-2) & , x-2<0 \\ (x-2) & , x-2 \ge 0 \end{cases}$$
$$= \begin{cases} 2-x & , x<2 \\ x-2 & , x \ge 2 \end{cases}$$

$$\text{LHD at } x=2 = \lim_{x \to 2^-} \frac{f(x)-f(2)}{x-2}$$
$$= \lim_{h \to 0} \frac{f(2-h)-f(2)}{(2-h)-2}$$
$$= \lim_{h \to 0} \frac{(2-(2-h))-(2-2)}{-h}$$
$$= \lim_{h \to 0} \frac{h-0}{-h}$$
$$= -1$$

$$\text{RHD at } x=2 = \lim_{x \to 2^+} \frac{f(x)-f(2)}{x-2}$$
$$= \lim_{h \to 0} \frac{f(2+h)-f(2)}{2+h-2}$$
$$= \lim_{h \to 0} \frac{(2+h-2)-0}{h}$$
$$= 1$$

Since,

LHD at $x = 2 \neq$ RHD at $x = 2$

So,

$f'(2)$ does not exist.

Q7

Write the points where $f(x) = |\log_e x|$ is not differentiable.

Solution

$$f(x) = |\log_e x| = \begin{cases} -\log_e x & , 0 < x < 1 \\ \log_e x & , x \geq 1 \end{cases}$$

$$(\text{LHD at } x = 1) = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1}$$
$$= \lim_{h \to 0} \frac{f(1-h) - f(1)}{(1-h) - 1}$$
$$= \lim_{h \to 0} \frac{-\log_e(1-h) - 0}{-h}$$
$$= \lim_{h \to 0} \frac{-\log(1-h)}{h}$$
$$= -1$$

$$(\text{RHD at } x = 1) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1}$$
$$= \lim_{h \to 0} \frac{f(1+h) - f(1)}{(1+h) - 1}$$
$$= \lim_{h \to 0} \frac{\log(1+h) - 0}{1 + h - 1}$$
$$= \lim_{h \to 0} \frac{\log(1+h)}{h}$$
$$= 1$$

$$(\text{LHD at } x = 1) \neq (\text{RHD at } x = 1)$$

$\therefore$ $f(x)$ is not differentiable at $x = 1$.

Q8

Write the points of non-differentiability of $f(x) = |\log|x||$.

Solution

Here,

$f(x) = |\log|x||$

$f(x)$ will always positive and let two points $x = 1$ and $x = -1$

$f(x) = 0$

The function $f(x) = |\log|x||$ is not differentiable at $x = -1$ and 1.

Q9

Write the derivative of $f(x) = |x|^3$ at $x = 0$.

Solution

$$f(x) = |x|^3$$
$$= \begin{cases} -x^3 & , x < 0 \\ x^3 & , x \geq 0 \end{cases}$$

$$(\text{LHD at } x = 0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}$$
$$= \lim_{h \to 0} \frac{f(0-h) - f(0)}{(0-h) - 0}$$
$$= \lim_{h \to 0} \frac{-(-h)^3}{-h}$$
$$= \lim_{h \to 0} \frac{h^3}{-h}$$
$$= 0$$

$$(\text{RHD at } x = 0) = \lim_{x \to 1^+} \frac{f(x) - f(0)}{x - 0}$$
$$= \lim_{h \to 0} \frac{f(0+h) - f(0)}{(0+h) - 0}$$
$$= \lim_{h \to 0} \frac{h^3}{h}$$
$$= 0$$

$$\therefore \quad f'(0) = 0$$

Q10

Write the number of points where $f(x) = |x| + |x - 1|$ is continuous but not differentiable.

Solution

$$f(x) = |x| + |x-1|$$
$$= \begin{cases} -x-(x-1) & , \text{ if } x \le 0 \\ x-(x-1) & , \text{ if } 0 < x < 1 \\ x+(x-1) & , \text{ if } x \ge 1 \end{cases}$$
$$= \begin{cases} 1-2x & , \text{ if } x \le 0 \\ 1 & , \text{ if } 0 < x < 1 \\ 2x-1 & , \text{ if } x \ge 1 \end{cases}$$

For $n = 0$

$$f(0) = 1 - 2(0) = 1$$

LHL $= \lim_{x \to 0^-} f(x)$
$= \lim_{h \to 0} f(0-h)$
$= \lim_{h \to 0} [1 - 2(0-h)]$
$= \lim_{h \to 0} [1 + 2h]$
$= 1$

RHL $= \lim_{x \to 0^+} f(x)$
$= \lim_{h \to 0} f(0+h)$
$= \lim_{h \to 0} 1$
$= 1$

So,

LHL $= f(0) =$ RHL

$\therefore$ $f(x)$ is continuous at $x = 0$

(LHD at $x = 0$) $= \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}$
$= \lim_{h \to 0} \frac{f(0-h) - f(0)}{(0-h) - 0}$
$= \lim_{h \to 0} \frac{1 - 2(0-h) - 1}{-h}$
$= \lim_{h \to 0} \frac{2h}{-h}$
$= -2$

(RHD at $x = 0$) $= \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0}$
$= \lim_{h \to 0} \frac{f(0+h) - f(0)}{0 + h - 0}$
$= \lim_{h \to 0} \frac{1-1}{h}$
$=$ Not defined

(LHD at $x = 0$) $\neq$ (RHD at $x = 0$)

$\therefore$ $f(x)$ is not differentiable at $x = 0$

For $n = 1$,

$$f(1) = 2(1) - 1 = 1$$

LHL $= \lim_{x \to 1^-} f(x)$
$= \lim_{h \to 0} f(1-h)$
$= 1$

RHL $= \lim_{x \to 1^+} f(x)$
$= \lim_{h \to 0} f(1+h)$
$= \lim_{h \to 0} [2(1+h) - 1]$
$= 1$

Q11

If $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists finitely, write the value of $\lim_{x \to c} f(x)$.

Solution

$\text{LHL} = f(1) = \text{RHL}$

So, $f(x)$ is continuous at $x = 1$

Now,

$$(\text{LHD at } x = 1) = \lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1}$$
$$= \lim_{h \to 0} \frac{f(1-h) - f(1)}{(1-h) - 1}$$
$$= \lim_{h \to 0} \frac{1-1}{-h}$$
$$= \text{Not defined}$$

$$(\text{RHD at } x = 1) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1}$$
$$= \lim_{h \to 0} \frac{f(1+h) - f(1)}{(1+h) - 1}$$
$$= \lim_{h \to 0} \frac{2(1+h) - 1 - 1}{h}$$
$$= \lim_{h \to 0} \frac{2h}{h}$$
$$= 1$$

$(\text{LHD at } x = 1) \neq (\text{RHD at } x = 1)$

$\therefore$ $f(x)$ is not differentiable at $x = 1$

So,

$f(x)$ is continuous but not differentiable at $x = 0$ and 1.

$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists finitely

So,

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$
$$f'(c) \lim_{x \to c} (x - c) = \lim_{x \to c} f(x) - f(c)$$
$$f'(c)(c - c) = \lim_{x \to c} f(x) - f(c)$$
$$0 = \lim_{x \to c} f(x) - f(c)$$

$$\lim_{x \to c} f(x) = f(c)$$

Q12

Write the value of the derivative of $f(x) = |x - 1| + |x - 3|$ at $x = 2$.

Solution

Here,

$$f(x) = |x-1| + |x+3|$$

at $x = 2$

$$f(x) = (x-1) - (x-3)$$
$$= x - 1 - x + 3$$
$$f(x) = 2$$

Now,

$$f'(x) = 0$$

Q13

If $f(x) = \sqrt{x^2+9}$, write the value of $\lim_{x \to 4} \frac{f(x) - f(4)}{x-4}$.

Solution

$$f(x) = \sqrt{x^2+9}$$
$$f'(4) = \lim_{x \to 4} \frac{f(x) - f(4)}{x-4}$$
$$= \lim_{h \to 0} \frac{f(4+h) - f(4)}{4+h-4}$$
$$= \lim_{h \to 0} \frac{\sqrt{(4+h)^2+9} - \sqrt{16+9}}{h}$$
$$= \lim_{h \to 0} \frac{\sqrt{h^2+8h+25} - \sqrt{25}}{h}$$
$$= \lim_{h \to 0} \left(\frac{\sqrt{h^2+8h+25} - \sqrt{25}}{h}\right)\left(\frac{\sqrt{h^2+8h+25} + \sqrt{25}}{\sqrt{h^2+8h+25} + \sqrt{25}}\right)$$
$$= \lim_{h \to 0} \frac{h^2+8h+25-25}{h\left(\sqrt{h^2+8h+25} + \sqrt{25}\right)}$$
$$= \lim_{h \to 0} \frac{h^2+8h}{5\left(\sqrt{h^2+8h+25} + \sqrt{25}\right)}$$
$$= \lim_{h \to 0} \frac{h+8}{\left(\sqrt{h^2+8h+25} + 5\right)}$$
$$= \frac{8}{5+5}$$
$$= \frac{8}{10}$$

$$f'(4) = \frac{4}{5}$$