Exercise 4.1

Q1

Find the value of each of the following :

$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

Solution

Let $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = y$.

Then $\sin y = -\frac{\sqrt{3}}{2} = -\sin\left(\frac{\pi}{3}\right) = \sin\left(-\frac{\pi}{3}\right)$

We know that principal value of $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

and $\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$.

Therefore principal value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ is $-\frac{\pi}{3}$

Q2

Find the value of each of the following :

$\sin^{-1}\left(\cos\frac{2\pi}{3}\right)$

Solution

Let $\sin^{-1}\left(\cos\frac{2\pi}{3}\right) = y$.

Then $\sin y = \cos\left(\frac{2\pi}{3}\right) = -\sin\left(\frac{\pi}{2}+\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right)$

We know that principal value of $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

and $-\sin\left(\frac{\pi}{6}\right) = \cos\left(\frac{2\pi}{3}\right)$.

Therefore principal value of $\sin^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right)$ is $-\frac{\pi}{6}$.

Q3

Find the value of each of the following :

$\sin^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)$

Solution

$$\sin^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)=\sin^{-1}\left(\frac{\sqrt{3}}{2\sqrt{2}}-\frac{1}{2\sqrt{2}}\right)$$
$$=\sin^{-1}\left(\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}-\frac{1}{2}\times\frac{1}{\sqrt{2}}\right)$$
$$=\sin^{-1}\left(\frac{\sqrt{3}}{2}\times\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2}-\frac{1}{\sqrt{2}}\times\sqrt{1-\left(\frac{\sqrt{3}}{2}\right)^2}\right)$$
$$=\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)-\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$
$$=\frac{\pi}{3}-\frac{\pi}{4}$$
$$=\frac{\pi}{12}$$

Q4

Find the value of each of the following :

$$\sin^{-1}\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)$$

Solution

$$\sin^{-1}\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)=\sin^{-1}\left(\frac{\sqrt{3}}{2\sqrt{2}}+\frac{1}{2\sqrt{2}}\right)$$
$$=\sin^{-1}\left(\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}+\frac{1}{2}\times\frac{1}{\sqrt{2}}\right)$$
$$=\sin^{-1}\left(\frac{\sqrt{3}}{2}\times\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2}+\frac{1}{\sqrt{2}}\times\sqrt{1-\left(\frac{\sqrt{3}}{2}\right)^2}\right)$$
$$=\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)+\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$$
$$=\frac{\pi}{3}+\frac{\pi}{4}$$
$$=\frac{7\pi}{12}$$

Q5

Find the value of each of the following :

$$\sin^{-1}\left(\cos\frac{3\pi}{4}\right)$$

Solution

Let $\sin^{-1}\left(\cos\frac{3\pi}{4}\right) = y$.

Then $\sin y = \cos\left(\frac{3\pi}{4}\right) = -\sin\left(\pi - \frac{3\pi}{4}\right) = -\sin\left(\frac{\pi}{4}\right)$

We know that principal value of $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

and $-\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{3\pi}{4}\right)$.

Therefore principal value of $\sin^{-1}\left(\cos\left(\frac{3\pi}{4}\right)\right)$ is $-\frac{\pi}{4}$.

Q6

Find the value of each of the following :

$\sin^{-1}\left(\tan\frac{5\pi}{4}\right)$

Solution

Let $y = \sin^{-1}\left(\tan\frac{5\pi}{4}\right)$

Therefore $\sin y = \left(\tan\frac{5\pi}{4}\right) = \tan\left(\pi + \frac{\pi}{4}\right) = \tan\left(\frac{\pi}{4}\right) = 1 = \sin\left(\frac{\pi}{2}\right)$

We know that principal value of $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

and $\sin\left(\frac{\pi}{2}\right) = \tan\left(\frac{5\pi}{4}\right)$.

Therefore principal value of $\sin^{-1}\left(\tan\left(\frac{5\pi}{4}\right)\right)$ is $\frac{\pi}{2}$.

Q7

Find the value of each of the following :

$\sin^{-1}\frac{1}{2} - 2\sin^{-1}\frac{1}{\sqrt{2}}$

Solution

$$\sin^{-1}\frac{1}{2} - 2\sin^{-1}\frac{1}{\sqrt{2}} = \sin^{-1}\frac{1}{2} - \sin^{-1}\left(2 \times \frac{1}{\sqrt{2}}\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2}\right)$$

$$= \sin^{-1}\frac{1}{2} - \sin^{-1}(1)$$

$$= \frac{\pi}{6} - \frac{\pi}{2}$$

$$= -\frac{\pi}{3}$$

Q8

Find the value of each of the following :

$$\sin^{-1}\left\{\cos\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)\right\}$$

Solution

$$\sin^{-1}\left\{\cos\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)\right\} = \sin^{-1}\left\{\cos\left(\frac{\pi}{3}\right)\right\} = \sin^{-1}\left\{\frac{\sqrt{3}}{2}\right\} = \frac{\pi}{6}$$

Q9

Find the domain of each of the following functions:

$F(x) = \sin^{-1}x^2$

Solution

Domain of $\sin^{-1}$ lies in the interval $[-1, 1]$.

$\therefore$ Domain of $\sin^{-1}x^2$ lies in the interval $[-1, 1]$.

$\Rightarrow -1 \le x^2 \le 1$

but x^2 can not take negative values

$\Rightarrow 0 \le x^2 \le 1$

$\Rightarrow -1 \le x \le 1$

Domain of $\sin^{-1}x^2$ is $[-1, 1]$.

Q10

Find the domain of each of the following functions:

$F(x) = \sin^{-1}x + \sin x$

Solution

Domain of $\sin^{-1}x$ lies in the interval $[-1, 1]$.
$\Rightarrow -1 \le x \le 1$
Domain of sinx lies in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
$\Rightarrow -\frac{\pi}{2} \le x \le \frac{\pi}{2}$
$\Rightarrow -1.57 \le x \le 1.57$

Domain of $\sin^{-1}x$ + sinx is the intersection of the domains of $\sin^{-1}x$ and sinx.
So Domain of $\sin^{-1}x$ + sinx is $[-1, 1]$.

Q11

Find the domain of each of the following functions:

$f(x) = \sin^{-1}\sqrt{x^2 - 1}$

Solution

Domain of $\sin^{-1}x$ lies in the interval $[-1, 1]$.
$\therefore$ Domain of $\sin^{-1}\sqrt{x^2-1}$ lies in the interval $[-1, 1]$
$\Rightarrow -1 \le \sqrt{x^2-1} \le 1$
$\Rightarrow 0 \le x^2 - 1 \le 1$
$\Rightarrow 1 \le x^2 \le 2$
$\Rightarrow \pm\sqrt{1} \le x \le \pm\sqrt{2}$
$\Rightarrow -\sqrt{2} \le x \le -1$ and $1 \le x \le \sqrt{2}$
Domain of $\sin^{-1}\sqrt{x^2-1}$ is $\left[-\sqrt{2}, 1\right] \cup \left[1, \sqrt{2}\right]$.

Q12

Find the domain of each of the following functions:

$f(x) = \sin^{-1}x + \sin^{-1}2x$

Solution

Domain of $\sin^{-1}x$ lies in the interval [-1, 1].

$\Rightarrow -1 \le x \le 1$

$\therefore$ Domain of $\sin^{-1}2x$ lies in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\Rightarrow -1 \le 2x \le 1$

$\Rightarrow -\frac{1}{2} \le x \le \frac{1}{2}$

Domain of $\sin^{-1}x + \sin^{-1}2x$ is the intersection of the domains of $\sin^{-1}x$ and $\sin^{-1}2x$.

So Domain of $\sin^{-1}x + \sin x$ is $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

Q13

If $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z + \sin^{-1}t = 2\pi$, then find the value $x^2 + y^2 + z^2 + t^2$.

Solution

Range of $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Given that $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z + \sin^{-1}t = 2\pi$

$\Rightarrow$ Each of $\sin^{-1}x$, $\sin^{-1}y$, $\sin^{-1}z$ and $\sin^{-1}t$ takes value of $\frac{\pi}{2}$.

$\Rightarrow x = 1, y = 1, z = 1$ and $t = 1$

$x^2 + y^2 + z^2 + t^2 = 1 + 1 + 1 + 1 = 4$

Q14

If $\left(\sin^{-1}x\right)^2 + \left(\sin^{-1}y\right)^2 + \left(\sin^{-1}z\right)^2 = \frac{3}{4}\pi^2$,

find the value of $x^2 + y^2 + z^2$.

Solution

Range of $\sin^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Given that $\left(\sin^{-1}x\right)^2 + \left(\sin^{-1}y\right)^2 + \left(\sin^{-1}z\right)^2 = \frac{3}{4}\pi^2$

$\Rightarrow$ Each of $\sin^{-1}x$, $\sin^{-1}y$ and $\sin^{-1}z$ takes value of $\frac{\pi}{2}$.

$\Rightarrow x = 1, y = 1,$ and $z = 1$

$x^2 + y^2 + z^2 = 1 + 1 + 1 = 3$

Exercise 4.2

Q1

Find the domain of definition of $f(x) = \cos^{-1}(x^2 - 4)$.

Solution

Domain of $\cos^{-1}x$ lies in the interval $[-1, 1]$.

$\therefore$ Domain of $\cos^{-1}(x^2 - 4)$ lies in the interval $[-1, 1]$

$\Rightarrow -1 \le x^2 - 4 \le 1$

$\Rightarrow 3 \le x^2 \le 5$

$\Rightarrow \pm\sqrt{3} \le x \le \pm\sqrt{5}$

$\Rightarrow -\sqrt{5} \le x \le -\sqrt{3}$ and $\sqrt{3} \le x \le \sqrt{5}$

Domain of $\cos^{-1}(x^2 - 4)$ is $[-\sqrt{5}, -\sqrt{3}] \cup [\sqrt{3}, \sqrt{5}]$

Q2

Find the domain of $f(x) = 2\cos^{-1}2x + \sin^{-1} x$.

Solution

Domain of $\cos^{-1}x$ lies in the interval $[-1, 1]$.

$\therefore$ Domain of $\cos^{-1}(2x)$ lies in the interval $[-1, 1]$

$\Rightarrow -1 \le 2x \le 1$

$\Rightarrow -\frac{1}{2} \le x \le \frac{1}{2}$

Domain of $\cos^{-1}(2x)$ is $\left[-\frac{1}{2}, \frac{1}{2}\right]$

Domain of $\sin^{-1}x$ lies in the interval $[-1, 1]$.

$\therefore$ Domain of $2\cos^{-1}(2x) + \sin^{-1}x$ lies in the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

Q3

Find the domain of $f(x) = \cos^{-1}x + \cos x$.

Solution

Domain of $\cos^{-1}x$ lies in the interval $[-1, 1]$.

Domain of $\cos x$ lies in the interval $[0, \pi] = [0, 3.14]$

$\therefore$ Domain of $\cos^{-1}x + \cos x$ lies in the interval $[-1, 1]$.

Q4

Find the principal value of $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

Solution

We know that for any $x \in [-1, 1]$, $\cos^{-1} x$ represents angle in $[0, \pi]$

$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ = an angle in $[0, \pi]$ whose cosine is $\left(-\frac{\sqrt{3}}{2}\right)$

$= \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$\therefore \quad \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{5\pi}{6}$

Q5

Find the principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Solution

Let $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = y$. Then, $\cos y = -\frac{1}{\sqrt{2}} = -\cos\left(\frac{\pi}{4}\right) = \cos\left(\pi - \frac{\pi}{4}\right) = \cos\left(\frac{3\pi}{4}\right)$.

We know that the range of the principal value branch of $\cos^{-1}$ is $[0, \pi]$ and $\cos\left(\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

Therefore, the principal value of $\cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ is $\frac{3\pi}{4}$

Q6

Find the principal value of each of the following :

$\cos^{-1}\left(\sin\frac{4\pi}{3}\right)$

Solution

$\cos^{-1}\left(\sin\frac{4\pi}{3}\right)$

$= \cos^{-1}\left(\sin\left(\pi+\frac{\pi}{3}\right)\right)$

$= \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

For any $x \in [-1, 1]$, $\cos^{-1}x$ represents an angle in $[0, \pi]$ whose cosine is x.

$\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

$\therefore$ Principle value of $\cos^{-1}\left(\sin\frac{4\pi}{3}\right)$ is $\frac{5\pi}{6}$.

Q7

Find the principal value of each of the following :

$\cos^{-1}\left(\tan\frac{3\pi}{4}\right)$

Solution

$\cos^{-1}\left(\tan\frac{3\pi}{4}\right)$

$= \cos^{-1}\left(\tan\left(\frac{\pi}{2}+\frac{\pi}{4}\right)\right)$

$= \cos^{-1}(-1)$

For any $x \in [-1, 1]$, $\cos^{-1}x$ represents an angle in $[0, \pi]$ whose cosine is x.

$\cos^{-1}(-1) = \pi$

$\therefore$ Principle value of $\cos^{-1}\left(\tan\frac{3\pi}{4}\right)$ is π.

Q8

Find the value of $\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$

Solution

Let $\cos^{-1}\left(\frac{1}{2}\right) = x$. Then, $\cos x = \frac{1}{2} = \cos\left(\frac{\pi}{3}\right)$.

$\therefore \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

Let $\sin^{-1}\left(\frac{1}{2}\right) = y$. Then, $\sin y = \frac{1}{2} = \sin\left(\frac{\pi}{6}\right)$.

$\therefore \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

$\therefore \cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} + \frac{2\pi}{6} = \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$

Q9

Evaluate $\cos^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\left(-\frac{1}{2}\right)$

Solution

Let $\cos^{-1}\left(\frac{1}{2}\right) = x$. Then, $\cos x = \frac{1}{2} = \cos\left(\frac{\pi}{3}\right)$

$\therefore \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

Let $\sin^{-1}\left(-\frac{1}{2}\right) = y$. Then, $\sin y = -\frac{1}{2} = -\sin\left(\frac{\pi}{6}\right) = \sin\left(-\frac{\pi}{6}\right)$

$\therefore \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

$\therefore \cos^{-1}\left(\frac{1}{2}\right) - 2\sin^{-1}\left(-\frac{1}{2}\right) = \frac{\pi}{3} - \left(-\frac{2\pi}{6}\right) = \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$

Q10

For the principal values, evaluate $\sin^{-1}\left(-\frac{1}{2}\right) + 2\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

Solution

Let $\sin^{-1}\left(-\frac{1}{2}\right) = x$. Then, $\sin x = -\frac{1}{2} = \sin\left(-\frac{\pi}{6}\right)$

$\therefore \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

Let $\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) = y$. Then, $\cos y = \frac{-\sqrt{3}}{2} = \cos\left(\pi - \frac{\pi}{6}\right)$

$\therefore \cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) = \frac{5\pi}{6}$

$\therefore \sin^{-1}\left(-\frac{1}{2}\right) + 2\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right) = -\frac{\pi}{6} + \frac{10\pi}{6} = \frac{-\pi + 10\pi}{6} = \frac{9\pi}{6} = \frac{3\pi}{2}$

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Q11

For the principal values, evaluate $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)+\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Solution

$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)+\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$= -\frac{\pi}{3}+\frac{\pi}{6}$ $\left\{\begin{array}{l}\text{Since, } \sin^{-1}x = \text{An angle in } \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \text{ whose sine is } x \\ \text{and } \cos^{-1}x = \text{An angle in } [0, \pi] \text{ whose cosine is } x\end{array}\right\}$

$= -\frac{\pi}{6}$

Hence,

$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)+\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{6}$

Exercise 4.3

Q1

Find the principal value of $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Solution

We know that, for any $x \in R$, $\tan^{-1}x$ represents an angle in $\left(\frac{-\pi}{2},\frac{\pi}{2}\right)$ whose tangent is x.

So,

$$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \text{An angle in } \left(\frac{-\pi}{2},\frac{\pi}{2}\right) \text{ whose tangest is } \frac{1}{\sqrt{3}}$$

$$= \frac{\pi}{6}$$

$\therefore \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$.

Q2

Find the principal value of each of the following:

$\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

Solution

We know that for any $x \in R$, $\tan^{-1}x$ represents an angle in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ whose tangent is x.

$\therefore \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

$\therefore$ Principle value of $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$ is $-\frac{\pi}{6}$.

Q3

Find the principal value of each of the following:

$\tan^{-1}\left(\cos\frac{\pi}{2}\right)$

Solution

$\tan^{-1}\left(\cos\frac{\pi}{2}\right) = \tan^{-1}(0)$

We know that for any $x \in R$, $\tan^{-1}x$ represents an angle in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is x.

$\therefore \tan^{-1}(0) = 0$

$\therefore$ Principle value of $\tan^{-1}\left(\cos\frac{\pi}{2}\right)$ is 0.

Q4

Find the principal value of each of the following:

$\tan^{-1}\left(2\cos\frac{2\pi}{3}\right)$

Solution

$\tan^{-1}\left(2\cos\frac{2\pi}{3}\right) = \tan^{-1}\left(2 \times \frac{-1}{2}\right) = \tan^{-1}(-1)$

We know that for any $x \in R$, $\tan^{-1}x$ represents an angle in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is x.

$\therefore \tan^{-1}(-1) = -\frac{\pi}{4}$

$\therefore$ Principle value of $\tan^{-1}\left(2\cos\frac{2\pi}{3}\right)$ is $-\frac{\pi}{4}$

Q5

For the principal values, evaluate $\tan^{-1}(-1) + \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

Solution

Let $\tan^{-1}(-1) = x$. Then, $\tan x = -1 = -\tan\left(\frac{\pi}{4}\right) = \tan\left(\pi - \frac{\pi}{4}\right)$

$\therefore \tan^{-1}(-1) = \frac{3\pi}{4}$

Let $\cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = y$. Then, $\cos y = \frac{-1}{\sqrt{2}} = -\cos\left(\frac{\pi}{4}\right) = \cos\left(\pi - \frac{\pi}{4}\right)$

$\therefore \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \frac{3\pi}{4}$

$\therefore \tan^{-1}(-1) + \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right) = \frac{3\pi}{4} + \frac{3\pi}{4} = \frac{6\pi}{4} = \frac{3\pi}{2}$

Q6

For the principal values, evaluate each of the following :

$\tan^{-1}\left(2\sin\left(4\cos^{-1}\frac{\sqrt{3}}{2}\right)\right)$

Solution

$\tan^{-1}\left(2\sin\left(4\cos^{-1}\frac{\sqrt{3}}{2}\right)\right) = \tan^{-1}\left(2\sin\left(4\times\frac{\pi}{6}\right)\right) = \tan^{-1}\left(2\times\frac{\sqrt{3}}{2}\right) = \tan^{-1}\left(\sqrt{3}\right)$

We know that for any $x \in R$, $\tan^{-1}x$ represents an angle in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ whose tangent is x.

$\therefore \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{6}$

$\therefore$ Principle value of $\tan^{-1}\left(2\sin\left(4\cos^{-1}\frac{\sqrt{3}}{2}\right)\right)$ is $\frac{\pi}{3}$.

Q7

Find the value of $\tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)$

Solution

Let $\tan^{-1}(1) = x$. Then, $\tan x = 1 = \tan\frac{\pi}{4}$

$\therefore \tan^{-1}(1) = \frac{\pi}{4}$

Let $\cos^{-1}\left(-\frac{1}{2}\right) = y$. Then, $\cos y = -\frac{1}{2} = -\cos\left(\frac{\pi}{3}\right) = \cos\left(\pi-\frac{\pi}{3}\right) = \cos\left(\frac{2\pi}{3}\right)$.

$\therefore \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

Let $\sin^{-1}\left(-\frac{1}{2}\right) = z$. Then, $\sin z = -\frac{1}{2} = -\sin\left(\frac{\pi}{6}\right) = \sin\left(-\frac{\pi}{6}\right)$.

$\therefore \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$

$\therefore \tan^{-1}(1)+\cos^{-1}\left(-\frac{1}{2}\right)+\sin^{-1}\left(-\frac{1}{2}\right)$

$= \frac{\pi}{4}+\frac{2\pi}{3}-\frac{\pi}{6}$

$= \frac{3\pi+8\pi-2\pi}{12} = \frac{9\pi}{12} = \frac{3\pi}{4}$

Q8

Evaluate each of the following:

$$\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)+\tan^{-1}\left(-\sqrt{3}\right)+\tan^{-1}\left(\sin\left(-\frac{\pi}{2}\right)\right)$$

Solution

$$\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)+\tan^{-1}\left(-\sqrt{3}\right)+\tan^{-1}\left(\sin\left(-\frac{\pi}{2}\right)\right)$$

$$=\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)+\tan^{-1}\left(-\sqrt{3}\right)+\tan^{-1}(-1)$$

We know that for any $x\in R$, $\tan^{-1}x$ represents an angle in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ whose tangent is x.

$$\therefore \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)=-\frac{\pi}{6}$$

$$\tan^{-1}\left(-\sqrt{3}\right)=-\frac{\pi}{3}$$

$$\tan^{-1}(-1)=-\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)+\tan^{-1}\left(-\sqrt{3}\right)+\tan^{-1}(-1)=-\frac{\pi}{6}-\frac{\pi}{3}-\frac{\pi}{4}=-\frac{3\pi}{4}$$

$\therefore$ Principle value of $\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)+\tan^{-1}\left(-\sqrt{3}\right)+\tan^{-1}\left(\sin\left(-\frac{\pi}{2}\right)\right)$ is $-\frac{3\pi}{4}$.

Q9

Evaluate each of the following:

$$\tan^{-1}\left(\tan\frac{5\pi}{6}\right)+\cos^{-1}\left\{\cos\left(\frac{13\pi}{6}\right)\right\}$$

Solution

$\tan^{-1}\left(\tan\frac{5\pi}{6}\right)+\cos^{-1}\left\{\cos\left(\frac{13\pi}{6}\right)\right\}$

$=\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)+\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$

We know that for any $x\in R$, $\tan^{-1}x$ represents an angle in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ whose tangent is x.

$\therefore \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)=-\frac{\pi}{6}$

We know that for any $x\in[-1, 1]$, $\cos^{-1}x$ represents an angle in $[0,\pi]$ whose cosine is x.

$\therefore \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{6}$

$\therefore \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)+\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)=-\frac{\pi}{6}+\frac{\pi}{6}=0$

$\therefore$ Principle value of $\tan^{-1}\left(\tan\frac{5\pi}{6}\right)+\cos^{-1}\left\{\cos\left(\frac{13\pi}{6}\right)\right\}$ is 0.

Exercise 4.4

Q1

Find the principal value of $Sec^{-1}\left(-\sqrt{2}\right)$

Solution

We know that, for $x \in R$, $sec^{-1}x$ represents an angle in $[0, \pi] - \left\{\frac{\pi}{2}\right\}$.

$\sec^{-1}\left(-\sqrt{2}\right)$ = An angle in $[0, \pi] - \left\{\frac{\pi}{2}\right\}$ whose secant is $\left(-\sqrt{2}\right)$

$$= \pi - \frac{\pi}{4}$$

$$= \frac{3\pi}{4}$$

$\sec^{-1}\left(-\sqrt{2}\right) = \frac{3\pi}{4}$.

Q2

Find the principal value of $\sec^{-1}(2)$

Solution

We know that, for any $x \in R$, $\sec^{-1}x$ represents an angle in $[0, \pi] - \left\{\frac{\pi}{2}\right\}$.

$\sec^{-1}(2)$ = An angle is $[0, \pi] - \left\{\frac{\pi}{2}\right\}$ whose secant is 2

$$= \frac{\pi}{3}$$

$\therefore \sec^{-1}(2) = \frac{\pi}{3}$.

Q3

Find the principal value of each of the following:

$$\sec^{-1}\left(2\sin\frac{3\pi}{4}\right)$$

Solution

$\sec^{-1}\left(2\sin\frac{3\pi}{4}\right) = \sec^{-1}\left(2\times\left(\frac{1}{\sqrt{2}}\right)\right) = \sec^{-1}\left(\sqrt{2}\right)$

We know that for any $x \in R-(-1, 1)$, $\sec^{-1}x$ represents an angle in $[0,\pi]-\left\{\frac{\pi}{2}\right\}$ whose secant is x.

$\therefore \sec^{-1}\left(\sqrt{2}\right) = \frac{\pi}{4}$

$\therefore$ Principle value of $\sec^{-1}\left(2\sin\frac{3\pi}{4}\right)$ is $\frac{\pi}{4}$.

Q4

Find the principal value of each of the following:

$\sec^{-1}\left(2\tan\frac{3\pi}{4}\right)$

Solution

$\sec^{-1}\left(2\tan\frac{3\pi}{4}\right) = \sec^{-1}\left(2\times(-1)\right) = \sec^{-1}(-2)$

We know that for any $x \in R-(-1, 1)$, $\sec^{-1}x$ represents an angle in $[0,\pi]-\left\{\frac{\pi}{2}\right\}$ whose secant is x.

$\therefore \sec^{-1}(-2) = \frac{2\pi}{3}$

$\therefore$ Principle value of $\sec^{-1}\left(2\tan\frac{3\pi}{4}\right)$ is $\frac{2\pi}{3}$.

Q5

For the principal values, evaluate $\tan^{-1}\left(\sqrt{3}\right) - \sec^{-1}(-2)$

Solution

Let $\tan^{-1}\left(\sqrt{3}\right) = x$. Then, $\tan x = \sqrt{3} = \tan\left(\frac{\pi}{3}\right)$

$\therefore \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$

Let $\sec^{-1}(-2) = y$. Then, $\sec y = -2 = \sec\left(\pi - \frac{\pi}{3}\right)$

$\therefore \sec^{-1}(-2) = \frac{2\pi}{3}$

$\therefore \tan^{-1}\left(\sqrt{3}\right) - \sec^{-1}(-2) = \frac{\pi}{3} - \frac{2\pi}{3} = \frac{\pi - 2\pi}{3} = -\frac{\pi}{3}$

Q6

Find the principal value of each of the following:

$$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)-2\sec^{-1}\left(2\tan\frac{\pi}{6}\right)$$

Solution

$$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)-2\sec^{-1}\left(2\tan\frac{\pi}{6}\right)=\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)-2\sec^{-1}\left(2\times\frac{1}{\sqrt{3}}\right)=\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)-2\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$$

We know that for any $x\in[-1,1]$, $\sin^{-1}x$ represents an angle in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ whose secant is x.

$$\therefore \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)=-\frac{\pi}{3}$$

We know that for any $x\in R-(-1,\ 1)$, $\sec^{-1}x$ represents an angle in $[0,\pi]-\left\{\frac{\pi}{2}\right\}$ whose secant is x.

$$\therefore \sec^{-1}\left(\frac{2}{\sqrt{3}}\right)=\frac{\pi}{6}$$

$$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)-2\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)=-\frac{\pi}{3}-2\times\frac{\pi}{6}=-\frac{2\pi}{3}$$

$\therefore$ Principle value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)-2\sec^{-1}\left(2\tan\frac{\pi}{6}\right)$ is $-\frac{2\pi}{3}$

Q7

Find the domain of

$\sec^{-1}(3x-1)$

Solution

Domain of $\sec^{-1}x$ lies in the interval $(-\infty,-1]\cup[1,\infty)$.

$\therefore$ Domain of $\sec^{-1}(3x-1)$ lies in the interval $(-\infty,-1]\cup[1,\infty)$

$\Rightarrow -\infty\le 3x-1\le -1$ and $1\le 3x-1\le\infty$

$\Rightarrow -\infty\le 3x\le 0$ and $2\le 3x\le\infty$

$\Rightarrow -\infty\le x\le 0$ and $\frac{2}{3}\le x\le\infty$

Domain of $\sec^{-1}x$ lies in the interval $(-\infty,0]\cup\left[\frac{2}{3},\infty\right)$.

Q8

Find the domain of

$\sec^{-1}x-\tan^{-1}x$

Solution

Domain of $\sec^{-1}x$ lies in the interval $(-\infty, -1] \cup [1, \infty)$.

Domain of $\tan^{-1}x$ lies is R

Domain of $\sec^{-1}x - \tan^{-1}x\left(x^2 - 4\right)$ is $(-\infty, -1] \cup [1, \infty)$.

Exercise 4.5

Q1

Find the principal value of $\operatorname{cosec}^{-1}(-\sqrt{2})$

Solution

Let $\operatorname{cosec}^{-1}(-\sqrt{2}) = y$. Then, $\operatorname{cosec} y = -\sqrt{2} = -\operatorname{cosec}\left(\frac{\pi}{4}\right) = \operatorname{cosec}\left(-\frac{\pi}{4}\right)$.

We know that the range of the principal value branch of $\operatorname{cosec}^{-1}$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ and $\operatorname{cosec}\left(-\frac{\pi}{4}\right) = -\sqrt{2}$.

Therefore, the principal value of $\operatorname{cosec}^{-1}(-\sqrt{2})$ is $-\frac{\pi}{4}$.

Q2

Find the principal value of each of the following:

$\operatorname{cosec}^{-1}(-2)$

Solution

$\operatorname{cosec}^{-1}x$ represents an angle in $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ whose cosecant is x.

Let $x = \operatorname{cosec}^{-1}(-2)$

$\Rightarrow \operatorname{cosec} x = -2 = \operatorname{cosec}\left(-\frac{\pi}{6}\right)$

$\Rightarrow x = -\frac{\pi}{6}$

$\therefore$ Principal value of $\operatorname{cosec}^{-1}(-2)$ is $-\frac{\pi}{6}$

Q3

Find the principal value of $\operatorname{cosec}^{-1}\left(\frac{2}{\sqrt{3}}\right)$

Solution

We know that, for any $x \in R$, $\operatorname{cosec}^{-1}x$ is an angle in $\left[\frac{-\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

$\operatorname{cosec}^{-1}\left(\frac{2}{\sqrt{3}}\right)$ = An angle is $\left[\frac{-\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ whose cosecant is $\left(\frac{2}{\sqrt{3}}\right)$

$= \frac{\pi}{3}$

$\therefore \operatorname{cosec}^{-1}\left(\frac{2}{\sqrt{3}}\right) = \frac{\pi}{3}$.

Q4

Find the principal value of each of the following:

$\operatorname{cosec}^{-1}\left(2\cos\frac{2\pi}{3}\right)$

Solution

$\operatorname{cosec}^{-1}x$ represents an angle in $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ whose cosecant is x.

Let $x = \operatorname{cosec}^{-1}\left(2\cos\frac{2\pi}{3}\right)$

$\Rightarrow \operatorname{cosec} x = 2\cos\frac{2\pi}{3} = 2\times\left(-\frac{1}{2}\right) = -1 = \operatorname{cosec}\left(-\frac{\pi}{2}\right)$

$\Rightarrow x = -\frac{\pi}{2}$

$\therefore$ Principal value of $\operatorname{cosec}^{-1}\left(2\cos\frac{2\pi}{3}\right)$ is $-\frac{\pi}{2}$.

Q5

Find the set of values of

$\operatorname{cosec}^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Solution

$\operatorname{cosec}^{-1}x$ represents an angle in $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ whose cosecant is x.

Domain of $\operatorname{cosec}^{-1}x$ is $(-\infty, -1] \cup [1, \infty)$.

$\frac{\sqrt{3}}{2} \notin (-\infty, -1] \cup [1, \infty)$

Hence $\operatorname{cosec}^{-1}\left(\frac{\sqrt{3}}{2}\right)$ does not exist or its ϕ.

Q6

For the principal value evaluate the following:

$$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)+\operatorname{cosec}^{-1}\left(-\frac{2}{\sqrt{3}}\right)$$

Solution

$\sin^{-1}x$ represents an angle in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ whose sine is x.

Let $x = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

$\Rightarrow \sin x = -\frac{\sqrt{3}}{2} = \sin\left(-\frac{\pi}{3}\right)$

$\operatorname{cosec}^{-1}x$ represents an angle in $\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$ whose cosecant is x.

Let $x = \operatorname{cosec}^{-1}\left(-\frac{2}{\sqrt{3}}\right)$

$\Rightarrow \operatorname{cosec}x = -\frac{2}{\sqrt{3}} = \operatorname{cosec}\left(-\frac{\pi}{3}\right)$

$\therefore \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)+\operatorname{cosec}^{-1}\left(-\frac{2}{\sqrt{3}}\right)$

$= -\frac{\pi}{3} - -\frac{\pi}{3}$

$= -\frac{2\pi}{3}$

$\therefore$ Principal value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)+\operatorname{cosec}^{-1}\left(-\frac{2}{\sqrt{3}}\right)$ is $-\frac{2\pi}{3}$.

Q7

For the principal value evaluate the following:

$$\sec^{-1}\left(\sqrt{2}\right)+2\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)$$

Solution

$\sec^{-1}x$ represents an angle in $[0, \pi]-\left\{\frac{\pi}{2}\right\}$ whose secant is x.

Let $x = \sec^{-1}\left(\sqrt{2}\right)$

$\Rightarrow \sec x = \sqrt{2} = \sec\left(\frac{\pi}{4}\right)$

$\operatorname{cosec}^{-1}x$ represents an angle in $\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$ whose cosecant is x.

Let $x = \operatorname{cosec}^{-1}\left(-\sqrt{2}\right)$

$\Rightarrow \operatorname{cosec} x = -\sqrt{2} = \operatorname{cosec}\left(-\frac{\pi}{4}\right)$

$\therefore \sec^{-1}\left(\sqrt{2}\right) + 2\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)$

$= \frac{\pi}{4} + 2 \times \left(-\frac{\pi}{4}\right)$

$= -\frac{\pi}{4}$

$\therefore$ Principal value of $\sec^{-1}\left(\sqrt{2}\right) + 2\operatorname{cosec}^{-1}\left(-\sqrt{2}\right)$ is $-\frac{\pi}{4}$.

Q8

For the principal value evaluate the following:

$\sin^{-1}\left[\cos\left\{2\operatorname{cosec}^{-1}(-2)\right\}\right]$

Solution

$cosec^{-1}x$ represents an angle in $\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$ whose cosecant is x.

Let $x = cosec^{-1}(-2)$

$\Rightarrow cosecx = -2 = cosec\left(-\frac{\pi}{6}\right)$

$\Rightarrow x = -\frac{\pi}{6}$

$\sin^{-1}\left[\cos\{2cosec^{-1}(-2)\}\right] = \sin^{-1}\left[\cos\left\{2\times\left(-\frac{\pi}{6}\right)\right\}\right] = \sin^{-1}\left[\cos\left(-\frac{\pi}{3}\right)\right] = \sin^{-1}\left[\frac{1}{2}\right]$

$\sin^{-1}x$ represents an angle in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ whose sine is x.

Let $x = \sin^{-1}\left[\frac{1}{2}\right]$

$\Rightarrow \sin x = \frac{1}{2} = \sin\left(\frac{\pi}{6}\right)$

$\Rightarrow x = \frac{\pi}{6}$

$\therefore$ Principal value of $\sin^{-1}\left[\cos\{2cosec^{-1}(-2)\}\right]$ is $\frac{\pi}{6}$.

Q9

For the principal value evaluate the following:

$cosec^{-1}\left(2\tan\frac{11\pi}{6}\right)$

Solution

$cosec^{-1}\left(2\tan\frac{11\pi}{6}\right) = cosec^{-1}\left(2\tan\left(\frac{11\pi}{6}\right)\right) = cosec^{-1}\left(2\tan\left(\frac{3\pi}{2}+\frac{\pi}{3}\right)\right) = cosec^{-1}\left(2\times\left(-\frac{1}{\sqrt{3}}\right)\right)$

$cosec^{-1}x$ represents an angle in $\left[-\frac{\pi}{2},0\right)\cup\left(0,\frac{\pi}{2}\right]$ whose cosecant is x.

Let $x = cosec^{-1}\left(-\frac{2}{\sqrt{3}}\right)$

$\Rightarrow cosecx = -\frac{2}{\sqrt{3}} = cosec\left(-\frac{\pi}{3}\right)$

$\Rightarrow x = -\frac{\pi}{3}$

$\therefore$ Principal value of $cosec^{-1}\left(2\tan\frac{11\pi}{6}\right)$ is $-\frac{\pi}{3}$.

Exercise 4.6

Q1

Find the principal value of $cot^{-1}\left(-\sqrt{3}\right)$

Solution

We know that, for any $x \in R$, $cot^{-1}x$ represents an angle in $(0, \pi)$

$cot^{-1}\left(-\sqrt{3}\right)$ = An angle in $(0, \pi)$ whose contangent is $\left(-\sqrt{3}\right)$

$= \pi - \frac{\pi}{6}$

$= \frac{5\pi}{6}$

$\therefore cot^{-1}\left(-\sqrt{3}\right) = \frac{5\pi}{6}$.

Q2

Find the principal value of each of the following:

$\cot^{-1}\left(\sqrt{3}\right)$

Solution

$\cot^{-1}x$ represents an angle in $(0, \pi)$ whose cotangent is x.

Let $x = \cot^{-1}(\sqrt{3})$

$\Rightarrow \cot\ x = \sqrt{3} = \cot\left(\frac{\pi}{6}\right)$

$\Rightarrow x = \frac{\pi}{6}$

$\therefore$ Principal value of $\cot^{-1}(\sqrt{3})$ is $\frac{\pi}{6}$.

Q3

Find the principal value of each of the following:

$\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

Solution

$\cot^{-1}x$ represents an angle in $(0, \pi)$ whose cotangent is x.

Let $x = \cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

$\Rightarrow \cot x = -\frac{1}{\sqrt{3}} = \cot\left(\frac{2\pi}{3}\right)$

$\Rightarrow x = \frac{2\pi}{3}$

$\therefore$ Principal value of $\cot^{-1}(\sqrt{3})$ is $\frac{2\pi}{3}$.

Q4

Find the principal value of each of the following:

$\cot^{-1}\left(\tan\frac{3\pi}{4}\right)$

Solution

$\cot^{-1}\left(\tan\frac{3\pi}{4}\right) = \cot^{-1}(-1)$

$\cot^{-1}x$ represents an angle in $(0, \pi)$ whose cotangent is x.

Let $x = \cot^{-1}(-1)$

$\Rightarrow \cot x = -1 = \cot\left(\frac{3\pi}{4}\right)$

$\Rightarrow x = \frac{3\pi}{4}$

$\therefore$ Principal value of $\cot^{-1}(-1)$ is $\frac{3\pi}{4}$.

Q5

Find the domain of $f(x) = \cot x + \cot^{-1} x$.

Solution

Domain of $\cot x$ is $(0, \pi)$.

Domain of $\cot^{-1}x$ is R.

So domain of $\cot x + \cot^{-1}x$ is R.

Q6

Evaluate each of the following:

$\cot^{-1}\frac{1}{\sqrt{3}} - \operatorname{cosec}^{-1}(-2) + \sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

Solution

$$\cot^{-1}\frac{1}{\sqrt{3}}-\operatorname{cosec}^{-1}(-2)+\sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$$
$$=\frac{\pi}{6}-\left(-\frac{\pi}{6}\right)+\frac{\pi}{3}$$
$$=\frac{\pi}{6}+\frac{\pi}{6}+\frac{\pi}{3}$$
$$=\frac{4\pi}{6}$$
$$=\frac{2\pi}{3}$$

Q7

Evaluate each of the following

$$\cot^{-1}\left\{2\cos\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)\right\}$$

Solution

$$\cot^{-1}\left\{2\cos\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)\right\}$$
$$=\cot^{-1}\left\{2\cos\left(\frac{\pi}{3}\right)\right\}$$
$$=\cot^{-1}\left\{2\times\frac{1}{2}\right\}$$
$$=\cot^{-1}(1)$$
$$=\frac{\pi}{4}$$

Q8

Evaluate each of the following:

$$\operatorname{cosec}^{-1}\left(-\frac{2}{\sqrt{3}}\right)+2\cot^{-1}(-1)$$

Solution

$$\operatorname{cosec}^{-1}\left(-\frac{2}{\sqrt{3}}\right)+2\cot^{-1}(-1)$$
$$=-\frac{\pi}{3}+2\times\left(\frac{3\pi}{4}\right)$$
$$=-\frac{\pi}{3}+\frac{3\pi}{2}$$
$$=\frac{7\pi}{6}$$

Q9

Evaluate each of the following

$$\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)+\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)+\tan^{-1}\left(\sin\left[-\frac{\pi}{2}\right]\right)$$

Solution

$$\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)+\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)+\tan^{-1}\left(\sin\left[-\frac{\pi}{2}\right]\right)$$
$$=-\frac{\pi}{6}+\frac{\pi}{3}+\tan^{-1}(-1)$$
$$=-\frac{\pi}{6}+\frac{\pi}{3}-\frac{\pi}{4}$$
$$=-\frac{\pi}{12}$$

Exercise 4.7

Q1

Evaluate $\sin^{-1}\left(\sin\frac{5\pi}{6}\right)$

Solution

We know that,

$$\sin^{-1}(\sin\theta) = \begin{cases} -\pi-\theta, & \text{if } \theta \in \left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right] \\ \theta, & \text{if } \theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \\ \pi-\theta, & \text{if } \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \\ -2\pi+\theta, & \text{if } \theta \in \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right] \end{cases}$$

$\therefore \sin^{-1}\left(\sin\frac{5\pi}{6}\right)$

$= \pi - \frac{5\pi}{6}$ $\quad\left[\text{since } \frac{5\pi}{6} \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]\right]$

$= \frac{\pi}{6}$

Hence,

$$\sin^{-1}\left(\sin\frac{5\pi}{6}\right) = \frac{\pi}{6}.$$

Q2

Evaluate the following :

$\sin^{-1}\left(\sin\frac{7\pi}{6}\right)$

Solution

$\sin^{-1}\left(\sin\frac{7\pi}{6}\right)$

$= \sin^{-1}\left(\sin\left(\pi+\frac{\pi}{6}\right)\right)$

$= \sin^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right)$

$= -\frac{\pi}{6}$

Q3

Evaluate the following :

$\sin^{-1}\left(\sin\frac{5\pi}{6}\right)$

Solution

$\sin^{-1}\left(\sin\frac{5\pi}{6}\right)$

$= \sin^{-1}\left(\sin\left(\pi - \frac{\pi}{6}\right)\right)$

$= \sin^{-1}\left(\sin\left(\frac{\pi}{6}\right)\right)$

$= \frac{\pi}{6}$

Q4

Evaluate the following :

$\sin^{-1}\left(\sin\frac{13\pi}{7}\right)$

Solution

$\sin^{-1}\left(\sin\frac{13\pi}{7}\right)$

$= \sin^{-1}\left(\sin\left(2\pi - \frac{\pi}{7}\right)\right)$

$= \sin^{-1}\left(\sin\left(-\frac{\pi}{7}\right)\right)$

$= -\frac{\pi}{7}$

Q5

Evaluate the following :

$\sin^{-1}\left(\sin\frac{17\pi}{8}\right)$

Solution

$\sin^{-1}\left(\sin\frac{17\pi}{8}\right)$

$=\sin^{-1}\left(\sin\left(2\pi+\frac{\pi}{8}\right)\right)$

$=\sin^{-1}\left(\sin\left(\frac{\pi}{8}\right)\right)$

$=\frac{\pi}{8}$

Q6

Evaluate the following :

$\sin^{-1}\left\{\left(\sin-\frac{17\pi}{8}\right)\right\}$

Solution

$\sin^{-1}\left(\sin-\frac{17\pi}{8}\right)$

$=\sin^{-1}\left(\sin\left(-2\pi-\frac{\pi}{8}\right)\right)$

$=\sin^{-1}\left(-\sin\left(\frac{\pi}{8}\right)\right)$

$=-\frac{\pi}{8}$

Q7

Evaluate the following :

$\sin^{-1}(\sin 3)$

Solution

$\sin^{-1}(\sin 3)$

$=\pi-3 \quad \ldots\ldots\ldots\left(\because \sin^{-1}(\sin\theta)=\pi-\theta,\ \text{if } \theta\in\left[\frac{\pi}{2},\frac{3\pi}{2}\right]\right)$

Q8

Evaluate the following :

$\sin^{-1}(\sin 4)$

Solution

$\sin^{-1}(\sin 4)$

$= \pi - 4 \quad \ldots\ldots\ldots \left(\because \sin^{-1}(\sin\theta) = \pi - \theta,\ \text{if } \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]\right)$

Q9

Evaluate the following :

$\sin^{-1}(\sin 12)$

Solution

$\sin^{-1}(\sin 12)$

$= -4\pi + 12 \quad \ldots\ldots\ldots \left(\because \sin^{-1}(\sin\theta) = -4\pi + \theta,\ \text{if } \theta \in \left[\frac{7\pi}{2}, \frac{9\pi}{2}\right]\right)$

Q10

Evaluate $\sin^{-1}(\sin 2)$

Solution

We know that,

$$\sin^{-1}(\sin\theta) = \begin{cases} -\pi - \theta, & \text{if } \theta \in \left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right] \\ \theta, & \text{if } \theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \\ \pi - \theta, & \text{if } \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \\ -2\pi + \theta, & \text{if } \theta \in \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right] \end{cases}$$

$\therefore \sin^{-1}(\sin 2)$

$= \pi - 2 \qquad \left[\text{since } 2 \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]\right]$

Hence,

$\sin^{-1}(\sin 2) = \pi - 2.$

Q11

Evaluate $\cos^{-1}\left\{\cos\left(\frac{-\pi}{4}\right)\right\}$

Solution

We know that,

$$\cos^{-1}(\cos\theta) = \begin{cases} -\theta & , \text{ if } \theta \in [-\pi, 0] \\ \theta & , \text{ if } \theta \in [0, \pi] \\ 2\pi - \theta & , \text{ if } \theta \in [\pi, 2\pi] \\ -2\pi + \theta & , \text{ if } \theta \in [2\pi, 3\pi] \end{cases}$$

$$\therefore \cos^{-1}\left\{ocs\left(-\frac{\pi}{4}\right)\right\}$$

$$= -\left(-\frac{\pi}{4}\right) \qquad \left\{\text{since } -\frac{\pi}{4} \in [-\pi, 0]\right\}$$

$$= \frac{\pi}{4}$$

Hence,

$$\cos^{-1}\left\{\cos\left(-\frac{\pi}{4}\right)\right\} = \frac{\pi}{4}.$$

Q12

Evaluate $\cos^{-1}\left\{\cos\left(\frac{4\pi}{3}\right)\right\}$

Solution

We know that,

$$\cos^{-1}(\cos\theta) = \begin{cases} -\theta & , \text{ if } \theta \in [-\pi, 0] \\ \theta & , \text{ if } \theta \in [0, \pi] \\ 2\pi - \theta & , \text{ if } \theta \in [\pi, 2\pi] \\ -2\pi - \theta & , \text{ if } \theta \in [2\pi, 3\pi] \end{cases}$$

$$\therefore \cos^{-1}\left\{\cos\left(\frac{4\pi}{3}\right)\right\}$$

$$= 2\pi - \frac{4\pi}{3} \qquad \left\{\text{since } \frac{4\pi}{3} \in [\pi, 2\pi]\right\}$$

$$= \frac{2\pi}{3}$$

Hence,

$$\cos^{-1}\left\{\cos\left(\frac{4\pi}{3}\right)\right\} = \frac{2\pi}{3}$$

Q13

Evaluate the following :

$\cos^{-1}(\cos 3)$

Solution

$\cos^{-1}(\cos 3)$

$= 3 \quad \ldots\ldots\ldots\left(\because \cos^{-1}(\cos\theta) = \theta,\ \text{if } \theta \in [0, \pi]\right)$

Q14

Evaluate the following :

$\cos^{-1}(\cos 4)$

Solution

$\cos^{-1}(\cos 4)$

$= 2\pi - 4 \quad \ldots\ldots\ldots\left(\because \cos^{-1}(\cos\theta) = 2\pi - \theta,\ \text{if } \theta \in [\pi, 2\pi]\right)$

Q15

Evaluate the following :

$\cos^{-1}(\cos 5)$

Solution

$\cos^{-1}(\cos 5)$

$= 2\pi - 5 \quad \ldots\ldots\ldots\left(\because \cos^{-1}(\cos\theta) = 2\pi - \theta,\ \text{if } \theta \in [\pi, 2\pi]\right)$

Q16

Evaluate the following :

$\cos^{-1}(\cos 12)$

Solution

$\cos^{-1}(\cos 12)$

$= 4\pi - 12 \quad \ldots\ldots\ldots\left(\because \cos^{-1}(\cos\theta) = 4\pi - \theta,\ \text{if } \theta \in [3\pi, 4\pi]\right)$

Q17

Evaluate the following :

$\tan^{-1}\left(\tan\frac{\pi}{3}\right)$

Solution

$\tan^{-1}\left(\tan\frac{\pi}{3}\right)$

$= \frac{\pi}{3} \quad \ldots\ldots\ldots \left(\because \tan^{-1}(\tan\theta) = \theta, \text{ if } \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

Q18

Evaluate the following :

$\tan^{-1}\left(\tan\frac{6\pi}{7}\right)$

Solution

$\tan^{-1}\left(\tan\frac{6\pi}{7}\right)$

$= \frac{6\pi}{7} - \pi \quad \ldots\ldots\ldots \left(\because \tan^{-1}(\tan\theta) = \theta - \pi, \text{ if } \theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)\right)$

$= -\frac{\pi}{7}$

Q19

Evaluate $\tan^{-1}\left\{\tan\frac{7\pi}{6}\right\}$

Solution

$\tan^{-1}\left\{\tan\frac{7\pi}{6}\right\}$

We know that,

$$\tan^{-1}(\tan\theta) = \begin{cases} \pi-\theta & ;\ \text{if } \theta \in \left[\frac{-3\pi}{2}, \frac{-\pi}{2}\right] \\ \theta & ;\ \text{if } \theta \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \\ \theta-\pi & ;\ \text{if } \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right] \\ \theta-2\pi & ;\ \text{if } \theta \in \left[\frac{3\pi}{2}, \frac{5\pi}{2}\right] \end{cases}$$

$\therefore \tan^{-1}\left\{\tan\frac{7\pi}{6}\right\}$

$= \frac{7\pi}{6} - \pi \qquad \left\{\text{since } \frac{7\pi}{6} \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]\right\}$

$= \frac{\pi}{6}$

Hence,

$\tan^{-1}\left(\tan\frac{7\pi}{6}\right) = \frac{\pi}{6}$

Q20

Evaluate the following

$\tan^{-1}\left(\tan\frac{9\pi}{4}\right)$

Solution

$\tan^{-1}\left(\tan\frac{9\pi}{4}\right)$

$= \frac{9\pi}{4} - 2\pi \quad \ldots\ldots\ldots \left(\because \tan^{-1}(\tan\theta) = \theta - 2\pi,\ \text{if } \theta \in \left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)\right)$

$= \frac{\pi}{4}$

Q21

Evaluate the following :

$\tan^{-1}$ (tan 1)

Solution

$\tan^{-1}(\tan 1)$

$= 1 \quad \ldots\ldots\ldots \left(\because \tan^{-1}(\tan\theta) = \theta,\ \text{if } \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right)$

Q22

Evaluate the following :

$\tan^{-1}(\tan 2)$

Solution

$\tan^{-1}(\tan 2)$

$= 2 - \pi \quad \ldots\ldots\ldots \left(\because \tan^{-1}(\tan\theta) = \theta - \pi,\ \text{if } \theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)\right)$

Q23

Evaluate the following :

$\tan^{-1}(\tan 4)$

Solution

$\tan^{-1}(\tan 4)$

$= 4 - \pi \quad \ldots\ldots\ldots \left(\because \tan^{-1}(\tan\theta) = \theta - \pi,\ \text{if } \theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)\right)$

Q24

Evaluate the following :

$\tan^{-1}(\tan 12)$

Solution

$\tan^{-1}(\tan 12)$

$= 12 - 4\pi \quad \ldots\ldots\ldots \left(\because \tan^{-1}(\tan\theta) = \theta - 4\pi,\ \text{if } \theta \in \left(\frac{7\pi}{2}, \frac{9\pi}{2}\right)\right)$

Q25

Evaluate the following :

$\sec^{-1}\left(\sec\frac{\pi}{3}\right)$

Solution

$\sec^{-1}\left(\sec\frac{\pi}{3}\right)$

$= \sec^{-1}\left(\frac{2}{\sqrt{3}}\right)$

$= \frac{\pi}{3}$

Q26

Evaluate the following :

$\sec^{-1}\left(\sec\frac{2\pi}{3}\right)$

Solution

$\sec^{-1}\left(\sec\frac{2\pi}{3}\right)$

$= \sec^{-1}\left(\sec\left(\frac{\pi}{2}+\frac{\pi}{6}\right)\right)$

$= \sec^{-1}\left(-\operatorname{cosec}\left(\frac{\pi}{6}\right)\right)$

$= \sec^{-1}(-2)$

$= \frac{2\pi}{3}$

Q27

Evaluate the following :

$\sec^{-1}\left(\sec\frac{5\pi}{4}\right)$

Solution

$\sec^{-1}\left(\sec\frac{5\pi}{4}\right)$

$= \sec^{-1}\left(\sec\left(\pi+\frac{\pi}{4}\right)\right)$

$= \sec^{-1}\left(-\sec\left(\frac{\pi}{4}\right)\right)$

$= \sec^{-1}(-\sqrt{2})$

$= \frac{3\pi}{4}$

Q28

Evaluate the following :

$\sec^{-1}\left(\sec\frac{7\pi}{3}\right)$

Solution

$\sec^{-1}\left(\sec\frac{7\pi}{3}\right)$

$=\sec^{-1}\left(\sec\left(2\pi+\frac{\pi}{3}\right)\right)$

$=\sec^{-1}\left(\sec\left(\frac{\pi}{3}\right)\right)$

$=\sec^{-1}(2)$

$=\frac{\pi}{3}$

Q29

Evaluate the following :

$\sec^{-1}\left(\sec\frac{9\pi}{5}\right)$

Solution

$\sec^{-1}\left(\sec\frac{9\pi}{5}\right)$

$=\sec^{-1}\left(\sec\left(2\pi-\frac{\pi}{5}\right)\right)$

$=\sec^{-1}\left(\sec\left(\frac{\pi}{5}\right)\right)$

$=\frac{\pi}{5}$

Q30

Evaluate the following :

$\sec^{-1}\left\{\sec\left(-\frac{7\pi}{3}\right)\right\}$

Solution

$\sec^{-1}\left\{\sec\left(-\frac{7\pi}{3}\right)\right\}$

$= \sec^{-1}\left\{\sec\left(-2\pi - \frac{\pi}{3}\right)\right\}$

$= \sec^{-1}\left(\sec\left(\frac{\pi}{3}\right)\right)$

$= \frac{\pi}{3}$

Q31

Evaluate the following :

$\sec^{-1}\left(\sec\frac{13\pi}{4}\right)$

Solution

$\sec^{-1}\left(\sec\frac{13\pi}{4}\right)$

$= \sec^{-1}\left\{\sec\left(3\pi + \frac{\pi}{4}\right)\right\}$

$= \sec^{-1}\left(-\sec\left(\frac{\pi}{4}\right)\right)$

$= \sec^{-1}(-1)$

$= \frac{3\pi}{4}$

Q32

Evaluate the following :

$\sec^{-1}\left(\sec\frac{25\pi}{6}\right)$

Solution

$\sec^{-1}\left(\sec\frac{25\pi}{6}\right)$

$= \sec^{-1}\left\{\sec\left(4\pi + \frac{\pi}{6}\right)\right\}$

$= \sec^{-1}\left(\sec\left(\frac{\pi}{6}\right)\right)$

$= \frac{\pi}{6}$

Q33

Evaluate the following :

$$\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{\pi}{4}\right)$$

Solution

$$\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{\pi}{4}\right)$$
$$= \operatorname{cosec}^{-1}\left(\sqrt{2}\right)$$
$$= \frac{\pi}{4}$$

Q34

Evaluate the following :

$$\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{3\pi}{4}\right)$$

Solution

$$\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{3\pi}{4}\right)$$
$$= \operatorname{cosec}^{-1}\left(-\sqrt{2}\right)$$
$$= \operatorname{cosec}^{-1}\left(\sqrt{2}\right)$$
$$= \frac{\pi}{4}$$

Q35

Evaluate the following :

$$\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{6\pi}{5}\right)$$

Solution

$$\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{6\pi}{5}\right)$$
$$= \operatorname{cosec}^{-1}\left(\operatorname{cosec}\left(\pi + \frac{\pi}{5}\right)\right)$$
$$= \operatorname{cosec}^{-1}\left(-\operatorname{cosec}\left(\frac{\pi}{5}\right)\right)$$
$$= -\frac{\pi}{5}$$

Q36

Evaluate the following :

$$\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{11\pi}{6}\right)$$

Solution

$$\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{11\pi}{6}\right)$$

$$= \operatorname{cosec}^{-1}\left(\operatorname{cosec}\left(2\pi - \frac{\pi}{6}\right)\right)$$

$$= \operatorname{cosec}^{-1}\left(\operatorname{cosec}\left(-\frac{\pi}{6}\right)\right)$$

$$= \operatorname{cosec}^{-1}\left(\operatorname{cosec}\left(\frac{\pi}{6}\right)\right)$$

$$= \frac{\pi}{6}$$

Q37

Evaluate the following :

$$\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{13\pi}{6}\right)$$

Solution

$$\operatorname{cosec}^{-1}\left(\operatorname{cosec}\frac{13\pi}{6}\right)$$

$$= \operatorname{cosec}^{-1}\left(\operatorname{cosec}\left(2\pi + \frac{\pi}{6}\right)\right)$$

$$= \operatorname{cosec}^{-1}\left(\operatorname{cosec}\left(\frac{\pi}{6}\right)\right)$$

$$= \frac{\pi}{6}$$

Q38

Evaluate the following :

$$\operatorname{cosec}^{-1}\left\{\operatorname{cosec}\left(-\frac{9\pi}{4}\right)\right\}$$

Solution

$$\text{cosec}^{-1}\left\{\text{cosec}\left(-\frac{9\pi}{4}\right)\right\}$$

$$= \text{cosec}^{-1}\left(\text{cosec}\left(-2\pi-\frac{\pi}{4}\right)\right)$$

$$= \text{cosec}^{-1}\left(-\text{cosec}\left(\frac{\pi}{4}\right)\right)$$

$$= -\frac{\pi}{4}$$

Q39

Evaluate the following :

$$\cot^{-1}\left(\cot\frac{\pi}{3}\right)$$

Solution

$$\cot^{-1}\left(\cot\frac{\pi}{3}\right)$$

$$= \cot^{-1}\left(\sqrt{3}\right)$$

$$= \frac{\pi}{3}$$

Q40

Evaluate the following :

$$\cot^{-1}\left(\cot\frac{4\pi}{3}\right)$$

Solution

$$\cot^{-1}\left(\cot\frac{4\pi}{3}\right)$$

$$= \cot^{-1}\left(\cot\left(\pi+\frac{\pi}{3}\right)\right)$$

$$= \cot^{-1}\left(\cot\left(\frac{\pi}{3}\right)\right)$$

$$= \cot^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{\pi}{3}$$

Q41

Evaluate the following :

$$\cot^{-1}\left(\cot\frac{9\pi}{4}\right)$$

Solution

$$\cot^{-1}\left(\cot\frac{9\pi}{4}\right)$$

$$=\cot^{-1}\left(\cot\left(2\pi+\frac{\pi}{4}\right)\right)$$

$$=\cot^{-1}\left(\cot\left(\frac{\pi}{4}\right)\right)$$

$$=\frac{\pi}{4}$$

Q42

Evaluate the following :

$$\cot^{-1}\left(\cot\frac{19\pi}{6}\right)$$

Solution

$$\cot^{-1}\left(\cot\frac{19\pi}{6}\right)$$

$$=\cot^{-1}\left(\cot\left(3\pi+\frac{\pi}{6}\right)\right)$$

$$=\cot^{-1}\left(\cot\left(\frac{\pi}{6}\right)\right)$$

$$=\frac{\pi}{6}$$

Q43

Evaluate the following :

$$\cot^{-1}\left\{\cot\left(-\frac{8\pi}{3}\right)\right\}$$

Solution

$\cot^{-1}\left\{\cot\left(-\frac{8\pi}{3}\right)\right\}$

$\cot^{-1}\left\{-\cot\left(\frac{8\pi}{3}\right)\right\}$

$= \cot^{-1}\left\{-\cot\left(3\pi-\frac{\pi}{3}\right)\right\}$

$= \cot^{-1}\left(-\cot\left(-\frac{\pi}{3}\right)\right)$

$= \cot^{-1}\left(\cot\left(\frac{\pi}{3}\right)\right)$

$= \frac{\pi}{3}$

Q44

Evaluate the following :

$\cot^{-1}\left\{\cot\left(\frac{21\pi}{4}\right)\right\}$

Solution

$\cot^{-1}\left\{\cot\left(\frac{21\pi}{4}\right)\right\}$

$\cot^{-1}\left\{\cot\left(5\pi+\frac{\pi}{4}\right)\right\}$

$= \cot^{-1}\left\{\cot\left(\frac{\pi}{4}\right)\right\}$

$= \frac{\pi}{4}$

Q45

Write $\cot^{-1}\frac{a}{\sqrt{x^2-a^2}}$, $|x| > a$ in the simplest form.

Solution

$\cot^{-1}\frac{a}{\sqrt{x^2-a^2}}$, $|x| > a$

Let, $x = a\sec\theta$

$$\cot^{-1}\left(\frac{a}{\sqrt{a^2\sec^2\theta - a^2}}\right)$$
$$= \cot^{-1}\left(\frac{a}{\sqrt{a^2\left(\sec^2\theta - 1\right)}}\right)$$
$$= \cot^{-1}\frac{1}{\sqrt{\tan^2\theta}} \qquad \left\{\text{Since, } \sec^2\theta - 1 = \tan^2\theta\right\}$$
$$= \cot^{-1}\left(\cot\theta\right)$$
$$= \theta$$
$$= \sec^{-1}\left(\frac{x}{a}\right)$$

Hence,

$$\cot^{-1}\left(\frac{a}{\sqrt{x^2-a^2}}\right) = \sec^{-1}\left(\frac{x}{a}\right)$$

Q46

Write $\tan^{-1}\left\{x + \sqrt{1+x^2}\right\}$ in the simplest form

Solution

$\tan^{-1}\left\{x+\sqrt{1+x^2}\right\}$

Let, $x = \cot\theta$

$\tan^{-1}\left[\cot+\sqrt{1+\cot^2\theta}\right]$

$= \tan^{-1}\left[\cot\theta+\sqrt{\operatorname{cosec}^2\theta}\right]$ $\quad\left\{\text{Since, } 1+\cot^2\theta = \operatorname{cosec}^2\theta\right\}$

$= \tan^{-1}\left[\cot\theta+\operatorname{cosec}\theta\right]$

$= \tan^{-1}\left[\frac{1+\cos\theta}{\sin\theta}\right]$ $\quad\left\{\text{Since, } \cot\theta = \frac{\cos\theta}{\sin\theta},\ \operatorname{cosec}\theta = \frac{1}{\sin\theta}\right\}$

$= \tan^{-1}\left\{\frac{\frac{2\cos^2\theta}{2}}{\frac{2\cos\theta}{2}\frac{\sin\theta}{2}}\right\}$ $\quad\left\{\text{Since, } 1+\cos\theta = \frac{2\cos^2\theta}{2},\ \sin\theta = \frac{2\sin\theta}{2}\frac{\cos\theta}{2}\right\}$

$= \tan^{-1}\left\{\frac{\frac{\cos\theta}{2}}{\frac{\sin\theta}{2}}\right\}$

$= \tan^{-1}\left\{\frac{\cot\theta}{2}\right\}$ $\quad\left\{\text{Since, } \cot\theta = \frac{\cos\theta}{\sin\theta}\right\}$

$= \tan^{-1}\left\{\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right\}$ $\quad\left\{\text{Since, } \cot\theta = \tan\left(\frac{\pi}{2}-\theta\right)\right\}$

$= \frac{\pi}{2}-\frac{\theta}{2}$

$= \frac{\pi}{2}-\frac{1}{2}\cot^{-1}x$ $\quad\left\{\text{Since, } \cot\theta = x \Rightarrow \theta = \cot^{-1}x\right\}$

Hence,

$\tan^{-1}\left[x+\sqrt{1+x^2}\right] = \frac{\pi}{2}-\frac{1}{2}\cot^{-1}x$

Q47

Write $\tan^{-1}\left\{\sqrt{1+x^2}-x\right\},\ x \in R$ in the simplest form.

Solution

$\tan^{-1}\left\{\sqrt{1+x^2}-x\right\},\ x \in R$

Let, $x = \cot\theta$

$\tan^{-1}\left\{\sqrt{1+\cot^2\theta}-\cot\theta\right\}$

$= \tan^{-1}\{\operatorname{cosec}\theta - \cot\theta\}$ $\{\text{Since, } 1+\cot^2\theta = \operatorname{cosec}^2\theta\}$

$= \tan^{-1}\left\{\frac{1-\cos\theta}{\sin\theta}\right\}$ $\left\{\text{Since, } \operatorname{cosec}\theta = \frac{1}{\sin\theta},\ \cot\theta = \frac{\cos\theta}{\sin\theta}\right\}$

$= \tan^{-1}\left\{\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right\}$ $\left\{\text{Since, } 1-\cos\theta = 2\sin^2\frac{\theta}{2},\ \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right\}$

$= \tan^{-1}\left\{\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right\}$

$= \tan^{-1}\left\{\tan\frac{\theta}{2}\right\}$ $\left\{\text{Since, } \tan\theta = \frac{\sin\theta}{\cos\theta}\right\}$

$= \frac{\theta}{2}$

$= \frac{1}{2}\cot^{-1}x$ $\{\text{Since, } \cot\theta = x \Rightarrow \theta = \cot^{-1}x\}$

Hence,

$\tan^{-1}\left\{\sqrt{1+x^2}-x\right\} = \frac{1}{2}\cot^{-1}x$

Q48

Write $\tan^{-1}\left\{\frac{\sqrt{1+x^2}-1}{x}\right\},\ x \neq 0$ in the simplest form.

Solution

$$\tan^{-1}\left\{\frac{\sqrt{1+x^2}-1}{x}\right\},\ x \neq 0$$

Let, $x = \tan\theta$

$$\tan^{-1}\left\{\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right\}$$

$$= \tan^{-1}\left\{\frac{\sec\theta-1}{\tan\theta}\right\} \qquad \left\{\text{Since, } 1+\tan^2\theta = \sec^2\theta\right\}$$

$$= \tan^{-1}\left\{\frac{1-\cos\theta}{\sin\theta}\right\} \qquad \left\{\text{Since, } \sec\theta = \frac{1}{\cos\theta},\ \tan\theta = \frac{\sin\theta}{\cos\theta}\right\}$$

$$= \tan^{-1}\left\{\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right\} \qquad \left\{\text{Since, } 1-\cos\theta = 2\sin^2\frac{\theta}{2},\ \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right\}$$

$$= \tan^{-1}\left\{\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right\}$$

$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right) \qquad \left\{\text{Since, } \frac{\sin\theta}{\cos\theta} = \tan\theta\right\}$$

$$= \frac{\theta}{2}$$

$$= \frac{1}{2}\tan^{-1}x \qquad \left\{\text{Since, } \tan\theta = x \Rightarrow \theta = \tan^{-1}x\right\}$$

Hence,

$$\tan^{-1}\left\{\frac{\sqrt{1+x^2}-1}{x}\right\} = \frac{1}{2}\tan^{-1}x$$

Q49

Write $\tan^{-1}\left\{\frac{\sqrt{1+x^2}+1}{x}\right\},\ x \neq 0$ in the simplest form.

Solution

$$\tan^{-1}\left\{\frac{\sqrt{1+x^2}+1}{x}\right\},\ x \neq 0$$

Let, $x = \tan\theta$

$$\tan^{-1}\left\{\frac{\sqrt{1+\tan^2\theta}+1}{\tan\theta}\right\}$$

$$= \tan^{-1}\left\{\frac{\sec\theta+1}{\tan\theta}\right\} \qquad \left\{\text{Since, } 1+\tan^2\theta = \sec^2\theta\right\}$$

$$= \tan^{-1}\left\{\frac{1+\cos\theta}{\sin\theta}\right\} \qquad \left\{\text{Since, } \sec\theta = \frac{1}{\cos\theta},\ \tan\theta = \frac{\sin\theta}{\cos\theta}\right\}$$

$$= \tan^{-1}\left\{\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right\} \qquad \left\{\text{Since, } 1+\cos\theta = 2\cos^2\frac{\theta}{2},\ 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = \sin\theta\right\}$$

$$= \tan^{-1}\left\{\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\right\}$$

$$= \tan^{-1}\left\{\cot\frac{\theta}{2}\right\} \qquad \left\{\text{Since, } \cot\theta = \frac{\cos\theta}{\sin\theta}\right\}$$

$$= \tan^{-1}\left\{\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right\} \qquad \left\{\text{Since, } \cot\theta = \tan\left(\frac{\pi}{2}-\theta\right)\right\}$$

$$= \frac{\pi}{2}-\frac{\theta}{2}$$

$$= \frac{\pi}{2}-\frac{1}{2}\tan^{-1}x \qquad \left\{\text{Since, } \tan\theta = x \Rightarrow \theta = \tan^{-1}x\right\}$$

$$\tan^{-1}\left\{\frac{\sqrt{1+x^2}+1}{x}\right\} = \frac{\pi}{2}-\frac{1}{2}\tan^{-1}x$$

Q50

Write $\tan^{-1}\sqrt{\frac{a-x}{a+x}},\ -a < x < a$ in the simplest form.

Solution

$\tan^{-1}\sqrt{\dfrac{a-x}{a+x}}, \quad -a < x < a$

Let, $x = a\cos\theta$

$$\tan^{-1}\sqrt{\frac{a-a\cos\theta}{a+a\cos\theta}}$$

$$= \tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

$$= \tan^{-1}\sqrt{\frac{\frac{2\sin^2\theta}{2}}{\frac{2\cos^2\theta}{2}}} \qquad \left\{\text{Since, } 1-\cos\theta = \frac{2\sin^2\theta}{2},\ 1+\cos\theta = \frac{2\cos^2\theta}{2}\right\}$$

$$= \tan^{-1}\left(\frac{\frac{\sin\theta}{2}}{\frac{\cos\theta}{2}}\right)$$

$$= \tan^{-1}\left(\frac{\tan}{2}\right) \qquad \left\{\text{Since, } \frac{\sin}{\cos} = \tan\right\}$$

$$= \frac{\theta}{2}$$

$$= \frac{1}{2}\cos^{-1}\left(\frac{x}{a}\right) \qquad \left\{\text{Since, } x = a\cos\theta \Rightarrow \theta = \cos^{-1}\left(\frac{x}{a}\right)\right\}$$

Hence,

$$\tan^{-1}\sqrt{\frac{a-x}{a+x}} = \frac{1}{2}\cos^{-1}\left(\frac{x}{a}\right)$$

Q51

Write $\tan^{-1}\left\{\dfrac{x}{a+\sqrt{a^2-x^2}}\right\}, \quad -a < x < a$ in the simplest form.

Solution

$\tan^{-1}\left\{\dfrac{x}{a+\sqrt{a^2-x^2}}\right\},\ -a<x<a$

Let, $x = a\sin\theta$

$$\tan^{-1}\left\{\frac{a\sin\theta}{1+\sqrt{a^2-a^2\sin^2\theta}}\right\}$$

$$= \tan^{-1}\left\{\frac{a\sin\theta}{a+a\sqrt{1-\sin^2\theta}}\right\}$$

$$= \tan^{-1}\left\{\frac{a\sin\theta}{a(1+\cos\theta)}\right\} \qquad \left\{\text{Since, } 1-\sin^2\theta=\cos^2\theta\right\}$$

$$= \tan^{-1}\left\{\frac{\sin\theta}{1+\cos\theta}\right\}$$

$$= \tan^{-1}\left\{\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\right\} \qquad \left\{\text{Since, } \sin\theta=2\sin\frac{\theta}{2}\cos\frac{\theta}{2},\ 1+\cos\theta=2\cos^2\frac{\theta}{2}\right\}$$

$$= \tan^{-1}\left\{\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right\}$$

$$= \tan^{-1}\left\{\tan\frac{\theta}{2}\right\} \qquad \left\{\text{Since, } \frac{\sin\theta}{\cos\theta}=\tan\theta\right\}$$

$$= \frac{\theta}{2}$$

$$= \frac{1}{2}\sin^{-1}x \qquad \left\{\text{Since, } x=a\sin\theta \Rightarrow \theta=\sin^{-1}\left(\frac{x}{a}\right)\right\}$$

Hence,

$$\tan^{-1}\left\{\frac{x}{a+\sqrt{a^2-x^2}}\right\} = \frac{1}{2}\sin^{-1}\frac{x}{a}$$

Q52

Write each of the following in the simplest form

$\sin-1\left\{\dfrac{x+\sqrt{1-x2}}{\sqrt{2}}\right\}, -\dfrac{1}{2} < x < \dfrac{1}{\sqrt{2}}$

Solution

$$\sin^{-1}\left\{\frac{x+\sqrt{1-x^2}}{\sqrt{2}}\right\}$$

Let, $x = \sin\theta$

$$\sin^{-1}\left\{\frac{\sin\theta+\sqrt{1-\sin^2\theta}}{\sqrt{2}}\right\}$$

$$= \sin^{-1}\left\{\frac{\sin\theta+\cos\theta}{\sqrt{2}}\right\} \quad \left\{\text{Since, } 1-\sin^2\theta = \cos^2\theta\right\}$$

$$= \sin^{-1}\left\{\frac{1}{\sqrt{2}}\sin\theta+\frac{1}{\sqrt{2}}\cos\theta\right\}$$

$$= \sin^{-1}\left\{\sin\theta\cos\frac{\pi}{4}+\cos\theta\sin\frac{\pi}{4}\right\}$$

$$= \sin^{-1}\left\{\sin\left(\theta+\frac{\pi}{4}\right)\right\} \quad \left\{\text{Since, } \sin x\cos y+\cos x\sin y = \sin(x+y)\right\}$$

$$= \theta+\frac{\pi}{4}$$

$$= \frac{\pi}{4}+\sin^{-1}x \quad \left\{\text{Since, } \sin\theta = x \Rightarrow \theta = \sin^{-1}x\right\}$$

Hence,

$$\sin^{-1}\left\{\frac{x+\sqrt{1-x^2}}{\sqrt{2}}\right\} = \frac{\pi}{4}+\sin^{-1}x$$

Q53

Write $\sin^{-1}\left\{\frac{\sqrt{1+x}+\sqrt{1-x}}{2}\right\}$, $0 < x < 1$ in the simplest form.

Solution

$\sin^{-1}\left\{\frac{\sqrt{1+x}+\sqrt{1-x}}{2}\right\}, \qquad 0 < x < 1$

Let, $x = \cos 2\theta$

$$\sin^{-1}\left\{\frac{\sqrt{1+\cos^2\theta}+\sqrt{1-\cos^2\theta}}{2}\right\}$$

$$= \sin^{-1}\left\{\frac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{2}\right\} \qquad \left\{\text{Since, } 1+\cos^2\theta = 2\cos^2\theta, \; 1-\cos^2\theta = 2\sin^2\theta\right\}$$

$$= \sin^{-1}\left\{\frac{\sqrt{2}\cos\theta}{2}+\frac{\sqrt{2}\sin\theta}{2}\right\}$$

$$= \sin^{-1}\left\{\frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta\right\}$$

$$= \sin^{-1}\left\{\sin\frac{\pi}{4}\cos\theta+\cos\frac{\pi}{4}\sin\theta\right\}$$

$$= \sin^{-1}\left\{\sin\left(\frac{\pi}{4}+\theta\right)\right\} \qquad \left\{\text{Since, } \sin x\cos y+\cos x\sin y = \sin(x+y)\right\}$$

$$= \frac{\pi}{4}+\theta$$

$$= \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x \qquad \left\{\text{Since, } \cos 2\theta = x \Rightarrow \theta = \frac{1}{2}\cos^{-1}x\right\}$$

Hence,

$$\sin^{-1}\left\{\frac{\sqrt{1+x}+\sqrt{1-x}}{2}\right\} = \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x$$

Q54

Write $\sin\left\{2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right\}$ in the simplest form.

Solution

$$\sin\left\{2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right\}$$

$$= \sin\left\{\sin^{-1}\left(\frac{2\sqrt{\frac{1-x}{1+x}}}{1+\left(\sqrt{\frac{1-x}{1+x}}\right)^2}\right)\right\} \qquad \left\{\text{Since, } 2\tan^{-1}x = \sin^{-1}\frac{2x}{1+x^2}\right\}$$

$$= \sin\left\{\sin^{-1}\left(\frac{2\sqrt{\frac{1-x}{1+x}}}{\frac{1+x+1-x}{1+x}}\right)\right\}$$

$$= 2\sqrt{\frac{1-x}{1+x}} \times \frac{1+x}{2}$$

$$= \sqrt{1-x}\sqrt{1+x}$$

$$= \sqrt{1-x^2}$$

Hence,

$$\sin\left\{2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right\} = \sqrt{1-x^2}$$

Exercise 4.8

Q1

Evaluate the following:

$\sin\left(\sin^{-1}\frac{7}{25}\right)$

Solution

$\sin\left(\sin^{-1}\frac{7}{25}\right)$

$= \frac{7}{25}$ $\left[\because \sin\left(\sin^{-1}x\right) = x \text{ for all } x \in [-1, 1]\right]$

Q2

Evaluate the following:

$\sin\left(\cos^{-1}\frac{5}{13}\right)$

Solution

$\sin\left(\cos^{-1}\frac{5}{13}\right)$

$= \sin\left(\sin^{-1}\frac{12}{13}\right)$.........$\left[\because \cos^{-1}x = \sin^{-1}\sqrt{1-x^2}\right]$

$= \frac{12}{13}$

Q3

Evaluate the following:

$\sin\left(\tan^{-1}\frac{24}{7}\right)$

Solution

$\sin\left(\tan^{-1}\frac{24}{7}\right)$

$= \sin\left(\sin^{-1}\frac{24}{25}\right)$.........$\left[\because \tan^{-1}\left(\frac{b}{p}\right) = \sin^{-1}\left(\frac{b}{h}\right)\right]$

$= \frac{24}{25}$

Q4

Evaluate the following:

$$\sin\left(\sec^{-1}\frac{17}{8}\right)$$

Solution

$$\sin\left(\sec^{-1}\frac{17}{8}\right)$$
$$= \sin\left(\sin^{-1}\frac{15}{17}\right) \ldots\ldots \left[\because \sec^{-1}\left(\frac{h}{p}\right) = \sin^{-1}\left(\frac{b}{h}\right)\right]$$
$$= \frac{15}{17}$$

Q5

Evaluate the following:

$$\operatorname{cosec}\left(\cos^{-1}\frac{3}{5}\right)$$

Solution

$$\operatorname{cosec}\left(\cos^{-1}\frac{3}{5}\right)$$
$$= \operatorname{cosec}\left(\operatorname{cosec}^{-1}\frac{5}{4}\right) \ldots\ldots \left[\because \cos^{-1}\left(\frac{p}{h}\right) = \operatorname{cosec}^{-1}\left(\frac{h}{b}\right)\right]$$
$$= \frac{5}{4}$$

Q6

Evaluate the following:

$$\sec\left(\sin^{-1}\frac{12}{13}\right)$$

Solution

$$\sec\left(\sin^{-1}\frac{12}{13}\right)$$
$$= \sec\left(\sec^{-1}\frac{13}{5}\right)$$
$$= \frac{13}{5}$$

Q7

Evaluate $\tan\left(\cos^{-1}\frac{8}{17}\right)$

Solution

$\tan\left(\cos^{-1}\frac{8}{17}\right)$

$$= \tan\left(\tan^{-1}\frac{\sqrt{1-\left(\frac{8}{17}\right)^2}}{\frac{8}{17}}\right) \qquad \left\{\text{Since } \cos^{-1}x = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)\right\}$$

$$= \tan\left(\tan^{-1}\frac{\frac{15}{17}}{\frac{8}{17}}\right)$$

$$= \tan\left(\tan^{-1}\frac{15}{8}\right)$$

$$= \tan\left(\tan^{-1}\frac{15}{8}\right)$$

$$= \frac{15}{8} \qquad \left\{\text{Since } \tan\left(\tan^{-1}x\right) = x \text{ if } x \in R\right\}$$

Hence,

$$\tan\left(\cos^{-1}\frac{8}{17}\right) = \frac{15}{8}$$

Q8

Evaluate the following:

$\cot\left(\cos^{-1}\frac{3}{5}\right)$

Solution

$\cot\left(\cos^{-1}\frac{3}{5}\right)$

$= \cot\left(\cot^{-1}\frac{3}{4}\right)$

$= \frac{3}{4}$

Q9

Evaluate the following:

$\cos\left(\tan^{-1}\frac{24}{7}\right)$

Solution

$$\cos\left(\tan^{-1}\frac{24}{7}\right)$$

$$= \frac{1}{\sqrt{1+\left(\frac{24}{7}\right)^2}} \quad \ldots\ldots\ldots\left[\because \cos\left(\tan^{-1}x\right) = \frac{1}{\sqrt{1+x^2}}\right]$$

$$= \frac{7}{25}$$

Q10

Prove the following result:

$$\tan\left(\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right) = \frac{17}{6}$$

Solution

$$\tan\left(\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right)$$

$$= \tan\left(\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{2}{3}\right)\ldots\ldots\ldots\left[\cos^{-1}\left(\frac{b}{h}\right) = \tan^{-1}\left(\frac{p}{b}\right)\right]$$

$$= \tan\left(\tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\times\frac{2}{3}}\right)\right)\ldots\ldots\left[\tan^{-1}(x)+\tan^{-1}(y) = \tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$= \tan\left(\tan^{-1}\left(\frac{\frac{17}{12}}{\frac{1}{2}}\right)\right)$$

$$= \tan\left(\tan^{-1}\left(\frac{17}{6}\right)\right)$$

$$= \frac{17}{6}$$

Q11

Prove the following result:

$$\cos\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right) = \frac{6}{5\sqrt{13}}$$

Solution

$$\cos\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)$$

$$=\cos\left(\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{2}{3}\right)\ldots\ldots\left[\begin{array}{l}\sin^{-1}\left(\frac{p}{h}\right)=\tan^{-1}\left(\frac{p}{b}\right)\\ \cot^{-1}\left(\frac{b}{p}\right)=\tan^{-1}\left(\frac{p}{b}\right)\end{array}\right]$$

$$=\cos\left(\tan^{-1}\left(\frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\times\frac{2}{3}}\right)\right)\ldots\ldots\left[\tan^{-1}(x)+\tan^{-1}(y)=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$=\cos\left(\tan^{-1}\left(\frac{\frac{17}{12}}{\frac{1}{2}}\right)\right)$$

$$=\cos\left(\tan^{-1}\left(\frac{17}{6}\right)\right)$$

$$=\cos\left(\cos^{-1}\left(\frac{6}{5\sqrt{13}}\right)\right)\ldots\ldots\left[\tan^{-1}\left(\frac{p}{b}\right)=\cos^{-1}\left(\frac{b}{h}\right)\right]$$

$$=\frac{6}{5\sqrt{13}}$$

Q12

Evaluate the following:

$$\tan\left(\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}\right)=\frac{63}{16}$$

Solution

$$\tan\left(\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}\right)$$

$$=\tan\left(\tan^{-1}\frac{5}{12}+\tan^{-1}\frac{4}{3}\right)\ldots\ldots\left[\begin{array}{l}\sin^{-1}\left(\frac{p}{h}\right)=\tan^{-1}\left(\frac{p}{b}\right)\\ \cos^{-1}\left(\frac{b}{h}\right)=\tan^{-1}\left(\frac{p}{b}\right)\end{array}\right]$$

$$=\tan\left(\tan^{-1}\left(\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}\times\frac{4}{3}}\right)\right)\ldots\ldots\left[\tan^{-1}(x)+\tan^{-1}(y)=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$=\tan\left(\tan^{-1}\left(\frac{\frac{21}{12}}{\frac{4}{9}}\right)\right)$$

$$=\tan\left(\tan^{-1}\left(\frac{63}{16}\right)\right)$$

$$=\frac{63}{16}$$

Q13

Evaluate the following:

$$\sin\left(\cos^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}\right)=\frac{63}{65}$$

Solution

$$\sin\left(\cos^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}\right)$$

$$=\sin\left(\tan^{-1}\frac{4}{3}+\tan^{-1}\frac{5}{12}\right)\ldots\ldots\ldots\begin{bmatrix}\sin^{-1}\left(\frac{p}{h}\right)=\tan^{-1}\left(\frac{p}{b}\right)\\ \cos^{-1}\left(\frac{b}{h}\right)=\tan^{-1}\left(\frac{p}{b}\right)\end{bmatrix}$$

$$=\sin\left(\tan^{-1}\left(\frac{\frac{4}{3}+\frac{5}{12}}{1-\frac{4}{3}\times\frac{5}{12}}\right)\right)\ldots\ldots\ldots\left[\tan^{-1}(x)+\tan^{-1}(y)=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right]$$

$$=\sin\left(\tan^{-1}\left(\frac{\frac{21}{12}}{\frac{4}{9}}\right)\right)$$

$$=\sin\left(\tan^{-1}\left(\frac{63}{16}\right)\right)$$

$$=\sin\left(\sin^{-1}\left(\frac{63}{65}\right)\right)\ldots\ldots\ldots\left[\tan^{-1}\left(\frac{p}{b}\right)=\sin^{-1}\left(\frac{p}{h}\right)\right]$$

$$=\frac{63}{65}$$

Q14

Solve:

$$\cos\left(\sin^{-1}x\right)=\frac{1}{6}$$

Solution

$$\frac{1}{6}=\cos\left(\sin^{-1}x\right)$$

$$\frac{1}{6}=\cos\left(\cos^{-1}\sqrt{1-x^2}\right)\ldots\ldots\ldots\left[\sin^{-1}x=\cos^{-1}\sqrt{1-x^2}\right]$$

$$\frac{1}{6}=\sqrt{1-x^2}$$

$$\frac{1}{36}=1-x^2$$

$$x^2=\frac{35}{36}$$

$$x=\pm\frac{\sqrt{35}}{6}$$

Q15

Solve:

$\cos\left\{2\sin^{-1}(-x)\right\} = 0$

Solution

$$0 = \cos\left\{2\sin^{-1}(-x)\right\}$$
$$0 = \cos\left(\sin^{-1}\left(-2x\sqrt{1-x^2}\right)\right) \ldots\ldots\ldots \left[2\sin^{-1}x = \sin^{-1}\left(2x\sqrt{1-x^2}\right)\right]$$
$$0 = \cos\left(\cos^{-1}\sqrt{1-\left(4x^2-4x^4\right)}\right) \ldots\ldots\ldots \left[\sin^{-1}x = \cos^{-1}\sqrt{1-x^2}\right]$$
$$0 = \sqrt{1-\left(4x^2-4x^4\right)}$$
$$0 = 1-\left(4x^2-4x^4\right)$$
$$4x^4 - 4x^2 + 1 = 0$$
$$4x^4 - 2x^2 - 2x^2 + 1 = 0$$
$$2x^2\left(2x^2-1\right) - 1\left(2x^2-1\right) = 0$$
$$\left(2x^2-1\right)^2 = 0$$
$$2x^2 - 1 = 0$$
$$x = \pm\frac{1}{\sqrt{2}}$$

Exercise 4.9

Q1

Evaluate:

$$\cos\left\{\sin^{-1}\left(-\frac{7}{25}\right)\right\}$$

Solution

$$\cos\left\{\sin^{-1}\left(-\frac{7}{25}\right)\right\}$$

$$= \cos\left\{-\sin^{-1}\left(\frac{7}{25}\right)\right\}\ldots\ldots\left(\sin^{-1}(-x) = -\sin^{-1}(x) \text{ for all } x \in [-1, 1]\right)$$

$$= \cos\left\{-\cos^{-1}\left(\frac{24}{25}\right)\right\}\ldots\ldots\left(\sin^{-1}\left(\frac{p}{h}\right) = \cos^{-1}\left(\frac{b}{h}\right)\right)$$

$$= \cos\left\{\cos^{-1}\left(\frac{24}{25}\right)\right\}\ldots\ldots\left(\cos(-x) = \cos x\right)$$

$$= \frac{24}{25}$$

Q2

Evaluate:

$$\sec\left\{\cot^{-1}\left(-\frac{5}{12}\right)\right\}$$

Solution

$$\sec\left\{\cot^{-1}\left(-\frac{5}{12}\right)\right\}$$

$$= \sec\left\{-\cot^{-1}\left(\frac{5}{12}\right)\right\}\ldots\ldots\left(\cot^{-1}(-x) = -\cot^{-1}(x) \text{ for all } x \in (-1, 1)\right)$$

$$= \sec\left\{-\sec^{-1}\left(\frac{13}{5}\right)\right\}\ldots\ldots\left(\cot^{-1}\left(\frac{b}{p}\right) = \sec^{-1}\left(\frac{h}{b}\right)\right)$$

$$= \sec\left\{\sec^{-1}\left(\frac{13}{5}\right)\right\}\ldots\ldots\left(\sec(-x) = \sec x\right)$$

$$= \frac{13}{5}$$

Q3

Evaluate:

$$\cot\left\{\sec^{-1}\left(-\frac{13}{25}\right)\right\}$$

Solution

$$\cot\left\{\sec^{-1}\left(-\frac{13}{25}\right)\right\}$$

$$= \cot\left\{-\sec^{-1}\left(\frac{13}{25}\right)\right\}\text{.........}\left(\sec^{-1}(-x) = -\sec^{-1}(x)\right)$$

$$= \cot\left\{-\cot^{-1}\left(\frac{5}{12}\right)\right\}\text{.........}\left(\cot^{-1}\left(\frac{b}{p}\right) = \sec^{-1}\left(\frac{h}{b}\right)\right)$$

$$= -\cot\left\{\cot^{-1}\left(\frac{5}{12}\right)\right\}\text{..........}\left(\cot(-x) = \cot x\right)$$

$$= \frac{5}{12}$$

Q4

Evaluate:

$$\tan\left\{\cos^{-1}\left(-\frac{7}{25}\right)\right\}$$

Solution

$$\tan\left\{\cos^{-1}\left(-\frac{7}{25}\right)\right\}$$

$$= \tan\left\{\cos^{-1}\left(\frac{7}{25}\right)\right\}\text{.........}\left(\cos^{-1}(-x) = \cos^{-1}(x)\right)$$

$$= \tan\left\{\tan^{-1}\left(\frac{24}{7}\right)\right\}\text{.........}\left(\tan^{-1}\left(\frac{p}{b}\right) = \cos^{-1}\left(\frac{b}{h}\right)\right)$$

$$= \frac{24}{7}$$

Q5

Evaluate:

$$\operatorname{cosec}\left\{\cot^{-1}\left(-\frac{12}{5}\right)\right\}$$

Solution

$$\operatorname{cosec}\left\{\cot^{-1}\left(-\frac{12}{5}\right)\right\}$$
$$= \operatorname{cosec}\left\{-\cot^{-1}\left(\frac{12}{5}\right)\right\}\ldots\ldots\left(\cot^{-1}(-x) = -\cot^{-1}(x)\right)$$
$$= \operatorname{cosec}\left\{-\operatorname{cosec}^{-1}\left(\frac{13}{12}\right)\right\}\ldots\ldots\left(\cot^{-1}\left(\frac{b}{p}\right) = \operatorname{cosec}^{-1}\left(\frac{h}{p}\right)\right)$$
$$= -\operatorname{cosec}\left\{\operatorname{cosec}^{-1}\left(\frac{13}{12}\right)\right\}\ldots\ldots\left(\operatorname{cosec}(-x) = -\operatorname{cosec}x\right)$$
$$= -\frac{13}{12}$$

Q6

Evaluate:

$$\cos\left\{\tan^{-1}\left(-\frac{3}{4}\right)\right\}$$

Solution

$$\cos\left\{\tan^{-1}\left(-\frac{3}{4}\right)\right\}$$
$$= \cos\left\{-\tan^{-1}\left(\frac{3}{4}\right)\right\}\ldots\ldots\left(\tan^{-1}(-x) = -\tan^{-1}(x)\right)$$
$$= \cos\left\{-\cos^{-1}\left(\frac{4}{5}\right)\right\}\ldots\ldots\left(\tan^{-1}\left(\frac{p}{b}\right) = \cos^{-1}\left(\frac{b}{h}\right)\right)$$
$$= \cos\left\{\cos^{-1}\left(\frac{4}{5}\right)\right\}\ldots\ldots\left(\cos(-x) = \cos x\right)$$
$$= \frac{4}{5}$$

Q7

Evaluate:

$$\sin\left\{\cos^{-1}\left(-\frac{3}{5}\right) + \cot^{-1}\left(-\frac{5}{12}\right)\right\}.$$

Solution

$$\sin\left\{\cos^{-1}\left(-\frac{3}{5}\right)+\cot^{-1}\left(-\frac{5}{12}\right)\right\}$$

$$=\sin\left\{\cos^{-1}\left(\frac{3}{5}\right)-\cot^{-1}\left(\frac{5}{12}\right)\right\}\ldots\ldots\begin{pmatrix}\cos^{-1}(-x)=\cos^{-1}(x)\\ \cot^{-1}(-x)=-\cot^{-1}(x)\end{pmatrix}$$

$$=\sin\left\{\tan^{-1}\left(\frac{4}{3}\right)-\tan^{-1}\left(\frac{12}{5}\right)\right\}\ldots\ldots\begin{pmatrix}\cot^{-1}\left(\frac{b}{p}\right)=\tan^{-1}\left(\frac{p}{b}\right)\\ \tan^{-1}\left(\frac{p}{b}\right)=\cos^{-1}\left(\frac{b}{h}\right)\end{pmatrix}$$

$$=\sin\left\{\tan^{-1}\left(\frac{\frac{4}{3}-\frac{12}{5}}{1+\frac{4}{3}\times\frac{12}{5}}\right)\right\}\ldots\ldots\left(\tan^{-1}(x)-\tan^{-1}(y)=\tan^{-1}\left(\frac{x-y}{1+xy}\right)\right)$$

$$=\sin\left\{\tan^{-1}\left(-\frac{16}{63}\right)\right\}$$

$$=\sin\left\{-\tan^{-1}\left(-\frac{16}{63}\right)\right\}\ldots\ldots\left(\tan^{-1}(-x)=-\tan^{-1}(x)\right)$$

$$=\sin\left\{-\sin^{-1}\left(\frac{16}{65}\right)\right\}\ldots\ldots\left(\tan^{-1}\left(\frac{b}{p}\right)=\sin^{-1}\left(\frac{b}{h}\right)\right)$$

$$=-\sin\left\{\sin^{-1}\left(\frac{16}{65}\right)\right\}\ldots\ldots(\sin(-x)=-\sin(x))$$

$$=-\frac{16}{65}$$

Exercise 4.10

Q1

Evaluate:

$$\cot\left(\sin^{-1}\frac{3}{4}+\sec^{-1}\frac{4}{3}\right)$$

Solution

$$\cot\left(\sin^{-1}\frac{3}{4}+\sec^{-1}\frac{4}{3}\right)$$
$$=\cot\left(\sin^{-1}\frac{3}{4}+\cos^{-1}\frac{3}{4}\right)$$
$$=\cot\left(\frac{\pi}{2}\right)$$
$$=0$$

Q2

Evaluate:

$$\sin\left(\tan^{-1}x+\tan^{-1}\frac{1}{x}\right) \text{ for } x<0$$

Solution

$$\sin\left(\tan^{-1}x+\tan^{-1}\frac{1}{x}\right)$$
$$=\sin\left(-\pi+\tan^{-1}\left(\frac{x+\frac{1}{x}}{1-\frac{x}{x}}\right)\right)$$
$$=\sin\left(-\pi+\tan^{-1}(\infty)\right)$$
$$=\sin\left(-\pi+\frac{\pi}{2}\right)$$
$$=\sin\left(-\frac{\pi}{2}\right)$$
$$=-1$$

Q3

Evaluate:

$$\sin\left(\tan^{-1}x+\tan^{-1}\frac{1}{x}\right) \text{ for } x>0$$

Solution

$\sin\left(\tan^{-1}x + \tan^{-1}\frac{1}{x}\right)$

$= \sin\left(\pi + \tan^{-1}\left(\frac{x+\frac{1}{x}}{1-\frac{x}{x}}\right)\right)$

$= \sin\left(\pi + \tan^{-1}(\infty)\right)$

$= \sin\left(\pi + \frac{\pi}{2}\right)$

$= -\sin\left(\frac{\pi}{2}\right)$

$= -1$

Q4

Evaluate:

$\cot\left(\tan^{-1}a + \cot^{-1}a\right)$

Solution

$\cot\left(\tan^{-1}a + \cot^{-1}a\right)$

$= \cot\left(\frac{\pi}{2}\right)$

$= 0$

Q5

Evaluate:

$\cos\left(\sec^{-1}x + \operatorname{cosec}^{-1}x\right), |x| \geq 1$

Solution

$\cos\left(\sec^{-1}x + \operatorname{cosec}^{-1}x\right)$

$= \cos\left(\frac{\pi}{2}\right)$

$= 0$

Q6

If $\cos^{-1}x + \cos^{-1}y = \frac{\pi}{4}$, find the value of $\sin^{-1}x + \sin^{-1}y$.

Solution

$[\Pi/2 - \sin^{-1} x] + [\Pi/2 - \sin^{-1} y] = \Pi/4$

$\sin^{-1}x + \sin^{-1}y = \Pi - \Pi/4$

$\sin^{-1}x + \sin^{-1}y = 3\Pi/4$

Q7

If $\sin^{-1} x + \sin^{-1} y = \frac{\pi}{3}$ and $\cos^{-1} x - \cos^{-1} y = \frac{\pi}{6}$, find the values of x and y.

Solution

$\sin^{-1} x + \sin^{-1} y = \frac{\pi}{3}$..........(i)

$\cos^{-1} x - \cos^{-1} y = \frac{\pi}{6}$..........(ii)

On adding both the equation

$\Pi/2 + \sin^{-1}y - \cos^{-1}y = \Pi/2$

$[\Pi/2 - \cos^{-1}y] - \cos^{-1}y = 0$

$\cos^{-1}y = \Pi/4$

$y = 1/\sqrt{2}$

on putting $y=1/\sqrt{2}$ in 2nd equation

$\cos^{-1}x - \Pi/4 = \Pi/6$

$\cos^{-1}x = \Pi/4 + \Pi/6$

$x = \cos(\Pi/4 + \Pi/6)$

$x = \cos(\Pi/4)\cos(\Pi/6) - \sin(\Pi/4)\sin(\Pi/6)$

$x = (\sqrt{3}-1)/2\sqrt{2}$

Q8

If $\cot\left(\cos^{-1} \frac{3}{5} + \sin^{-1} x\right) = 0$, find the values of x.

Solution

cot(z) = 0 means z = Π/2, 3Π/2, 5Π/2 ....

$\cos^{-1}(3/5) + \sin^{-1}x = n\Pi + \Pi/2$

$\sin^{-1}x = n\Pi + \Pi/2 - \cos^{-1}(3/5)$

$\sin^{-1}x = n\Pi + \sin^{-1}(3/5)$

$x = \sin(n\Pi + \sin^{-1}(3/5)) = (-1)^n \sin(\sin^{-1}(3/5))$

$x = (-1)^n\, 3/5$

Q9

If $\left(\sin^{-1} x\right)^2 + \left(\cos^{-1} x\right)^2 = \frac{17\pi^2}{36}$, find x.

Solution

$[\Pi/2 - \cos^{-1}x]^2 + (\cos^{-1}x)^2 = 17\Pi^2/36$

$\Pi^2/4 - \Pi\cos^{-1}x + 2(\cos^{-1}x)^2 = 17\Pi^2/36$

Let, $\cos^{-1}x = u$

$2u^2 - \Pi u + \Pi^2/4 - 17\Pi^2/36 = 0$

$2u^2 - \Pi u - 2\Pi^2/9 = 0$

$18u^2 - 9\Pi u - 2\Pi^2 = 0$

On factorizing

$18u^2 - 12\Pi u + 3\Pi u - 2\Pi^2 = 0$

$6u(3u - 2\Pi) + \Pi(3u - 2\Pi) = 0$

$(3u - 2\Pi)(6u + \Pi) = 0$

$u = -\Pi/6,\ 2\Pi/3$

i.e. $\cos^{-1}x = -\Pi/6,\ 2\Pi/3$

but range of $\cos^{-1}x$ is $[0, \pi]$

$x = \cos(\Pi/2 + \Pi/6)$

$x = -1/2$

Q10

Solve:

$$\sin\left\{\sin^{-1}\frac{1}{5} + \cos^{-1} x\right\} = 1$$

Solution

$\sin^{-1}(1/5) + [\Pi/2 - \sin^{-1}x] = \sin^{-1}1$

$\sin^{-1}(1/5) + \Pi/2 - \sin^{-1}x = \Pi/2$

$\sin^{-1}(1/5) - \sin^{-1}x = 0$

$x = 1/5$

Q11

Solve:

$$\sin^{-1} x = \frac{\pi}{6} + \cos^{-1} x$$

Solution

$\Pi/2 - \cos^{-1}x = \Pi/6 + \cos^{-1}x$

$\Pi/3 = 2\cos^{-1}x$

$\cos^{-1}x = \Pi/6$

$x = \sqrt{3}/2$

Q12

Solve:

$4 \sin^{-1}x = \Pi - \cos^{-1}x$

Solution

$4\sin^{-1}x+\cos^{-1}x=\Pi$
$3\sin^{-1}x+\sin^{-1}x+\cos^{-1}x=\Pi$
$3\sin^{-1}x=\Pi/2$ $[\sin^{-1}x+\cos^{-1}x=\Pi/2]$
$\sin^{-1}x=\Pi/6$
$x = \sin\Pi/6=0.5$

Q13

Solve:

$$\tan^{-1} x + 2 \cot^{-1} x = \frac{2\pi}{3}$$

Solution

$\tan^{-1}x+\cot^{-1}x=\Pi/2$ so the above equation reduces to
$\cot^{-1}x =2\Pi/3-\Pi/2 =\Pi/6$
$x= \cot\Pi/6 =\sqrt{3}$

Q14

Solve:

$5 \tan^{-1}x + 3 \cot^{-1}x = 2\Pi$

Solution

$2\tan^{-1}x+3(\Pi/2)=2\Pi$

$2\tan^{-1}x=2\Pi-3\Pi/2=\Pi/3$

$\tan^{-1}x=\Pi/6$

$x=\tan\Pi/6=1/\sqrt{3}$

Exercise 4.11

Q1

Prove that $\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right)=\tan^{-1}\left(\frac{2}{9}\right)$

Solution

$\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right)=\tan^{-1}\left(\frac{2}{9}\right)$

LHS $=\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{1}{13}\right)$

$=\tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{7}\times\frac{1}{13}}\right)$ $\left\{\text{Since } \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right\}$

$=\tan^{-1}\left(\frac{\frac{13+7}{91}}{\frac{91-1}{91}}\right)$

$=\tan^{-1}\left(\frac{20}{91}\times\frac{91}{90}\right)$

$=\tan^{-1}\left(\frac{2}{9}\right)$

$=$ RHS

Hence proved.

Q2

Prove that $\sin^{-1}\left(\frac{12}{13}\right)+\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{63}{16}\right)=\pi$

Solution

$$\sin^{-1}\left(\frac{12}{13}\right)+\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{63}{16}\right)=\pi$$

$$\text{LHS} = \sin^{-1}\left(\frac{12}{13}\right)+\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{63}{16}\right)$$

$$=\tan^{-1}\left(\frac{\frac{12}{13}}{\sqrt{1-\left(\frac{12}{13}\right)^2}}\right)+\tan^{-1}\left(\frac{\sqrt{1-\left(\frac{4}{5}\right)^2}}{\frac{4}{5}}\right)+\tan^{-1}\left(\frac{63}{16}\right)$$

$$\left\{\text{Since } \sin^{-1}x=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\cos^{-1}x=\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)\right\}$$

$$=\tan^{-1}\left(\frac{\frac{12}{13}}{\frac{5}{13}}\right)+\tan^{-1}\left(\frac{\frac{3}{5}}{\frac{4}{5}}\right)+\tan^{-1}\left(\frac{63}{16}\right)$$

$$=\tan^{-1}\left(\frac{12}{5}\right)+\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{63}{16}\right)$$

$$=\pi+\tan^{-1}\left(\frac{\frac{12}{5}+\frac{3}{4}}{1-\frac{12}{5}\times\frac{3}{4}}\right)+\tan^{-1}\left(\frac{63}{16}\right)$$

$$\left\{\text{Since } \tan^{-1}x+\tan^{-1}y=\pi+\tan^{-1}\left(\frac{x+y}{1-xy}\right) \text{ if } x>0,\ y>0 \text{ and } xy>0\right\}$$

$$=\pi+\tan^{-1}\left(\frac{\frac{63}{20}}{-\frac{16}{20}}\right)+\tan^{-1}\left(\frac{63}{16}\right)$$

$$=\pi+\tan^{-1}\left(-\frac{63}{16}\right)+\tan^{-1}\left(\frac{63}{16}\right)$$

$$=\pi-\tan^{-1}\left(\frac{63}{16}\right)+\tan^{-1}\left(\frac{63}{16}\right)$$

$$\left\{\text{Since } \tan^{-1}(-x)=-\tan^{-1}x\right\}$$

$$=\pi$$

Hence,

$$\sin^{-1}\left(\frac{12}{13}\right)+\cos^{-1}\left(\frac{4}{5}\right)+\tan^{-1}\left(\frac{63}{16}\right)=\pi$$

Q3

Prove the following result

$$\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\sin^{-1}\frac{1}{\sqrt{5}}$$

Solution

$$\tan^{-1}1/4+\tan^{-1}2/9=\frac{\tan^{-1}(1/4+2/9)}{(1-1/4\times 2/9)}$$

$$=\tan^{-1}\frac{(17/36)}{(34/36)}$$

$$=\tan^{-1}(1/2)$$

Let $\tan^{-1}(1/2)=\theta$

$\tan\theta=1/2$

we know that $\tan\theta=\frac{\text{opposite side}}{\text{adjacent side}}$

so if opposite side = 1 unit

adjacent side = 2 unit, then hypotenuse = $\sqrt{5}$ unit

so $\sin\theta=\frac{\text{opposite side}}{\text{hypotenuse}}$

so $\sin\theta=1/\sqrt{5}$

so $\theta=\sin^{-1}(1/\sqrt{5})=\tan^{-1}(1/2)$

Q4

Find the value of $\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{x-y}{x+y}\right)$

Solution

We know that, $\tan^{-1}A-\tan^{-1}B=\tan^{-1}\left(\frac{A-B}{1+AB}\right)$, *if* $AB>-1$

Consider the given expression $\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{x-y}{x+y}\right)$;

$$\tan^{-1}\left(\frac{x}{y}\right)-\tan^{-1}\left(\frac{x-y}{x+y}\right)=\tan^{-1}\left(\frac{\frac{x}{y}-\frac{x-y}{x+y}}{1+\left(\frac{x}{y}\right)\left(\frac{x-y}{x+y}\right)}\right)$$

$$=\tan^{-1}\left(\frac{x(x+y)-y(x-y)}{y(x+y)+x(x-y)}\right)$$

$$=\tan^{-1}\left(\frac{x^2+y^2}{x^2+y^2}\right)$$

$$=\tan^{-1}(1)$$

$$=\frac{\pi}{4}$$

Q5

Solve the equation for x :

$$\tan^{-1}2x+\tan^{-1}3x=n\pi+\frac{3\pi}{4}$$

Solution

Given

$$\tan^{-1}2x + \tan^{-1}3x = n\pi + \frac{3\pi}{4} \quad \text{---(i)}$$

$$\Rightarrow \tan^{-1}\left(\frac{2x+3x}{1-2x\times 3x}\right) = n\pi + \frac{3\pi}{4} \quad \left\{\text{Since } \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right) \text{ if } xy < 1\right\}$$

$$\Rightarrow \tan^{-1}\left(\frac{5x}{1-6x^2}\right) = n\pi + \frac{3\pi}{4},\ 6x^2 < 1$$

$$\Rightarrow \frac{5x}{1-6x^2} = \tan\left(n\pi + \frac{3\pi}{4}\right),\ 6x^2 < 1$$

$$\Rightarrow \frac{5x}{1-6x^2} = -1,\ 6x^2 < 1$$

$$\Rightarrow 5x = -1 + 6x^2,\ 6x^2 < 1$$

$$\Rightarrow 6x^2 - 5x - 1 = 0,\ x^2 < \frac{1}{6}$$

$$\Rightarrow 6x^2 - 6x + x - 1 = 0,\ -\frac{1}{\sqrt{6}} < x < \frac{1}{\sqrt{6}}$$

$$\Rightarrow 6x(x-1) + 1(x-1) = 0,\ -\frac{1}{\sqrt{6}} < x < \frac{1}{\sqrt{6}}$$

$$\Rightarrow (6x+1)(x-1) = 0$$

$$\Rightarrow 6x + 1 = 0 \quad \text{or} \quad x - 1 = 0$$

$$\Rightarrow x = -\frac{1}{6} \text{ or } \quad x = 1$$

Since $x = 1 \notin \left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

So, $x = 1$ is not root of the given equation (i).

Since,

$$x = -\frac{1}{6} \in \left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

So,

$x = -\frac{1}{6}$ is the root of the gien equation (i).

Hence,

$$x = -\frac{1}{6}$$

Q6

Solve the equation for x :

$$\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\frac{8}{31}$$

Solution

Given,

$$\tan^{-1}(x+1)+\tan^{-1}(x-1)=\tan^{-1}\frac{8}{31} \quad ---(i)$$

$$\Rightarrow \tan^{-1}\left[\frac{(x+1)+(x-1)}{1-(x+1)(x-1)}\right]=\tan^{-1}\frac{8}{31},\ (x+1)(x-1)<1$$

$$\left\{\text{Since } \tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy},\ \text{if } xy<1\right\}$$

$$\Rightarrow \tan^{-1}\left[\frac{2x}{1-(x^2-1)}\right]=\tan^{-1}\frac{8}{31}, \quad (x^2-1)<1$$

$$\Rightarrow \tan^{-1}\left[\frac{2x}{1-x^2+1}\right]=\tan^{-1}\frac{8}{31},\ x^2<2$$

$$\Rightarrow \frac{2x}{2-x^2}=\frac{8}{31}, \quad -\sqrt{2}<x<\sqrt{2}$$

$$\Rightarrow 8x^2+62x-16=0, \quad -\sqrt{2}<x<\sqrt{2}$$

$$\Rightarrow 4x^2+31x-8=0, \quad -\sqrt{2}<x<\sqrt{2}$$

$$\Rightarrow 4x(x+8)-1(x+8)=0, \quad -\sqrt{2}<x<\sqrt{2}$$

$$\Rightarrow (4x-1)(x+8)=0, \quad -\sqrt{2}<x<\sqrt{2}$$

$$\Rightarrow x=\frac{1}{4} \quad \text{or} \quad x=-8$$

But, $x=-8\notin\left(-\sqrt{2},\ \sqrt{2}\right)$

$\Rightarrow$ $x=8$ is not root of the given equation (i)

For $x=\frac{1}{4}\in\left(-\sqrt{2},\ \sqrt{2}\right)$

$\Rightarrow$ $x=\frac{1}{4}$ is a root of the equation (i)

Hence,

$$x=\frac{1}{4}$$

Q7

Solve the equation for x:

$$\tan^{-1}(x-1)+\tan^{-1}x+\tan^{-1}(x+1)=\tan^{-1}3x$$

Solution

Given,

$$\tan^{-1}(x-1)+\tan^{-1}x+\tan^{-1}(x+1)=\tan^{-1}3x$$

$$\Rightarrow \tan^{-1}(x-1)+\tan^{-1}(x+1)+\tan^{-1}x=\tan^{-1}3x$$

$$\Rightarrow \tan^{-1}\left[\frac{(x-1)+(x+1)}{1-(x-1)(x+1)}\right]+\tan^{-1}x=\tan^{-1}3x$$

$$\left\{\text{Since, } \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right),\ \text{if } xy<1\right\}$$

$$\Rightarrow \tan^{-1}\left(\frac{2x}{1-x^2+1}\right)+\tan^{-1}x=\tan^{-1}3x, \quad x^2-1<1$$

$$\Rightarrow \tan^{-1}\left(\frac{2x}{2-x^2}\right)+\tan^{-1}x=\tan^{-1}3x, \quad x^2<2$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{2x}{2-x^2}+x}{1-\left(\frac{2x}{2-x^2}\right)^x}\right]=\tan^{-1}3x, \quad \frac{2x^2}{2-x^2}<1$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{2x+2x-x^3}{2-x^2}}{\frac{2-x^2-2x^2}{2-x^2}}\right]=\tan^{-1}3x$$

$$\Rightarrow \tan^{-1}\left[\frac{4x-x^3}{2-3x^2}\right]=\tan^{-1}3x, \quad 2x^2<2-x^2$$

$$\Rightarrow \frac{4x-x^3}{2-3x^2}=3x, \quad 3x^2<2$$

$$\Rightarrow 4x-x^3=6x-9x^3, \quad x^2<\frac{2}{3}$$

$$\Rightarrow 9x^3-x^3+4x-6x=0$$

$$\Rightarrow 8x^3-2x=0$$

$$\Rightarrow 2x\left(4x^2-1\right)=0$$

$$\Rightarrow x=0,\ x=\frac{1}{2},\ x=-\frac{1}{2} \text{ all satisfies } x^2<\frac{2}{3} \text{ ed}$$

$$x=0,\ \pm\frac{1}{2}$$

Q8

Solve the following equations for x :

$$\tan^{-1}\left(\frac{1-x}{1+x}\right)-\frac{1}{2}\tan^{-1}x=0, \text{ where } x>0$$

Solution

$$\tan^{-1}\frac{1-x}{1+x}=\frac{1}{2}\tan^{-1}x$$

$$\Rightarrow \tan^{-1}1-\tan^{-1}x=\frac{1}{2}\tan^{-1}x \qquad \left[\tan^{-1}x-\tan^{-1}y=\tan^{-1}\frac{x-y}{1+xy}\right]$$

$$\Rightarrow \frac{\pi}{4}=\frac{3}{2}\tan^{-1}x$$

$$\Rightarrow \tan^{-1}x=\frac{\pi}{6}$$

$$\Rightarrow x=\tan\frac{\pi}{6}$$

$$\therefore x=\frac{1}{\sqrt{3}}$$

Q9

Solve the equation for x :

$\cot^{-1}x-\cot^{-1}(x+2)=\frac{\pi}{12}$, where $x>0$

Solution

Given,

$$\cot^{-1}x - \cot^{-1}(x+2) = \frac{\pi}{12}, \text{ where } x > 0$$

$$\Rightarrow \tan^{-1}\left(\frac{1}{x}\right) - \tan^{-1}\left(\frac{1}{x+2}\right) = \frac{\pi}{12} \qquad \left\{\text{Since } \cot^{-1}x = \tan^{-1}\frac{1}{x}\right\}$$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{1}{x} - \frac{1}{x+2}}{1 + \frac{1}{x} \times \frac{1}{x+2}}\right) = \frac{\pi}{12} \qquad \left\{\text{Since, } \tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}\right\}$$

$$\Rightarrow \left(\frac{\frac{x+2-x}{x(x+2)}}{\frac{x(x+2)+1}{x(x+2)}}\right) = \tan\frac{\pi}{12}$$

$$\Rightarrow \frac{2}{x^2+2x+1} = \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$\Rightarrow \frac{2}{(x+1)^2} = \frac{\tan\frac{\pi}{3} - \tan\frac{\pi}{4}}{1 + \tan\frac{\pi}{4} \times \tan\frac{\pi}{3}} \qquad \left\{\text{Since, } \tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}\right\}$$

$$\Rightarrow \frac{2}{(x+1)^2} = \frac{\sqrt{3}-1}{1+\sqrt{3}}$$

$$\Rightarrow \frac{2}{(x+1)^2} = \frac{\sqrt{3}-1}{\sqrt{3}+1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$$

$$\Rightarrow \frac{2}{(x+1)^2} = \frac{3-1}{\left(\sqrt{3}+1\right)^2}$$

$$\Rightarrow (x+1)^2 = \left(\sqrt{3}+1\right)^2$$

$$\Rightarrow x+1 = \pm\left(\sqrt{3}+1\right)$$

$$\Rightarrow x+1 = \sqrt{3}+1 \text{ or } x+1 = -\sqrt{3}-1$$

$$\Rightarrow x = \sqrt{3}+1-1 \text{ or } x = -\sqrt{3}-2$$

$$\Rightarrow x = \sqrt{3} \text{ or } x = -\left(\sqrt{3}+2\right)$$

Given, $x > 0$, so

$$x = \sqrt{3}$$

Q10

Solve the equation for x :

$$\tan^{-1}(x+2) + \tan^{-1}(x-2) = \tan^{-1}\left(\frac{8}{79}\right), \; x > 0$$

Solution

Given,

$$\tan^{-1}(x+2)+\tan^{-1}(x-2)=\tan^{-1}\left(\frac{8}{79}\right),\ x>0$$

$$\Rightarrow \tan^{-1}\left[\frac{(x+2)+(x-2)}{1-(x+2)(x-2)}\right]=\tan^{-1}\frac{8}{79} \qquad \left\{\text{Since, } \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right\}$$

$$\Rightarrow \tan^{-1}\left[\frac{2x}{1-x^2+4}\right]=\tan^{-1}\frac{8}{79}$$

$$\Rightarrow \frac{2x}{5-x^2}=\frac{8}{79}$$

$$\Rightarrow 40-8x^2=158x$$

$$\Rightarrow 8x^2+158x-40=0$$

$$\Rightarrow 4x^2+79x-20=0$$

$$\Rightarrow 4x^2+80x-x-20=0$$

$$\Rightarrow 4x(x+20)-1(x+20)=0$$

$$\Rightarrow (4x-1)(x+20)=0$$

$$\Rightarrow (4x-1)=0 \quad \text{or} \quad x+20=0$$

$$\Rightarrow x=\frac{1}{4} \quad \text{or} \quad x=-20$$

Since, $x>0$, so

$$x=\frac{1}{4}$$

Q11

Solve the equation for x :

$$\tan^{-1}\frac{x}{2}+\tan^{-1}\frac{x}{3}=\frac{\pi}{4},\ 0<x<\sqrt{6}$$

Solution

Given,

$$\tan^{-1}\frac{x}{2}+\tan^{-1}\frac{x}{3}=\frac{\pi}{4},\quad 0<x<\sqrt{6}$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{x}{2}+\frac{x}{3}}{1-\frac{x}{2}\times\frac{x}{3}}\right]=\frac{\pi}{4} \qquad \left\{\text{Since, } \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right\}$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{5x}{6}}{\frac{\left(6-x^2\right)}{16}}\right]=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left[\frac{5x}{6-x^2}\right]=\frac{\pi}{4}$$

$$\Rightarrow \frac{5x}{6-x^2}=\tan\frac{\pi}{4}$$

$$\Rightarrow \frac{5x}{6-x^2}=1$$

$$\Rightarrow 5x=6-x^2$$

$$\Rightarrow x^2+5x-6=0$$

$$\Rightarrow x^2+6x-x-6=0$$

$$\Rightarrow x(x+6)-1(x+6)=0$$

$$\Rightarrow (x+6)(x-1)=0$$

$$\Rightarrow x=-6 \text{ or } x=1$$

But, $0<x<\sqrt{6}$, so
$x=1$

Q12

Solve the following equations for x:

$$\tan^{-1}\left(\frac{x-2}{x-4}\right)+\tan^{-1}\left(\frac{x+2}{x+4}\right)=\frac{\pi}{4}$$

Solution

$$\tan^{-1}\left(\frac{x-2}{x-4}\right)+\tan^{-1}\left(\frac{x+2}{x+4}\right)=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{x-2}{x-4}+\frac{x+2}{x+4}}{1-\left(\frac{x-2}{x-4}\right)\left(\frac{x+2}{x+4}\right)}\right)=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{(x-2)(x+4)+(x+2)(x-4)}{(x-4)(x+4)}}{\frac{(x-4)(x+4)-(x-2)(x+2)}{(x-4)(x+4)}}\right)=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{(x-2)(x+4)+(x+2)(x-4)}{(x-4)(x+4)-(x-2)(x+2)}\right)=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{x^2+2x-8+x^2-2x-8}{(x^2-16)-(x^2-4)}\right)=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{2x^2-16}{-12}\right)=\frac{\pi}{4}$$

$$\Rightarrow \left(\frac{x^2-8}{-6}\right)=\tan\frac{\pi}{4}$$

$$\Rightarrow \left(\frac{x^2-8}{-6}\right)=1$$

$$\Rightarrow x^2-8=-6$$

$$\Rightarrow x^2=2$$

$$\therefore x=\pm\sqrt{2}$$

Q13

Solve the following equations for x:

$$\tan^{-1}(2+x)+\tan^{-1}(2-x)=\tan^{-1}\frac{2}{3}, \text{where } x<-\sqrt{3} \text{ or}, x>\sqrt{3}$$

Solution

$$\tan^{-1}\frac{(2+x+2-x)}{[1-(2+x)(2-x)]}$$

$$\tan^{-1}\frac{4}{(x^2-3)}=\tan^{-1}2/3$$

$$4/(x^2-3)=2/3$$

$$x^2-3=6$$

$$x=3,-3$$

Q14

Sum the following series:

$$\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{2}{9}+\tan^{-1}\frac{4}{33}+\ldots+\tan^{-1}\frac{2^{n-1}}{1+2^{2n-1}}$$

Solution

Let $T_n = \frac{\tan^{-1} 2^{n-1}}{1 + 2^{2n-1}}$

$T_n = \tan^{-1} \frac{(2^n - 2^{n-1})}{1 + 2^n 2^{n-1}}$

$= \tan^{-1} 2^n - \tan^{-1} 2^{n-1}$

So, $T_1 = \tan^{-1} 2 - \tan^{-1} 1$

$T_2 = \tan^{-1} 4 - \tan^{-1} 2$

$T_n = \tan^{-1} 2^n - \tan^{-1} 2^{n-1}$

Adding all the terms we get

$\tan^{-1} 2^n - \tan^{-1} 1$

$\tan^{-1} 2^n - \pi / 4$

Exercise 4.12

Q1

Evaluate $\cos\left(\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}\right)$

Solution

$$\cos\left(\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}\right)$$

$$=\cos\left[\sin^{-1}\left(\frac{3}{5}\sqrt{1-\left(\frac{5}{13}\right)^2}+\frac{5}{13}\sqrt{1-\left(\frac{3}{5}\right)^2}\right)\right] \quad \left\{\text{Since } \sin^{-1}x+\sin^{-1}y=\sin^{-1}\left[x\sqrt{1-y^2}+y\sqrt{1-x^2}\right]\right\}$$

$$=\cos\left[\sin^{-1}\left(\frac{3}{5}\times\frac{12}{13}+\frac{5}{13}\times\frac{4}{5}\right)\right]$$

$$=\cos\left[\sin^{-1}\left(\frac{56}{65}\right)\right]$$

$$=\cos\left[\cos^{-1}\left(\sqrt{1-\left(\frac{56}{65}\right)^2}\right)\right] \quad \left\{\text{Since } \sin^{-1}x=\cos^{-1}\left(\sqrt{1-x^2}\right)\right\}$$

$$=\cos\left[\cos^{-1}\left(\frac{33}{65}\right)\right]$$

$$=\frac{33}{65} \quad \left\{\text{Since } \cos\left(\cos^{-1}x\right)=x \text{ as } x\in[0,\ 1]\right\}$$

Hence,

$$\cos\left(\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}\right)=\frac{33}{65}$$

Q2

Prove that $\sin^{-1}\frac{63}{65}=\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}$

Solution

$$\text{RHS} = \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5}$$

$$= \sin^{-1}\frac{5}{13} + \sin^{-1}\sqrt{1-\frac{9}{25}} \qquad \left[\text{Since } \cos^{-1}x = \sin^{-1}\sqrt{1-x^2}\right]$$

$$= \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{4}{5}$$

$$= \tan^{-1}\left(\frac{\frac{5}{13}}{\sqrt{1-\frac{25}{169}}}\right) + \tan^{-1}\left(\frac{\frac{4}{5}}{\sqrt{1-\frac{16}{25}}}\right)$$

$$= \tan^{-1}\left(\frac{5}{12}\right) + \tan^{-1}\left(\frac{4}{3}\right)$$

$$= \tan^{-1}\left(\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}\cdot\frac{4}{3}}\right)$$

$$= \tan^{-1}\left(\frac{63}{16}\right) \qquad \ldots\ldots(1)$$

Again, $\sin^{-1}\frac{63}{65} = \tan^{-1}\frac{\frac{63}{65}}{\sqrt{1-\left(\frac{63}{65}\right)^2}}$

$$= \tan^{-1}\frac{63}{16} \qquad \ldots\ldots(2)$$

Hence from (1) and (2), $\sin^{-1}\frac{63}{65} = \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5}$

Q3

Prove $\tan^{-1}\frac{63}{16} = \sin^{-1}\frac{5}{13} + \cos^{-1}\frac{3}{5}$

Solution

Let $\sin^{-1}\frac{5}{13}=x$. Then, $\sin x=\frac{5}{13}\Rightarrow\cos x=\frac{12}{13}$.

$\therefore \tan x=\frac{5}{12}\Rightarrow x=\tan^{-1}\frac{5}{12}$

$\therefore \sin^{-1}\frac{5}{13}=\tan^{-1}\frac{5}{12}$ ...(1)

Let $\cos^{-1}\frac{3}{5}=y$. Then, $\cos y=\frac{3}{5}\Rightarrow\sin y=\frac{4}{5}$.

$\therefore \tan y=\frac{4}{3}\Rightarrow y=\tan^{-1}\frac{4}{3}$

$\therefore \cos^{-1}\frac{3}{5}=\tan^{-1}\frac{4}{3}$ ...(2)

Using (1) and (2), we have

$$\begin{aligned}\text{R.H.S.}&=\sin^{-1}\frac{5}{13}+\cos^{-1}\frac{3}{5}\\&=\tan^{-1}\frac{5}{12}+\tan^{-1}\frac{4}{3}\\&=\tan^{-1}\left(\frac{\frac{5}{12}+\frac{4}{3}}{1-\frac{5}{12}\times\frac{4}{3}}\right)\quad\left[\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right]\\&=\tan^{-1}\left(\frac{15+48}{36-20}\right)\\&=\tan^{-1}\frac{63}{16}\\&=\text{L.H.S.}\end{aligned}$$

Concept Insight:

As L.H.S is $\tan^{-1}$ express the terms in R.H.S in the form of $\tan^{-1}$

Q4

Prove $\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}=\frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}$

Solution

$$\begin{aligned}\text{L.H.S.}&=\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}\\&=\frac{9}{4}\left(\frac{\pi}{2}-\sin^{-1}\frac{1}{3}\right)\\&=\frac{9}{4}\left(\cos^{-1}\frac{1}{3}\right)\quad ....(1)\quad\left[\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\right]\end{aligned}$$

Now, let $\cos^{-1}\frac{1}{3}=x$. Then, $\cos x=\frac{1}{3}\Rightarrow\sin x=\sqrt{1-\left(\frac{1}{3}\right)^2}=\frac{2\sqrt{2}}{3}$.

$\therefore x=\sin^{-1}\frac{2\sqrt{2}}{3}\Rightarrow\cos^{-1}\frac{1}{3}=\sin^{-1}\frac{2\sqrt{2}}{3}$

$\therefore \text{L.H.S.}=\frac{9}{4}\sin^{-1}\frac{2\sqrt{2}}{3}=\text{R.H.S.}$

Q5

Solve the following:

$$\sin^{-1} x + \sin^{-1} 2x = \frac{\pi}{3}$$

Solution

$$\sin^{-1} x + \sin^{-1} 2x = \frac{\pi}{3}$$
$$\Rightarrow \sin^{-1} x + \sin^{-1} 2x = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$
$$\Rightarrow \sin^{-1} 2x = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1} x$$
$$\Rightarrow \sin^{-1} 2x = \sin^{-1}\left(\frac{\sqrt{3}}{2}\sqrt{1-x^2} - x\sqrt{1-\frac{3}{4}}\right)$$
$$\Rightarrow 2x = \left(\frac{\sqrt{3}}{2}\sqrt{1-x^2} - \frac{x}{2}\right)$$
$$\Rightarrow \frac{5}{2}x = \frac{\sqrt{3}}{2}\sqrt{1-x^2}$$
$$\Rightarrow 25x^2 = 3\left(1-x^2\right)$$
$$\Rightarrow 28x^2 = 3$$
$$\Rightarrow x^2 = \frac{3}{28}$$
$$\Rightarrow x = \frac{1}{2}\sqrt{\frac{1}{7}}$$

Q6

Solve the following :

$$\cos^{-1} x + \sin^{-1} \frac{x}{2} = \frac{\pi}{6}$$

Solution

$$\cos^{-1}x + \sin^{-1}\frac{x}{2} = \frac{\pi}{6}$$

$$\Rightarrow \sin^{-1}\frac{x}{2} = \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}\left(\sqrt{1-x^2}\right)$$

$$\Rightarrow \sin^{-1}\frac{x}{2} = \sin^{-1}\left[\frac{1}{2}\sqrt{1-1+x^2} - \sqrt{1-x^2}\sqrt{1-\frac{1}{4}}\right]$$

$$\Rightarrow \frac{x}{2} = \frac{x}{2} - \frac{\sqrt{3}\sqrt{1-x^2}}{2}$$

$$\Rightarrow \frac{\sqrt{3}\sqrt{1-x^2}}{2} = 0$$

$$\Rightarrow \sqrt{1-x^2} = 0$$

$$\Rightarrow x = \pm 1$$

Exercise 4.13

Q1

If $\cos^{-1}\frac{x}{2}+\cos^{-1}\frac{y}{3}=\alpha$, then prove that $9x^2-12xy\cos\alpha+4y^2=36\sin^2\alpha$.

Solution

Given

$$\cos^{-1}\frac{x}{2}+\cos^{-1}\frac{y}{3}=\alpha$$

$$\cos\left[\frac{x}{2}\times\frac{y}{3}-\sqrt{1-\left(\frac{x}{2}\right)^2}\sqrt{1-\left(\frac{y}{3}\right)^2}\right]=\alpha \qquad \left\{\text{Since } \cos^{-1}x+\cos^{-1}y=\cos^{-1}\left[xy-\sqrt{1-x^2}\sqrt{1-y^2}\right]\right\}$$

$$\frac{xy}{6}-\frac{\sqrt{4-x^2}}{2}\frac{\sqrt{9-y^2}}{3}=\cos\alpha$$

$$xy-\sqrt{4-x^2}\sqrt{9-y^2}=6\cos\alpha$$

$$xy-6\cos\alpha=\sqrt{4-x^2}\sqrt{9-y^2}$$

Squaring both the sides,

$$(xy-6\cos\alpha)^2=\left(4-x^2\right)\left(9-y^2\right)$$

$$x^2y^2+36\cos^2\alpha-12xy\cos\alpha=36-9x^2-4y^2+x^2y^2$$

$$9x^2+4y^2-x^2y^2-36+x^2y^2+36\cos^2\alpha-12xy\cos\alpha=0$$

$$9x^2+4y^2-12xy\cos\alpha-36\left(1-\cos^2\alpha\right)=0$$

$$9x^2+4y^2-12xy\cos\alpha-36\sin^2\alpha=0$$

$$9x^2+4y^2-12xy\cos\alpha=36\sin^2\alpha$$

Q2

Solve the equation:

$$\cos^{-1}\frac{a}{x}-\cos^{-1}\frac{b}{x}=\cos^{-1}\frac{1}{b}-\cos^{-1}\frac{1}{a}$$

Solution

$$\cos^{-1}\frac{a}{x}-\cos^{-1}\frac{b}{x}=\cos^{-1}\frac{1}{b}-\cos^{-1}\frac{1}{a}$$
$$\cos^{-1}\frac{a}{x}+\cos^{-1}\frac{1}{a}=\cos^{-1}\frac{b}{x}+\cos^{-1}\frac{1}{b}$$
$$\cos^{-1}\left[\frac{1}{x}-\sqrt{1-\frac{a^2}{x^2}}\sqrt{1-\frac{1}{a^2}}\right]=\cos^{-1}\left[\frac{1}{x}-\sqrt{1-\frac{b^2}{x^2}}\sqrt{1-\frac{1}{b^2}}\right]$$
$$\frac{1}{x}-\sqrt{1-\frac{a^2}{x^2}}\sqrt{1-\frac{1}{a^2}}=\frac{1}{x}-\sqrt{1-\frac{b^2}{x^2}}\sqrt{1-\frac{1}{b^2}}$$
$$\sqrt{1-\frac{a^2}{x^2}}\sqrt{1-\frac{1}{a^2}}=\sqrt{1-\frac{b^2}{x^2}}\sqrt{1-\frac{1}{b^2}}$$
$$\left(1-\frac{a^2}{x^2}\right)\left(1-\frac{1}{a^2}\right)=\left(1-\frac{b^2}{x^2}\right)\left(1-\frac{1}{b^2}\right)$$
$$1-\frac{1}{a^2}-\frac{a^2}{x^2}+\frac{1}{x^2}=1-\frac{1}{b^2}-\frac{b^2}{x^2}+\frac{1}{x^2}$$
$$\frac{b^2}{x^2}-\frac{a^2}{x^2}=\frac{1}{a^2}-\frac{1}{b^2}$$
$$\left(b^2-a^2\right)a^2b^2=x^2\left(b^2-a^2\right)$$
$$x^2=a^2b^2$$
$$x=ab$$

Q3

Solve :

$$\cos^{-1}\sqrt{3}x+\cos^{-1}x=\frac{\pi}{2}$$

Solution

$$\cos^{-1}\sqrt{3}x+\cos^{-1}x=\frac{\pi}{2}$$
$$\cos^{-1}\left[\sqrt{3}x^2-\sqrt{1-3x^2}\sqrt{1-x^2}\right]=\frac{\pi}{2}$$
$$\sqrt{3}x^2-\sqrt{1-3x^2}\sqrt{1-x^2}=0$$
$$\sqrt{3}x^2=\sqrt{1-3x^2}\sqrt{1-x^2}$$
$$3x^4=1-x^2-3x^2+3x^4$$
$$4x^2-1=0$$
$$x^2=\frac{1}{4}$$
$$x=\pm\frac{1}{2}$$

Q4

Prove that:

$$\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}=\cos^{-1}\frac{33}{65}$$

Solution

$$\cos^{-1}\frac{4}{5}+\cos^{-1}\frac{12}{13}$$

$$=\cos^{-1}\left[\frac{4}{5}\times\frac{12}{13}-\sqrt{1-\frac{16}{25}}\sqrt{1-\frac{144}{169}}\right]$$

$$=\cos^{-1}\left[\frac{48}{65}-\frac{3}{5}\times\frac{5}{13}\right]$$

$$=\cos^{-1}\left[\frac{48}{65}-\frac{15}{65}\right]$$

$$=\cos^{-1}\left[\frac{33}{65}\right]$$

Exercise 4.14

Q1

Evaluate $\tan\left\{2\tan^{-1}\frac{1}{5}-\frac{\pi}{4}\right\}$

Solution

$\tan\left\{2\tan^{-1}\frac{1}{5}-\frac{\pi}{4}\right\}$

$$=\tan\left\{\tan^{-1}\frac{2\times\left(\frac{1}{5}\right)}{1-\left(\frac{1}{5}\right)^2}-\tan^{-1}(1)\right\} \qquad \left\{\text{Sicne } 2\tan^{-1}=\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right\}$$

$$=\tan\left\{\tan^{-1}\frac{\frac{2}{5}}{\frac{24}{25}}-\tan^{-1}(1)\right\}$$

$$=\tan\left\{\tan^{-1}\frac{5}{12}-\tan^{-1}1\right\}$$

$$=\tan\left\{\tan^{-1}\left(\frac{\frac{5}{12}-1}{1+\frac{5}{12}\times 1}\right)\right\} \qquad \left\{\text{Since } \tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right)\right\}$$

$$=\tan\left\{\tan^{-1}\left(\frac{\frac{-7}{12}}{\frac{17}{12}}\right)\right\}$$

$$=\tan\left\{\tan^{-1}\left(-\frac{7}{17}\right)\right\}$$

$$=-\frac{7}{17} \qquad \left\{\text{Since } \tan\left(\tan^{-1}x\right)=x \text{ if } x\in R\right\}$$

Hence,

$$\tan\left\{2\tan^{-1}\frac{1}{5}-\frac{\pi}{4}\right\}=-\frac{7}{17}$$

Q2

Evaluate the following

$\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)$

Solution

$$Let\ \frac{1}{2}\sin^{-1}\frac{3}{4}=x$$
$$\sin^{-1}\frac{3}{4}=2x$$
$$\sin 2x=\frac{3}{4}$$
$$\cos 2x=\frac{\sqrt{7}}{4}$$

$$\tan\left(\frac{1}{2}\sin^{-1}\frac{3}{4}\right)$$
$$=\tan x$$
$$=\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}$$
$$=\sqrt{\frac{1-\frac{\sqrt{7}}{4}}{1+\frac{\sqrt{7}}{4}}}$$
$$=\sqrt{\frac{4-\sqrt{7}}{4+\sqrt{7}}}$$
$$=\sqrt{\frac{(4-\sqrt{7})(4-\sqrt{7})}{(4+\sqrt{7})(4-\sqrt{7})}}$$
$$=\sqrt{\frac{(4-\sqrt{7})^2}{9}}$$
$$=\frac{4-\sqrt{7}}{3}$$

Q3

Evaluate $\sin\left(\frac{1}{2}\cos^{-1}\frac{4}{5}\right)$

Solution

$$\sin\left(\frac{1}{2}\cos^{-1}\frac{4}{5}\right)$$

$$=\sin\left\{\frac{1}{2}2\sin^{-1}\left(\pm\sqrt{\frac{1-\frac{4}{5}}{2}}\right)\right\}\qquad\left\{\text{Since } \cos^{-1}x=2\sin^{-1}\left(\pm\sqrt{\frac{1-x}{2}}\right)\right\}$$

$$=\sin\left\{\sin^{-1}\left(\pm\frac{1}{\sqrt{10}}\right)\right\}$$

$$=\pm\frac{1}{\sqrt{10}}\qquad\left\{\text{Since } \sin\left(\sin^{-1}x\right)=x \text{ as } n\in[-1,\ 1]\right\}$$

Hence,

$$\sin\left(\frac{1}{2}\cos^{-1}\frac{4}{5}\right)=\pm\frac{1}{\sqrt{10}}$$

Q4

Evaluate the following:

$$\sin\left(2\tan^{-1}\frac{2}{3}\right)+\cos\left(\tan^{-1}\sqrt{3}\right)$$

Solution

$$\sin\left(2\tan^{-1}\frac{2}{3}\right)+\cos\left(\tan^{-1}\sqrt{3}\right)$$

$$=\sin\left(\sin^{-1}\left(\frac{\frac{4}{3}}{1+\frac{4}{9}}\right)\right)+\cos\left(\cos^{-1}\left(\frac{1}{\sqrt{1+3}}\right)\right)$$

$$=\sin\left(\sin^{-1}\left(\frac{12}{13}\right)\right)+\cos\left(\cos^{-1}\left(\frac{1}{2}\right)\right)$$

$$=\frac{12}{13}+\frac{1}{2}$$

$$=\frac{37}{26}$$

Q5

Prove that $2\sin^{-1}\frac{3}{5}=\tan^{-1}\left(\frac{24}{7}\right)$

Solution

$$2\sin^{-1}\frac{3}{5} = \tan^{-1}\left(\frac{24}{7}\right)$$

LHS $= 2\sin^{-1}\frac{3}{5}$

$$= 2\times\tan^{-1}\left(\frac{\frac{3}{5}}{\sqrt{1-\left(\frac{3}{5}\right)^2}}\right) \quad \left\{\text{Since } \sin^{-1}x = \tan^{-1}\frac{x}{\sqrt{1-x^2}}\right\}$$

$$= 2\tan^{-1}\left(\frac{\frac{3}{5}}{\frac{4}{5}}\right)$$

$$= 2\tan^{-1}\left(\frac{3}{4}\right)$$

$$= \tan^{-1}\left(\frac{2\times\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2}\right) \quad \left\{\text{Since } 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right\}$$

$$= \tan^{-1}\left(\frac{\frac{3}{2}}{\frac{7}{16}}\right)$$

$$= \tan^{-1}\left(\frac{24}{7}\right)$$

$= \text{RHS}$

So,

$$2\sin^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{24}{7}\right)$$

Q6

Prove that $\tan^{-1}\left(\frac{1}{4}\right) + \tan^{-1}\left(\frac{2}{9}\right) = \frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right) = \frac{1}{2}\sin^{-1}\left(\frac{4}{5}\right)$

Solution

$$\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{2}{9}\right)=\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)=\frac{1}{2}\sin^{-1}\left(\frac{4}{5}\right)$$

LHS $=\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{2}{9}\right)$

$$=\tan^{-1}\left(\frac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}\times\frac{2}{9}}\right) \qquad \left\{\text{Since } \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right\}$$

$$=\tan^{-1}\left(\frac{\frac{9+8}{36}}{\frac{36-2}{36}}\right)$$

$$=\tan^{-1}\left(\frac{17}{36}\times\frac{36}{34}\right)$$

$$=\tan^{-1}\left(\frac{1}{2}\right)$$

Multiplying and dividing by 2,

$$=\frac{1}{2}\left\{2\tan^{-1}\left(\frac{1}{2}\right)\right\}$$

$$=\frac{1}{2}\left\{\cos^{-1}\left(\frac{1-\left(\frac{1}{2}\right)^2}{1+\left(\frac{1}{2}\right)^2}\right)\right\} \qquad \left\{\text{Since } 2\tan^{-1}x=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right\}$$

$$=\frac{1}{2}\left\{\cos^{-1}\left(\frac{1-\frac{1}{4}}{1+\frac{1}{4}}\right)\right\}$$

$$=\frac{1}{2}\left\{\cos^{-1}\left(\frac{3}{4}\times\frac{4}{5}\right)\right\}$$

$$=\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$$

So,

$$\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{2}{9}\right)=\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$$

$$=\frac{1}{2}\cos^{-1}\left(\frac{3}{5}\right)$$

$$=\frac{1}{2}\sin^{-1}\left(\frac{4}{5}\right) \qquad \left\{\text{Since } \cos^{-1}x=\sin^{-1}\left(\sqrt{1-x^2}\right)\right\}$$

So,

$$\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{2}{9}\right)=\frac{1}{2}\sin^{-1}\left(\frac{4}{5}\right)$$

Q7

Prove that $\tan^{-1}\left(\frac{2}{3}\right)=\frac{1}{2}\tan^{-1}\left(\frac{12}{5}\right)$

Solution

$$\tan^{-1}\left(\frac{2}{3}\right) = \frac{1}{2}\tan^{-1}\left(\frac{12}{5}\right)$$

LHS $= \tan^{-1}\left(\frac{2}{3}\right)$

Dividing and multiplying by 2,

$$= \frac{1}{2}\left\{2\tan^{-1}\left(\frac{2}{3}\right)\right\}$$

$$= \frac{1}{2}\left\{\tan^{-1}\left(\frac{2\left(\frac{2}{3}\right)}{1-\left(\frac{2}{3}\right)^2}\right)\right\} \qquad \left\{\text{Since } 2\tan^{-1}x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right\}$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{\frac{4}{3}}{\frac{5}{9}}\right)$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{4}{3}\times\frac{9}{5}\right)$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{12}{5}\right)$$

$$\tan^{-1}\left(\frac{2}{3}\right) = \frac{1}{2}\tan^{-1}\left(\frac{12}{5}\right)$$

Q8

Prove that $\tan^{-1}\left(\frac{1}{7}\right) + 2\tan^{-1}\left(\frac{1}{3}\right) = \frac{\pi}{4}$

Solution

$$\tan^{-1}\left(\frac{1}{7}\right)+2\tan^{-1}\left(\frac{1}{3}\right)=\frac{\pi}{4}$$

$$\text{LHS} = \tan^{-1}\left(\frac{1}{7}\right)+2\tan^{-1}\left(\frac{1}{3}\right)$$

$$= \tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{2\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)^2}\right) \quad \left\{\text{Since } 2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right\}$$

$$= \tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{2}{3}\times\frac{9}{8}\right)$$

$$= \tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{3}{4}\right)$$

$$= \tan^{-1}\left(\frac{\frac{1}{7}+\frac{3}{4}}{1-\frac{1}{7}\times\frac{3}{4}}\right) \quad \left\{\text{Since } \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right\}$$

$$= \tan^{-1}\left(\frac{\frac{25}{28}}{\frac{25}{28}}\right)$$

$$= \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

= RHS Hence, proved

$$\tan^{-1}\left(\frac{1}{7}\right)+2\tan^{-1}\left(\frac{1}{3}\right)=\frac{\pi}{4}$$

Q9

Prove that $\sin^{-1}\frac{4}{5}+2\tan^{-1}\frac{1}{3}=\frac{\pi}{2}$

Solution

$\sin^{-1}\frac{4}{5}+2\tan^{-1}\frac{1}{3}=\frac{\pi}{2}$

LHS $=\sin^{-1}\frac{4}{5}+2\tan^{-1}\left(\frac{1}{3}\right)$

$=\tan^{-1}\left(\frac{\frac{4}{5}}{\sqrt{1-\left(\frac{4}{5}\right)^2}}\right)+\tan^{-1}\left(\frac{2\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)^2}\right)$

$\left\{\text{Since } \sin^{-1}x=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right) \text{ and } 2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right\}$

$=\tan^{-1}\left(\frac{\frac{4}{5}}{\frac{3}{5}9}\right)+\tan^{-1}\left(\frac{\frac{2}{3}}{\frac{8}{9}}\right)$

$=\tan^{-1}\left(\frac{4}{3}\right)+\tan^{-1}\left(\frac{3}{4}\right)$

$=\tan^{-1}\left(\frac{\frac{4}{3}+\frac{3}{4}}{1-\frac{4}{3}\times\frac{3}{4}}\right)$

$=\tan^{-1}\left(\frac{\frac{25}{12}}{0}\right)$

$=\tan^{-1}(\infty)$

$=\frac{\pi}{2}$

$=$ RHS

So,

$\sin^{-1}\frac{4}{5}+2\tan^{-1}\frac{1}{3}=\frac{\pi}{2}$

Q10

Prove that $2\sin^{-1}\frac{3}{5}-\tan^{-1}\frac{17}{31}=\frac{\pi}{4}$

Solution

$2\sin^{-1}\frac{3}{5}-\tan^{-1}\frac{17}{31}=\frac{\pi}{4}$

LHS $=2\sin^{-1}\left(\frac{3}{5}\right)-\tan^{-1}\left(\frac{17}{31}\right)$

$=2\tan^{-1}\left(\frac{\frac{3}{5}}{\sqrt{1-\left(\frac{3}{5}\right)^2}}\right)-\tan^{-1}\left(\frac{17}{31}\right)$

$=2\tan^{-1}\left(\frac{\frac{3}{5}}{\frac{4}{5}}\right)-\tan^{-1}\left(\frac{17}{31}\right)$ $\left\{\text{Since } \sin^{-1}x=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right\}$

$=2\tan^{-1}\left(\frac{3}{4}\right)-\tan^{-1}\left(\frac{17}{31}\right)$

$=\tan^{-1}\left(\frac{2\times\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2}\right)-\tan^{-1}\left(\frac{17}{31}\right)$ $\left\{\text{Since } 2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right\}$

$=\tan^{-1}\left(\frac{\frac{3}{2}}{\frac{7}{16}}\right)-\tan^{-1}\left(\frac{17}{31}\right)$

$=\tan^{-1}\left(\frac{24}{7}\right)-\tan^{-1}\left(\frac{17}{31}\right)$

$=\tan^{-1}\left(\frac{\frac{24}{7}-\frac{17}{31}}{1+\frac{24}{7}y\times\frac{17}{31}}\right)$ $\left\{\text{Since } \tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right)\right\}$

$=\tan^{-1}\left(\frac{\frac{744-119}{217}}{\frac{217+408}{217}}\right)$

$=\tan^{-1}\left(\frac{625}{625}\right)$

$=\tan^{-1}(1)$

$=\frac{\pi}{4}$

Hence,

$2\sin^{-1}\left(\frac{3}{5}\right)-\tan^{-1}\left(\frac{17}{31}\right)=\frac{\pi}{4}$

Q11

Prove the result $2\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}=\tan^{-1}\frac{4}{7}$

Solution

$$\text{LHS} = 2\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8}$$

$$= \tan^{-1}\frac{2\cdot\frac{1}{5}}{1-\left(\frac{1}{5}\right)^2} + \tan^{-1}\frac{1}{8} \qquad \left[\text{Since } 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$= \tan^{-1}\frac{\frac{2}{5}}{\frac{24}{25}} + \tan^{-1}\frac{1}{8}$$

$$= \tan^{-1}\frac{5}{12} + \tan^{-1}\frac{1}{8}$$

$$= \tan^{-1}\left(\frac{\frac{5}{12}+\frac{1}{8}}{1-\frac{5}{12}\cdot\frac{1}{8}}\right) \qquad \left[\text{Since } \tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\left(\frac{\frac{10+3}{24}}{\frac{96-5}{96}}\right)$$

$$= \tan^{-1}\left(\frac{13}{24}\cdot\frac{96}{91}\right)$$

$$= \tan^{-1}\left(\frac{4}{7}\right)$$

$$= \text{RHS}$$

Hence, $2\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{8} = \tan^{-1}\frac{4}{7}$

Q12

Prove that $2\tan^{-1}\frac{3}{4} - \tan^{-1}\frac{17}{31} = \frac{\pi}{4}$

Solution

$$\text{LHS} = 2\tan^{-1}\frac{3}{4} - \tan^{-1}\frac{17}{31}$$

$$= \tan^{-1}\frac{2\cdot\frac{3}{4}}{1-\left(\frac{3}{4}\right)^2} - \tan^{-1}\frac{17}{31} \qquad \left[\text{Since } 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$= \tan^{-1}\left(\frac{3}{2}\cdot\frac{16}{7}\right) - \tan^{-1}\frac{17}{31}$$

$$= \tan^{-1}\frac{24}{7} - \tan^{-1}\frac{17}{31}$$

$$= \tan^{-1}\left(\frac{\frac{24}{7}-\frac{17}{31}}{1+\frac{24}{7}\cdot\frac{17}{31}}\right) \qquad \left[\text{Since } \tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}\right]$$

$$= \tan^{-1}\left(\frac{\frac{744-119}{217}}{\frac{217+408}{217}}\right)$$

$$= \tan^{-1}\left(\frac{625}{625}\right)$$

$$= \tan^{-1}1$$

$$= \frac{\pi}{4}$$

$$= \text{RHS}$$

Hence, $2\tan^{-1}\frac{3}{4} - \tan^{-1}\frac{17}{31} = \frac{\pi}{4}$

Q13

Prove that $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$

Solution

$$\text{LHS} = 2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\frac{2\cdot\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2} + \tan^{-1}\frac{1}{7} \qquad \left[\text{Since } 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}\right]$$

$$= \tan^{-1}\frac{4}{3} + \tan^{-1}\frac{1}{7}$$

$$= \tan^{-1}\left(\frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\cdot\frac{1}{7}}\right) \qquad \left[\text{Since } \tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}\right]$$

$$= \tan^{-1}\left(\frac{\frac{28+3}{21}}{\frac{21-4}{21}}\right)$$

$$= \tan^{-1}\left(\frac{31}{17}\right)$$

$$= \text{RHS}$$

Hence, $2\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{7} = \tan^{-1}\frac{31}{17}$

Q14

Prove the following result :

$$4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} = \frac{\pi}{4}$$

Solution

$$4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$$

$$= \tan^{-1}\left[\frac{4\left(\frac{1}{5}\right) - 4\left(\frac{1}{5}\right)^3}{1-6\left(\frac{1}{5}\right)^2 + \left(\frac{1}{5}\right)^4}\right] - \tan^{-1}\frac{1}{239} \ldots\ldots\ldots \left[4\tan^{-1}(x) = \tan^{-1}\left(\frac{4x-4x^3}{1-6x^2+x^4}\right)\right]$$

$$= \tan^{-1}\left[\frac{120}{119}\right] - \tan^{-1}\frac{1}{239}$$

$$= \tan^{-1}\left(\frac{120\times 239 - 119}{119\times 239 + 120}\right) \ldots\ldots\ldots\ldots \left[\tan^{-1}(x) - \tan^{-1}(y) = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right]$$

$$= \tan^{-1}\left(\frac{28561}{28561}\right)$$

$$= \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

Q15

If $\sin^{-1}\frac{2a}{1+a^2}-\cos^{-1}\frac{1-b^2}{1+b^2}=\tan^{-1}\frac{2x}{1-x^2}$, then prove that $x=\frac{a-b}{1+ab}$

Solution

Given

$$\sin^{-1}\left(\frac{2a}{1+a^2}\right)-\cos^{-1}\left(\frac{1-b^2}{1+b^2}\right)=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

$\Rightarrow$ $2\tan^{-1}a-2\tan^{-1}b=2\tan^{-1}x$

$\left\{\text{Since, } 2\tan^{-1}x=\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)=\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right\}$

$\Rightarrow$ $2\left\{\tan^{-1}a-\tan^{-1}b\right\}=2\tan^{-1}x$

$\Rightarrow$ $\tan^{-1}a-\tan^{-1}b=\tan^{-1}x$

$\Rightarrow$ $\tan^{-1}\left(\frac{a-b}{1+ab}\right)=\tan^{-1}x$

$\left\{\text{Since } \tan^{-1}x-\tan^{-1}y=\tan^{-1}\left(\frac{x-y}{1+xy}\right)\right\}$

$\Rightarrow$ On comparing, we get

$$\frac{a-b}{1+ab}=x$$

Q16

Prove that

$$\tan^{-1}\left(\frac{1-x^2}{2x}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\frac{\pi}{2}$$

Solution

$$\tan^{-1}\left(\frac{1-x^2}{2x}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\frac{\pi}{2}$$

LHS $=\tan^{-1}\left(\frac{1-x^2}{2x}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right)$

$=\tan^{-1}\left(\frac{1-x^2}{2x}\right)+\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ $\left\{\text{Since } \cot^{-1}x=\tan^{-1}\left(\frac{1}{x}\right)\right\}$

$=\tan^{-1}\left[\frac{\frac{(1-x^2)}{2x}+\left(\frac{2x}{1-x^2}\right)}{1-\left(\frac{1-x^2}{2x}\right)\left(\frac{2x}{1-x^2}\right)}\right]$ $\left\{\text{Since } \tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right\}$

$=\tan^{-1}\left[\frac{\frac{1+x^4-2x^2+4x^2}{2x(1-x^2)}}{\frac{2x(1-x^2)-2x(1-x^2)}{2x(1-x^2)}}\right]$

$=\tan^{-1}\left[\frac{1+x^4+2x^2}{0}\right]$

$=\tan^{-1}(\infty)$

$=\frac{\pi}{2}$

$=$ RHS

$$\tan^{-1}\left(\frac{1-x^2}{2x}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\frac{\pi}{2}$$

Q17

Prove that

$$\sin\left[\tan^{-1}\left(\frac{1-x^2}{2x}\right)+\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]=1$$

Solution

$$\sin\left[\tan^{-1}\left(\frac{1-x^2}{2x}\right)+\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]=1$$

LHS $=\sin\left[\tan^{-1}\left(\frac{1-x^2}{2x}\right)+\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]$

$=\sin\left[\tan^{-1}\left(\frac{1-x^2}{2x}\right)+2\tan^{-1}x\right]$ $\left\{\text{Since } 2\tan^{-1}x=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right\}$

$=\sin\left[\tan^{-1}\right]\left(\frac{1-x^2}{2x}\right)+\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ $\left\{\text{Since } 2\tan^{-1}x=\tan\frac{2x}{1-x^2}\right\}$

$=\sin\left[\tan^{-1}\left(\frac{\frac{1-x^2}{2x}+\frac{2x}{1-x^2}}{1-\frac{1-x^2}{2x}\times\frac{2x}{1-x^2}}\right)\right]$ $\left\{\text{Since } \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right\}$

$=\sin\left[\tan^{-1}\left(\frac{\frac{1+x^4-2x^2+4x^2}{2x\left(1-x^2\right)}}{0}\right)\right]$

$=\sin\left[\tan^{-1}(\infty)\right]$

$=\sin\left[\frac{\pi}{2}\right]$

$=1$

$=$ RHS

Hence,

$$\sin\left[\tan^{-1}\left(\frac{1-x^2}{2x}\right)+\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]=1$$

Q18

If $\sin^{-1}\frac{2a}{1+a^2}+\sin^{-1}\frac{2b}{1+b^2}=2\tan^{-1}x$, prove that $x=\frac{a+b}{1-ab}$.

Solution

Given,

$$\sin^{-1}\left(\frac{2a}{1+a^2}\right)+\sin^{-1}\left(\frac{2b}{1+b^2}\right)=2\tan^{-1}x$$

$\Rightarrow\quad 2\tan^{-1}a+2\tan^{-1}b=2\tan^{-1}x$ $\left\{\text{Since, } 2\tan^{-1}x=\sin^{-1}\left(\frac{2x}{1+x}\right)\right\}$

$\Rightarrow\quad 2\left\{\tan^{-1}a+\tan^{-1}b\right\}=2\tan^{-1}x$

$\Rightarrow\quad \tan^{-1}\left(\frac{a+b}{1-ab}\right)=\tan^{-1}x$ $\left\{\text{Since, } \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right\}$

$\Rightarrow\quad \frac{a+b}{1-ab}=x$

Q19

Show that $2\tan^{-1}x + \sin^{-1}\frac{2x}{1+x^2}$ is constant for $x \geq 1$, find that constant.

Solution

$2\tan^{-1}x + \sin^{-1}\frac{2x}{1+x^2}$

$= 2\tan^{-1}x + 2\tan^{-1}x$ $\left\{\text{Since, } 2\tan^{-1}x = \sin\frac{2x}{1+x^2}\right\}$

$= 4\tan^{-1}x$

For $x \geq 1$

$= 4 \times \tan^{-1}(1)$

$= 4 \times \frac{\pi}{4}$

$= \pi$ (Constant)

Hence,

$$2\tan^{-1}x + \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \pi$$

Q20

Find the value of $\tan^{-1}\left\{2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right\}$

Solution

$\tan^{-1}\left\{2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right\}$

$= \tan^{-1}\left\{2\cos\left(2\times\frac{\pi}{6}\right)\right\}$

$= \tan^{-1}\left\{2\cos\frac{\pi}{3}\right\}$

$= \tan^{-1}\left\{2\times\frac{1}{2}\right\}$

$= \tan^{-1}(1)$

$= \frac{\pi}{4}$

Hence,

$$\tan^{-1}\left\{2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right\} = \frac{\pi}{4}$$

Q21

Find the value of $\cos\left(\sec^{-1}x + \operatorname{cosec}^{-1}x\right)$, $|x| \geq 1$

Solution

$\cos\left(\sec^{-1}x + \operatorname{cosec}^{-1}x\right), \quad |x| \geq 1$

$= \cos\left(\frac{\pi}{2}\right) \quad \left\{\text{Since, } \sec^{-1}x + \operatorname{cosec}^{-1}x = \frac{\pi}{2}\right\}$

$= 0$

Hence,

$\cos\left(\sec^{-1}x + \operatorname{cosec}^{-1}x\right) = 0$

Q22

Solve the equation for x :

$\tan^{-1}\frac{1}{4} + 2\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{6} + \tan^{-1}\frac{1}{x} = \frac{\pi}{4}$

Solution

Given,

$$\tan^{-1}\frac{1}{4}+2\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{6}+\tan^{-1}\frac{1}{x}=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\frac{1}{4}+\tan^{-1}\left(\frac{2\times\frac{1}{5}}{1-\left(\frac{1}{5}\right)^2}\right)+\tan^{-1}\frac{1}{6}+\tan\frac{1}{x}=\frac{\pi}{4} \qquad \left\{\text{Since, } 2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right\}$$

$$\Rightarrow \tan^{-1}\frac{1}{4}+\tan^{-1}\left(\frac{\frac{2}{5}}{\frac{24}{25}}\right)+\tan^{-1}\frac{1}{6}+\tan^{-1}\frac{1}{x}=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\frac{1}{4}+\tan^{-1}\frac{5}{12}+\tan^{-1}\frac{1}{6}+\tan^{-1}\frac{1}{x}=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{1}{4}+\frac{5}{12}}{1-\frac{1}{4}\times\frac{5}{12}}\right)+\tan^{-1}\frac{1}{6}+\tan^{-1}\frac{1}{x}=\frac{\pi}{4}$$

$$\left\{\text{Since } \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right) \text{ if } xy<1\right\}$$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{8}{12}}{\frac{43}{48}}\right)+\tan^{-1}\frac{1}{6}\tan^{-1}\frac{1}{x}=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{32}{43}\right)+\tan^{-1}\frac{1}{6}+\tan^{-1}\frac{1}{x}=\frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{32}{43}+\frac{1}{6}}{1-\frac{32}{43}\times\frac{1}{6}}\right)+\tan^{-1}\frac{1}{x}=\tan^{-1}1 \qquad \left\{\text{Since } \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right) \text{ if } xy<1\right\}$$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{235}{258}}{\frac{226}{258}}\right)+\tan^{-1}\frac{1}{x}=\tan^{-1}1$$

$$\Rightarrow \tan^{-1}\left(\frac{235}{226}\right)+\tan^{-1}\frac{1}{x}=\tan^{-1}1$$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{235}{226}+\frac{1}{x}}{1-\frac{235}{226x}}\right)=\tan^{-1}1,\quad \frac{235}{226x}<1 \qquad \left\{\text{Since } \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right) \text{ if } xy<1\right\}$$

$$\Rightarrow \frac{235x+226}{226x-235}=1 \qquad x>\frac{235}{226}$$

$$\Rightarrow 235x+226=226x-235 \qquad x>\frac{235}{226}$$

$$\Rightarrow 235x-226x=-235-226$$

$$\Rightarrow x=-\frac{461}{9}$$

Q23

Solve the equation for x :

$$3\sin^{-1}\frac{2x}{1+x^2}-4\cos^{-1}\frac{1-x^2}{1+x^2}+2\tan^{-1}\frac{2x}{1-x^2}=\frac{\pi}{3}$$

Solution

Given,

$$3\sin^{-1}\frac{2x}{1+x^2}-4\cos^{-1}\frac{1-x^2}{1+x^2}+2\tan^{-1}\frac{2x}{1-x^2}=\frac{\pi}{3}$$

$$\Rightarrow \quad 3\left(2\tan^{-1}x\right)-4\left(2\tan^{-1}x\right)+2\left(2\tan^{-1}x\right)=\frac{\pi}{3}$$

$$\left\{\text{Since, } 2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}=\sin^{-1}\frac{2x}{1+x^2}=\cos^{-1}\frac{1-x^2}{1+x^2}\right\}$$

$$\Rightarrow \quad 6\tan^{-1}x-8\tan^{-1}x+4\tan^{-1}x=\frac{\pi}{3}$$

$$\Rightarrow \quad 2\tan^{-1}x=\frac{\pi}{3}$$

$$\Rightarrow \quad \tan^{-1}x=\frac{\pi}{6}$$

$$\Rightarrow \quad \tan^{-1}x=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\Rightarrow \quad x=\frac{1}{\sqrt{3}}$$

Q24

Solve the equation for x :

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\frac{2\pi}{3},\ x>0$$

Solution

Given,

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right)=\frac{2\pi}{3},\ x>0$$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2x}{1-x^2}\right)+\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{2\pi}{3},\ x>0 \qquad \left\{\text{Since, } \cot^{-1}x=\tan^{-1}\frac{1}{x}\right\}$$

$$\Rightarrow \quad 2\tan^{-1}x+2\tan^{-1}x=\frac{2\pi}{3} \qquad \left\{\text{Since, } 2\tan^{-1}=\tan^{-1}\frac{2x}{1-x^2}\right\}$$

$$\Rightarrow \quad 4\tan^{-1}x=\frac{2\pi}{3}$$

$$\Rightarrow \quad \tan^{-1}x=\frac{2\pi}{12}$$

$$\Rightarrow \quad \tan^{-1}x=\frac{\pi}{6}$$

$$\Rightarrow \quad \tan^{-1}x=\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\Rightarrow \quad x=\frac{1}{\sqrt{3}}$$

Q25

Evaluate the equation $2\tan^{-1}(\sin x)=\tan^{-1}(2\sec x),\ x\neq\frac{\pi}{2}$ for x

Solution

$$2\tan^{-1}(\sin x) = \tan^{-1}(2\sec x)$$

$$\tan^{-1}\left(\frac{2\sin x}{1-\sin^2 x}\right) = \tan^{-1}(2\sec x) \qquad \left[\text{Since } 2\tan^{-1} = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$\frac{2\sin x}{\cos^2 x} = 2\sec x$$

$$\frac{\sin x}{\cos x \cdot \cos x} = \sec x$$

$$\tan x \sec x = \sec x$$

$$\tan x = 1$$

$$x = \frac{\pi}{4}$$

Hence, the value of x is $\frac{\pi}{4}$

Thus, the solution is $x = n\pi + \frac{\pi}{4}$

Q26

Prove that $2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{\theta}{2}\right) = \cos^{-1}\left(\frac{a\cos\theta + b}{a + b\cos\theta}\right)$

Solution

$$2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{\theta}{2}\right)=\cos^{-1}\left(\frac{a\cos\theta+b}{a+b\cos\theta}\right)$$

$$\text{LHS} = 2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{\theta}{2}\right)$$

$$=\cos^{-1}\left(\frac{1-\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{\theta}{2}\right)^2}{1+\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{\theta}{2}\right)^2}\right) \qquad \left\{\text{Since } 2\tan^{-1}x=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right\}$$

$$=\cos^{-1}\left(\frac{1-\left(\frac{a-b}{a+b}\right)\tan^2\frac{\theta}{2}}{1+\left(\frac{a-b}{a+b}\right)\tan^2\frac{\theta}{2}}\right)$$

$$=\cos^{-1}\left(\frac{a+b-(a-b)\tan^2\frac{\theta}{2}}{a+b+(a-b)\tan^2\frac{\theta}{2}}\right)$$

$$=\cos^{-1}\left(\frac{a\left(1-\tan^2\frac{\theta}{2}\right)+b\left(1+\tan^2\frac{\theta}{2}\right)}{a\left(1+\tan^2\frac{\theta}{2}\right)+b\left(1-\tan^2\frac{\theta}{2}\right)}\right)$$

Dividing numerator and denominator by $\left(1+\tan^2\frac{\theta}{2}\right)$, we get

$$=\cos^{-1}\left(\frac{a\left(\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right)+b}{a+b\left(\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right)}\right)$$

$$=\cos^{-1}\left(\frac{a\cos\theta+b}{a+b\cos\theta}\right) \qquad \left\{\text{Since } \cos\theta=\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right\}$$

= RHS

Hence,

$$2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{\theta}{2}\right)=\cos^{-1}\left(\frac{a\cos\theta+b}{a+b\cos\theta}\right)$$

Q27

Prove that:

$$\tan^{-1}\left(\frac{2ab}{a^2-b^2}\right)+\tan^{-1}\left(\frac{2xy}{x^2-y^2}\right)=\tan^{-1}\left(\frac{2\alpha\beta}{\alpha^2-\beta^2}\right)$$

Solution

$$\tan^{-1}\frac{2ab}{a^2-b^2}+\tan^{-1}\frac{2xy}{x^2-y^2}=\tan^{-1}\frac{2\alpha\beta}{\alpha^2-\beta^2} \text{ as } \alpha=ax-by,\ \beta=ay+bx$$

$$\text{LHS} = \tan^{-1}\frac{2ab}{a^2-b^2}+\tan^{-1}\frac{2xy}{x^2-y^2}$$

$$=\tan^{-1}\left[\frac{\frac{2ab}{a^2-b^2}+\frac{2xy}{x^2-y^2}}{1-\left(\frac{2ab}{a^2-b^2}\right)\left(\frac{2xy}{x^2-y^2}\right)}\right] \qquad \left\{\text{Since, } \tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}\right\}$$

$$=\tan^{-1}\left[\frac{\frac{2abx^2-2aby^2+2xya^2-2xyb^2}{(a^2-b^2)(x^2-y^2)}}{\frac{a^2x^2-a^2y^2-b^2x^2+b^2y^2-4abxy}{(a^2-b^2)(x^2-y^2)}}\right]$$

$$=\tan^{-1}\left[\frac{2(abx^2+xya^2-aby^2-xyb^2)}{a^2x^2+b^2y^2-2abxy-a^2y^2-b^2x^2-2abxy}\right]$$

$$=\tan^{-1}\left[\frac{2\{ax(bx+ay)-by(ay+bx)\}}{(ax-by)^2-(a^2y^2+b^2x^2+2abxy)}\right]$$

$$=\tan^{-1}\left[\frac{2(bx+ay)(ax-by)}{(ax-by)^2-(bx+ay)^2}\right]$$

$$=\tan^{-1}\left[\frac{2\alpha\beta}{\alpha^2-\beta^2}\right] \qquad \{\text{Since, } bx+ay=\alpha,\ ax-by=\beta\}$$

$$=\text{RHS}$$

Hence,

$$\tan^{-1}\left(\frac{2ab}{a^2-b^2}\right)+\tan^{-1}\left(\frac{2xy}{x^2-y^2}\right)=\tan^{-1}\left(\frac{2\alpha\beta}{\alpha^2-\beta^2}\right)$$

Q28

For any $a,b,x,y>0$, prove that:

$$\frac{2}{3}\tan^{-1}\left(\frac{3ab^2-a^3}{b^3-3a^2b}\right)+\frac{2}{3}\tan^{-1}\left(\frac{3xy^2-x^3}{y^3-3x^2y}\right)=\tan^{-1}\left(\frac{2\alpha\beta}{\alpha^2-\beta^2}\right)$$

where $\alpha=-ax+by,\ \beta=bx+ay$

Solution

$$\frac{2}{3}\tan^{-1}\left(\frac{3ab^2-a^3}{b^3-3a^2b}\right)+\frac{2}{3}\tan^{-1}\left(\frac{3xy^2-x^3}{y^3-3x^2y}\right)=\tan^{-1}\left(\frac{2\alpha\beta}{\alpha^2-\beta^2}\right) \text{ as } \alpha=-ax+by,\ \beta=bx-ay$$

LHS $=\frac{2}{3}\tan^{-1}\left(\frac{3ab^2-a^3}{b^3-3a^2b}\right)+\frac{2}{3}\tan^{-1}\left(\frac{3xy^2-x^3}{y^3-3x^2y}\right)$

$$=\frac{2}{3}\tan^{-1}\left(\frac{\frac{3ab^2-a^3}{b^3}}{\frac{b^3-3a^2b}{b^3}}\right)+\frac{2}{3}\tan^{-1}\left(\frac{\frac{3xy^2-x^3}{y^3}}{\frac{y^3-3x^2y}{y^3}}\right)$$

{Dividing Numerator and denominator of first function and second function by b^3 and y^3 respectively.}

$$=\frac{2}{3}\tan^{-1}\left(\frac{3\left(\frac{a}{b}\right)-\left(\frac{a}{b}\right)^3}{1-3\left(\frac{a}{b}\right)^2}\right)+\frac{2}{3}\tan^{-1}\left(\frac{3\left(\frac{x}{y}\right)-\left(\frac{x}{y}\right)^3}{1-3\left(\frac{x}{y}\right)^2}\right)$$

$$=\frac{2}{3}\left\{3\tan^{-1}\left(\frac{a}{b}\right)\right\}+\frac{2}{3}\left\{3\tan^{-1}\left(\frac{x}{y}\right)\right\} \quad \left\{\text{Since, } 3\tan^{-1}x=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)\right\}$$

$$=2\tan^{-1}\left(\frac{a}{b}\right)+2\tan^{-1}\left(\frac{x}{y}\right)$$

$$=2\left\{\tan^{-1}\left(\frac{a}{b}\right)+\tan^{-1}\left(\frac{x}{y}\right)\right\}$$

$$=2\tan^{-1}\left\{\frac{\frac{a}{b}+\frac{x}{y}}{1-\frac{a}{b}\times\frac{x}{y}}\right\} \quad \left\{\text{Since, } \tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)\right\}$$

$$=2\tan^{-1}\left\{\frac{\frac{ay+bx}{by}}{\frac{by-ax}{by}}\right\}$$

$$=2\tan^{-1}\left(\frac{ay+bx}{by-ax}\right)$$

$$=2\tan^{-1}\left(\frac{\beta}{\alpha}\right) \quad \{\text{Since, } ay+bx=\beta,\ -ax+by=\alpha\}$$

$$=\tan^{-1}\left[\frac{2\times\frac{\beta}{\alpha}}{1-\left(\frac{\beta}{\alpha}\right)^2}\right] \quad \left\{\text{Since, } 2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right\}$$

$$=\tan^{-1}\left[\frac{2\beta}{\alpha}\times\frac{\alpha^2}{\alpha^2-\beta^2}\right]$$

$$=\tan^{-1}\left[\frac{2\alpha\beta}{\alpha^2-\beta^2}\right]$$

= RHS

Hence,

$$\frac{2}{3}\tan^{-1}\left(\frac{3ab^2-a^3}{b^3-3a^2b}\right)+\frac{2}{3}\tan^{-1}\left(\frac{3xy^2-x^3}{y^3-3x^2y}\right)=\tan^{-1}\left(\frac{2\alpha\beta}{\alpha^2-\beta^2}\right)$$

as $\alpha=-ax+by,\ \beta=bx+ay$.

Exercise MCQ

Q1

If $\tan^{-1}\left\{\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\} = \alpha$, then $x^2 =$

a. sin 2α
b. sin α
c. cos 2α
d. cos α

Solution

Correct option: (a)

$$\tan^{-1}\left\{\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}\right\} = \alpha$$

$$\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}} = \tan\alpha$$

$$\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}} \times \frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}} = \tan\alpha$$

$$\frac{1+x^2-2\sqrt{1-x^2}\sqrt{1+x^2}+1-x^2}{1+x^2-1+x^2} = \tan\alpha$$

$$\frac{1-\sqrt{1-x^4}}{x^2} = \tan\alpha$$

$$1-\sqrt{1-x^4} = x^2\tan\alpha$$

$$\left(1-x^2\tan\alpha\right)^2 = 1-x^4$$

$$1-2x^2\tan\alpha+x^4\tan^2\alpha = 1-x^4$$

$$x^4-2x^2\tan\alpha+x^4\tan^2\alpha = 0$$

$$x^2\left(x^2-2\tan\alpha+x^2\tan^2\alpha\right) = 0$$

$$x^2 = \frac{2\tan\alpha}{1+\tan^2\alpha}$$

$$x^2 = \frac{2\tan\alpha}{\sec^2\alpha}$$

$$x^2 = 2\tan\alpha\cos^2\alpha$$

$$x^2 = 2\sin\alpha\cos\alpha = \sin 2\alpha$$

Q2

The value of $\tan\left\{\cos^{-1}\frac{1}{5\sqrt{2}} - \sin^{-1}\frac{4}{\sqrt{17}}\right\}$ is

a. $\frac{\sqrt{29}}{3}$
b. $\frac{29}{3}$
c. $\frac{\sqrt{3}}{29}$
d. $\frac{3}{29}$

Solution

Correct option: (d)

Given that to find $\tan\left\{\cos^{-1}\frac{1}{5\sqrt{2}} - \sin^{-1}\frac{4}{\sqrt{17}}\right\}$

Put, $\cos^{-1}\frac{1}{5\sqrt{2}} = u$ and $\sin^{-1}\frac{4}{\sqrt{17}} = v$

$\frac{1}{5\sqrt{2}} = \cos u$ and $\frac{4}{\sqrt{17}} = \sin v$

$\Rightarrow \tan u = 7$ and $\tan z = 4$

Using properties of trigonometry.

$\tan\left\{\cos^{-1}\frac{1}{5\sqrt{2}} - \sin^{-1}\frac{4}{\sqrt{17}}\right\} = \tan(y - z)$

$\tan(y - z) = \frac{\tan y - \tan z}{1 + \tan y \tan z} = \frac{7 - 4}{1 + 28} = \frac{3}{29}$

Q3

$2\tan^{-1}\{\operatorname{cosec}(\tan^{-1}x) - \tan(\cot^{-1}x)\}$ is equal to

a. $\cot^{-1}x$
b. $\cot^{-1}\frac{1}{x}$
c. $\tan^{-1}x$
d. None of these

Solution

Correct option: (c)

Put, $\tan^{-1}x = z$

$$2\tan^{-1}\left\{\operatorname{cosec}\left(\tan^{-1}x\right) - \tan\left(\cot^{-1}x\right)\right\}$$

$$= 2\tan^{-1}\left\{\operatorname{cosec}\left(\tan^{-1}x\right) - \tan\left(\tan^{-1}\frac{1}{x}\right)\right\}$$

$$= 2\tan^{-1}\left\{\operatorname{cosec}\left(\tan^{-1}x\right) - \frac{1}{x}\right\}$$

$$= 2\tan^{-1}\left\{\operatorname{cosec} z - \frac{1}{\tan z}\right\}$$

$$= 2\tan^{-1}\left\{\frac{1}{\sin z} - \frac{\cos z}{\sin z}\right\}$$

$$= 2\tan^{-1}\left\{\frac{1-\cos z}{\sin z}\right\}$$

$$= 2\tan^{-1}\left\{\frac{2\sin^2\frac{z}{2}}{2\sin\frac{z}{2}\cos\frac{z}{2}}\right\}$$

$$= 2\tan^{-1}\left(\tan\frac{z}{2}\right)$$

$$= 2\times\frac{z}{2}$$

$$= z$$

$$= \tan^{-1}x$$

Q4

If $\cos^{-1}\frac{x}{a} + \cos^{-1}\frac{y}{b} = \alpha$, then $\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\alpha + \frac{y^2}{b^2} =$

a. $\sin^2\alpha$
b. $\cos^2\alpha$
c. $\tan^2\alpha$
d. $\cot^2\alpha$

Solution

Correct option: (a)

$\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$

Consoder, $\cos^{-1}\frac{x}{a} + \cos^{-1}\frac{y}{b} = \alpha$

$\Rightarrow \cos^{-1}\left(\frac{x}{a} \times \frac{y}{b} - \sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{a^2}}\right) = \alpha$

$\Rightarrow \frac{x}{a} \times \frac{y}{b} - \sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{a^2}} = \cos\alpha$

$\Rightarrow \frac{x}{a} \times \frac{y}{b} - \cos\alpha = \sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{a^2}}$

Squaring on both sides,

$\Rightarrow \frac{x^2y^2}{a^2b^2} + \cos^2\alpha - \frac{2xy}{ab}\cos\alpha = \left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{a^2}\right)$

$\Rightarrow \frac{x^2y^2}{a^2b^2} + \cos^2\alpha - \frac{2xy}{ab}\cos\alpha = 1 - \frac{x^2}{a^2} - \frac{y^2}{a^2} + \frac{x^2y^2}{a^2b^2}$

$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{2xy}{ab}\cos\alpha = 1 - \cos^2\alpha$

$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{2xy}{ab}\cos\alpha = \sin^2\alpha$

Q5

The positive integral solution of the equation $\tan^{-1}x + \cos^{-1}\frac{y}{\sqrt{1+y^2}} = \sin^{-1}\frac{3}{\sqrt{10}}$ is

a. x = 1, y = 2
b. x = 2, y = 1
c. x = 3, y = 2
d. x = -2, y = -1

Solution

Correct option: (a)

$\tan^{-1} x + \cos^{-1}\frac{y}{\sqrt{1+y^2}} = \sin^{-1}\frac{3}{\sqrt{10}}$

Let, $\cos^{-1}\frac{y}{\sqrt{1+y^2}} = u \Rightarrow \cos u = \frac{y}{\sqrt{1+y^2}}$

Also, $\sin^{-1}\frac{3}{\sqrt{10}} = v \Rightarrow \sin v = \frac{3}{\sqrt{10}}$

Using trigonometric identities,

$\tan u = \frac{1}{y}$ and $\tan v = 3$

$\Rightarrow u = \tan^{-1}\frac{1}{y}$ and $v = \tan^{-1} 3$

Consider,

$\tan^{-1} x + \cos^{-1}\frac{y}{\sqrt{1+y^2}} = \sin^{-1}\frac{3}{\sqrt{10}}$

$\Rightarrow \tan^{-1} x + \tan^{-1}\frac{1}{y} = \tan^{-1} 3$

$\Rightarrow \tan^{-1}\left(\frac{x+\frac{1}{y}}{1-\frac{x}{y}}\right) = \tan^{-1} 3$

$\Rightarrow \frac{xy+1}{y-x} = 3$

$\Rightarrow xy + 1 = 3y - 3x$

$\Rightarrow x = \frac{3y-1}{3+y}$

Put, $y = 1 \Rightarrow x = \frac{1}{2}$

Put, $y = 2 \Rightarrow x = 1$

Put, $y = 3 \Rightarrow x = \frac{4}{3}$

and so on..........

$\Rightarrow$ Integral soluions are : $x = 1$, $y = 2$

Q6

If $\sin^{-1} x - \cos^{-1} x = \frac{\pi}{6}$, then $x =$

a. $\frac{1}{2}$

b. $\frac{\sqrt{3}}{2}$

c. $-\frac{1}{2}$

d. None of these

Solution

Correct option: (b)

$\sin^{-1}x - \cos^{-1}x = \frac{\pi}{6}$

$\frac{\pi}{2} - \cos^{-1}x - \cos^{-1}x = \frac{\pi}{6}$

$\frac{\pi}{2} - 2\cos^{-1}x = \frac{\pi}{6}$

$\frac{\pi}{2} - \frac{\pi}{6} = 2\cos^{-1}x$

$\frac{2\pi}{6} = 2\cos^{-1}x$

$\frac{\pi}{3} \times \frac{1}{2} = \cos^{-1}x$

$\frac{\pi}{6} = \cos^{-1}x$

$x = \cos\frac{\pi}{6}$

$x = \frac{\sqrt{3}}{2}$

Q7

$\sin\left[\cot^{-1}\left\{\tan\left(\cos^{-1}x\right)\right\}\right]$ is equal to

a. x
b. $\sqrt{1-x^2}$
c. $\frac{1}{x}$
d. None of these

Solution

Correct option: (a)

Put $\cos^{-1}x = u$

$\sin\left[\cot^{-1}\left\{\tan\left(\cos^{-1}x\right)\right\}\right]$

$= \sin\left[\cot^{-1}\left\{\tan(u)\right\}\right]$

$= \sin\left[\cot^{-1}\left\{\cot\left(\frac{\pi}{2} - u\right)\right\}\right]$

$= \sin\left[\frac{\pi}{2} - u\right]$

$= \cos u$

$= x \qquad \left(\because \cos^{-1}x = u \Rightarrow x = \cos u\right)$

Q8

The number of solutions of the equation

$\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$ is

a. 2
b. 3
c. 1
d. None of these

Solution

Correct option: (a)

$\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$

$\Rightarrow \tan^{-1}\left(\frac{2x+3x}{1-6x^2}\right) = \frac{\pi}{4}$

$\Rightarrow \frac{5x}{1-6x^2} = \tan\frac{\pi}{4}$

$\Rightarrow \frac{5x}{1-6x^2} = 1$

$\Rightarrow 5x = 1-6x^2$

$\Rightarrow 6x^2 + 5x - 1 = 0$

$\Rightarrow x = -1$ or $\frac{1}{6}$

Solutions of the given equation are 2.

Q9

If $\alpha = \tan^{-1}\left(\tan\frac{5\pi}{4}\right)$ and $\beta = \tan^{-1}\left(-\tan\frac{2\pi}{3}\right)$, then

a. $4\alpha = 3\beta$
b. $3\alpha = 4\beta$
c. $\alpha - \beta = \frac{7\pi}{12}$
d. None of these

Solution

Correct option: (a)

$\alpha = \tan^{-1}\left(\tan\frac{5\pi}{4}\right)$

$\Rightarrow \alpha = \tan^{-1}\left(\tan\left(\pi + \frac{\pi}{4}\right)\right)$

$\Rightarrow \alpha = \tan^{-1}\left(\tan\left(\frac{\pi}{4}\right)\right)$

$\Rightarrow \alpha = \frac{\pi}{4}$

and

$\beta = \tan^{-1}\left(-\tan\left(\pi - \frac{2\pi}{3}\right)\right)$

$\beta = \tan^{-1}\left(-\tan\left(\frac{\pi}{3}\right)\right)$

$\beta = \frac{\pi}{3}$

$4\alpha = 4 \times \frac{\pi}{4} = \pi \quad \ldots(i)$

$3\beta = 3 \times \frac{\pi}{3} = \pi \quad \ldots(ii)$

From (i) and (ii)

$4\alpha = 3\beta$

Q10

The number of real solutions of the equation $\sqrt{1 + \cos 2x} = \sqrt{2}\sin^{-1}(\sin x), -\pi \le x \le \pi$ is

a. 0
b. 1
c. 2
d. Infinite

Solution

Correct option: (c)

$\sqrt{1+\cos 2x} = \sqrt{2}\sin^{-1}(\sin x), -\pi \le x \le \pi$

$\Rightarrow \sqrt{2\cos^2 x} = \sqrt{2}(-\pi - x)$

$\Rightarrow |\cos x| = x$

If $\cos x$ is positive then $\cos x = -\pi - x$

It does not satisfy any value in the interval $\left(-\pi, -\frac{\pi}{2}\right)$

for the inetrval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\cos x = x$

It gives one value of x in the $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

For the interval $\left[-\frac{\pi}{2}, \pi\right]$.

$-\cos x = \pi - x$

$\cos x = x - \pi$

It gives one value of x in the inetrval $\left[\frac{\pi}{2}, \pi\right]$.

Two real solutions in the interval $[-\pi, \pi]$

Q11

If $x < 0$, $y < 0$ such that $xy = 1$, then $\tan^{-1}x + \tan^{-1}y$ equals

a. $\frac{\pi}{2}$

b. $-\frac{\pi}{2}$

c. $-\pi$

d. None of these

Solution

Correct option: (b)

Given that $xy = 1$

Consider,

$\tan^{-1}x + \tan^{-1}y$

$= \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$= \tan^{-1}(-\infty) \quad .....(\because x < 0, y < 0)$

$= -\frac{\pi}{2}$

Q12

If $u = \cot^{-1}\left\{\sqrt{\tan\theta}\right\} - \tan^{-1}\left\{\sqrt{\tan\theta}\right\}$ then, $\tan\left(\frac{\pi}{4} - \frac{u}{2}\right) =$

a. $\sqrt{\tan\theta}$
b. $\sqrt{\cot\theta}$
c. Tan θ
d. Cot θ

Solution

Correct option: (a)

$u = \cot^{-1}\left\{\sqrt{\tan\theta}\right\} - \tan^{-1}\left\{\sqrt{\tan\theta}\right\}$

Put, $\sqrt{\tan\theta} = z$

$\Rightarrow u = \cot^{-1} z - \tan^{-1} z$

$\Rightarrow u = \frac{\pi}{2} - \tan^{-1} z - \tan^{-1} z$

$\Rightarrow u = \frac{\pi}{2} - 2\tan^{-1} z$

$\Rightarrow 2\tan^{-1} z = \frac{\pi}{2} - u$

$\Rightarrow \tan^{-1} z = \frac{\pi}{4} - \frac{u}{2}$

$\Rightarrow z = \tan\left(\frac{\pi}{4} - \frac{u}{2}\right)$

$\Rightarrow \sqrt{\tan\theta} = \tan\left(\frac{\pi}{4} - \frac{u}{2}\right)$

Q13

If $\cos^{-1}\frac{x}{3} + \cos^{-1}\frac{y}{2} = \frac{\theta}{2}$, then $4x^2 - 12xy\cos\frac{\theta}{2} + 9y^2 =$

a. 36
b. 36 - 36 cos θ
c. 18 - 18 cos θ
d. 18 + 18 cos θ

Solution

Correct option: (c)

$\cos^{-1} x + \cos^{-1} y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$

$\Rightarrow \cos^{-1}\frac{x}{3} + \cos^{-1}\frac{y}{2} = \frac{\theta}{2}$

$\Rightarrow \cos^{-1}\left(\frac{x}{3}\times\frac{y}{2} - \sqrt{1-\left(\frac{x}{3}\right)^2}\sqrt{1-\left(\frac{y}{2}\right)^2}\right) = \frac{\theta}{2}$

$\Rightarrow \frac{xy}{6} - \sqrt{1-\left(\frac{x^2}{9}\right)}\sqrt{1-\left(\frac{y^2}{4}\right)} = \cos\frac{\theta}{2}$

$\Rightarrow \frac{xy - 6\cos\frac{\theta}{2}}{6} = \frac{\sqrt{9-x^2}\sqrt{4-y^2}}{6}$

$\Rightarrow xy - 6\cos\frac{\theta}{2} = \sqrt{9-x^2}\sqrt{4-y^2}$

Takings square on both sides,

$\Rightarrow x^2y^2 - 12xy\cos\frac{\theta}{2} + 36\cos^2\frac{\theta}{2} = (9-x^2)(4-y^2)$

$\Rightarrow x^2y^2 - 12xy\cos\frac{\theta}{2} + 36\cos^2\frac{\theta}{2} = 36 - 9y^2 - 4x^2 + x^2y^2$

$\Rightarrow 4x^2 + 9y^2 - 12xy\cos^2\frac{\theta}{2} = 36\left(1-\cos^2\frac{\theta}{2}\right)$

$\Rightarrow 4x^2 + 9y^2 - 12xy\cos^2\frac{\theta}{2} = 36\left(1-\frac{1+\cos\theta}{2}\right)$

$\Rightarrow 4x^2 + 9y^2 - 12xy\cos^2\frac{\theta}{2} = 18 - 18\cos\theta$

Q14

If $\alpha = \tan^{-1}\left(\frac{\sqrt{3}x}{2y-x}\right), \beta = \tan^{-1}\left(\frac{2x-y}{\sqrt{3}y}\right)$, then $\alpha - \beta =$

a. $\frac{\pi}{6}$

b. $\frac{\pi}{3}$

c. $\frac{\pi}{2}$

d. $-\frac{\pi}{3}$

Solution

Correct option: (a)

$\alpha = \tan^{-1}\left(\frac{\sqrt{3}x}{2y - x}\right), \beta = \tan^{-1}\left(\frac{2x - y}{\sqrt{3}y}\right)$

$\alpha - \beta = \tan^{-1}\left(\frac{\sqrt{3}x}{2y - x}\right) - \tan^{-1}\left(\frac{2x - y}{\sqrt{3}y}\right)$

$$\alpha - \beta = \tan^{-1}\left(\frac{\frac{\sqrt{3}x}{2y - x} - \frac{2x - y}{\sqrt{3}y}}{1 + \frac{\sqrt{3}x}{2y - x} \times \frac{2x - y}{\sqrt{3}y}}\right)$$

$\alpha - \beta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\alpha - \beta = \frac{\pi}{6}$

Q15

Let $f(x) = e^{\cos^{-1}\{\sin(x+\pi/3)\}}$. Then, $f(8\pi/9) =$

a. $e^{5\pi/18}$
b. $e^{13\pi/18}$
c. $e^{-2\pi/18}$
d. None of these

Solution

Correct option: (b)

$f(x) = e^{\cos^{-1}\{\sin(x + \pi/3)\}}$

$f\left(\frac{8\pi}{9}\right) = e^{\cos^{-1}\left\{\sin\left(\frac{8\pi}{9} + \frac{\pi}{3}\right)\right\}}$

$\Rightarrow f\left(\frac{8\pi}{9}\right) = e^{\cos^{-1}\left\{\sin\left(\frac{11\pi}{9}\right)\right\}}$

$\Rightarrow f\left(\frac{8\pi}{9}\right) = e^{\cos^{-1}\left\{\cos\left(\frac{\pi}{2} + \frac{4\pi}{18}\right)\right\}}$

$\Rightarrow f\left(\frac{8\pi}{9}\right) = e^{\frac{13\pi}{18}}$

Q16

$\tan^{-1}\frac{1}{11} + \tan^{-1}\frac{2}{11}$ is equal to

a. 0
b. 1/2
c. -1
d. none of these

Solution

Correct option: (d)

$\tan^{-1}\frac{1}{11}+\tan^{-1}\frac{2}{11}$

$=\tan^{-1}\left(\frac{\frac{1}{11}+\frac{2}{11}}{1-\frac{2}{11}\times\frac{1}{11}}\right)$

$=\tan^{-1}\left(\frac{\frac{3}{11}}{1-\frac{2}{121}}\right)$

$=\tan^{-1}\left(\frac{33}{119}\right)$

Q17

If $\cos^{-1}\frac{x}{2}+\cos^{-1}\frac{y}{3}=\theta$, then $9x^2-12xy\cos\theta+4y^2$ is equal to

a. 36
b. $-36\sin^2\theta$
c. $36\sin^2\theta$
d. $36\cos^2\theta$

Solution

Correct option: (c)

$\cos^{-1}\frac{x}{2}+\cos^{-1}\frac{y}{3}=\theta$

We know that

$\cos^{-1}x+\cos^{-1}y=\cos^{-1}\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right)$

$\cos^{-1}\left(\frac{x}{2}\times\frac{y}{3}-\sqrt{1-\frac{x^2}{4}}\sqrt{1-\frac{y^2}{9}}\right)=\theta$

$xy-\sqrt{1-x^2}\sqrt{1-y^2}=6\cos\theta$

$xy-6\cos\theta=\sqrt{1-x^2}\sqrt{1-y^2}$

$(xy-6\cos\theta)^2=(1-x^2)(1-y^2)$

Simplifying this you will get

$9x^2-12xy\cos\theta+4y^2=36\sin^2\theta$

Q18

If $\tan^{-1}3+\tan^{-1}x=\tan^{-1}8$, then x =

a. 5
b. 1/5
c. 5/14
d. 14/5

Solution

Correct option: (b)

$\tan^{-1}3 + \tan^{-1}x = \tan^{-1}8$

$\tan^{-1}\left(\frac{3+x}{1-3x}\right) = \tan^{-1}8$

$\frac{3+x}{1-3x} = 8$

$3+x = 8-24x$

$25x = 5$

$x = \frac{1}{5}$

Q19

The value of $\sin^{-1}\left(\cos\frac{33\pi}{5}\right)$ is

a. $\frac{3\pi}{5}$

b. $-\frac{\pi}{10}$

c. $\frac{\pi}{10}$

d. $\frac{7\pi}{5}$

Solution

Correct option: (b)

$\sin^{-1}\left(\cos\frac{33\pi}{5}\right)$

$= \sin^{-1}\left(\cos\left(6\pi + \frac{3\pi}{5}\right)\right)$

$= \sin^{-1}\left(\cos\left(\frac{3\pi}{5}\right)\right)$

$= \sin^{-1}\left(\sin\left(\frac{\pi}{2} - \frac{3\pi}{5}\right)\right)$

$= \frac{\pi}{2} - \frac{3\pi}{5}$

$= \frac{-\pi}{10}$

Q20

The value of $\cos^{-1}\left(\cos\frac{5\pi}{3}\right)+\sin^{-1}\left(\sin\frac{5\pi}{3}\right)$ is

a. $\frac{\pi}{2}$
b. $\frac{5\pi}{3}$
c. $\frac{10\pi}{3}$
d. 0

Solution

Correct option: (d)

$\cos^{-1}\left(\cos\frac{5\pi}{3}\right)+\sin^{-1}\left(\sin\frac{5\pi}{3}\right)$

$=\cos^{-1}\left(\cos\left(2\pi-\frac{\pi}{3}\right)\right)+\sin^{-1}\left(\sin\left(2\pi-\frac{\pi}{3}\right)\right)$

$=\cos^{-1}\left(\cos\left(\frac{\pi}{3}\right)\right)+\sin^{-1}\left(-\sin\left(\frac{\pi}{3}\right)\right)$

$=\cos^{-1}\left(\cos\left(\frac{\pi}{3}\right)\right)-\sin^{-1}\left(\sin\left(\frac{\pi}{3}\right)\right)$

$=\frac{\pi}{3}-\frac{\pi}{3}$

$=0$

Q21

$\sin\left\{2\cos^{-1}\left(\frac{-3}{5}\right)\right\}$ is equal to

a. $\frac{6}{25}$
b. $\frac{24}{25}$
c. $\frac{4}{5}$
d. $-\frac{24}{25}$

Solution

Correct option: (d)

To find $\sin\left\{2\cos^{-1}\left(\frac{-3}{5}\right)\right\}$

Let, $\cos^{-1}\left(\frac{-3}{5}\right) = y$

$\Rightarrow \frac{-3}{5} = \cos y$

$\Rightarrow \sin y = \sqrt{1-\cos^2 y}$

$\Rightarrow \sin y = \sqrt{1-\left(\frac{-3}{5}\right)^2}$

$\Rightarrow \sin y = \frac{4}{5}$

$\sin\left\{2\cos^{-1}\left(\frac{-3}{5}\right)\right\} = \sin 2y$

$\Rightarrow \sin\left\{2\cos^{-1}\left(\frac{-3}{5}\right)\right\} = 2\sin y\cos y$

$\Rightarrow \sin\left\{2\cos^{-1}\left(\frac{-3}{5}\right)\right\} = 2\times\frac{4}{5}\times\frac{-3}{5}$

$\Rightarrow \sin\left\{2\cos^{-1}\left(\frac{-3}{5}\right)\right\} = \frac{-24}{25}$

Q22

If $\theta = \sin^{-1}\{\sin(-600°)\}$, then one of the possible values of θ is

a. $\frac{\pi}{3}$

b. $\frac{\pi}{2}$

c. $\frac{2\pi}{3}$

d. $-\frac{2\pi}{3}$

Solution

Correct option: (a)

$\theta = \sin^{-1}\{\sin(-600°)\}$

$\theta = \sin^{-1}\left[-\sin(600°)\right]$

$\theta = \sin^{-1}\left[-\sin(180°\times 3+60)\right]$

$\theta = \sin^{-1}\left[-\{-\sin(60°)\}\right]$

$\theta = \sin^{-1}\left(\sin(60°)\right)$

$\theta = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$\theta = \frac{\pi}{3}$

Q23

If $3\sin^{-1}\left(\frac{2x}{1+x^2}\right)-4\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)+2\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{\pi}{3}$,

then x is equal to

a. $\frac{1}{\sqrt{3}}$

b. $-\frac{1}{\sqrt{3}}$

c. $\sqrt{3}$

d. $-\frac{\sqrt{3}}{4}$

Solution

Correct option: (a)

$$3\sin^{-1}\left(\frac{2x}{1+x^2}\right)-4\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)+2\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\frac{\pi}{3}$$

Let, $x=\tan\theta \Rightarrow \theta=\tan^{-1}x$

$$\Rightarrow 3\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)-4\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)+2\tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)=\frac{\pi}{3}$$

$$\Rightarrow 3\sin^{-1}(\sin 2\theta)-4\cos^{-1}(\cos 2\theta)+2\tan^{-1}(\tan 2\theta)=\frac{\pi}{3}$$

$$\Rightarrow 3\times 2\theta-4\times 2\theta+2\times 2\theta=\frac{\pi}{3}$$

$$\Rightarrow 2\theta=\frac{\pi}{3}$$

$$\Rightarrow \theta=\frac{\pi}{6}$$

$$\Rightarrow \tan^{-1}x=\frac{\pi}{6}$$

$$\Rightarrow x=\tan\frac{\pi}{6}$$

$$\Rightarrow x=\frac{1}{\sqrt{3}}$$

Q24

If $4\cos^{-1}x+\sin^{-1}x=\pi$, then the value of x is

a. $\frac{3}{2}$

b. $\frac{1}{\sqrt{2}}$

c. $\frac{\sqrt{3}}{2}$

d. $\frac{2}{\sqrt{3}}$

Solution

Correct option: (c)

$4\cos^{-1}x + \sin^{-1}x = \pi$

$\Rightarrow 3\cos^{-1}x + \cos^{-1}x + \sin^{-1}x = \pi$

$\Rightarrow 3\cos^{-1}x + \frac{\pi}{2} = \pi$

$\Rightarrow \cos^{-1}x = \frac{\pi}{6}$

$\Rightarrow x = \cos\frac{\pi}{6}$

$\Rightarrow x = \frac{\sqrt{3}}{2}$

Q25

If $\tan^{-1}\frac{x+1}{x-1} + \tan^{-1}\frac{x-1}{x} = \tan^{-1}(-7)$, then the value of x is

a. 0
b. -2
c. 1
d. 2

Solution

Correct option: (d)

$\tan^{-1}\frac{x+1}{x-1} + \tan^{-1}\frac{x-1}{x} = \tan^{-1}(-7)$

$\Rightarrow \tan^{-1}\left(\frac{\frac{x+1}{x-1}+\frac{x-1}{x}}{1-\frac{x+1}{x-1}\times\frac{x-1}{x}}\right) = \tan^{-1}(-7)$

$\Rightarrow \tan^{-1}\left(\frac{x^2+x+x^2-2x+1}{x^2-x-(x^2-1)}\right) = \tan^{-1}(-7)$

$\Rightarrow \frac{2x^2-x+1}{-x+1} = -7$

$\Rightarrow 2x^2-x+1 = 7x-7$

$\Rightarrow 2x^2-8x+8 = 0$

$\Rightarrow x^2-4x+4 = 0$

$\Rightarrow (x-2)^2 = 0$

$\Rightarrow x = 2$

Q26

If $\cos^{-1} x > \sin^{-1} x$, then

a. $\frac{1}{\sqrt{2}} < x \le 1$

b. $0 \le x < \frac{1}{\sqrt{2}}$

c. $-1 \le x < \frac{1}{\sqrt{2}}$

d. $X > 0$

Solution

Correct option: (a)

$\cos^{-1}x > \sin^{-1}x$

$\cos^{-1}x > \frac{\pi}{2} - \cos^{-1}x$

$2\cos^{-1}x > \frac{\pi}{2}$

$\cos^{-1}x > \frac{\pi}{4}$

$x > \cos\frac{\pi}{4}$

$x > \frac{1}{\sqrt{2}}$

Hence; $\frac{1}{\sqrt{2}} < x \le 1$

Q27

In a ΔABC, if C is a right angle, then $\tan^{-1}\left(\frac{a}{b+c}\right) + \tan^{-1}\left(\frac{b}{c+a}\right) =$

a. $\frac{\pi}{3}$

b. $\frac{\pi}{4}$

c. $\frac{5\pi}{2}$

d. $\frac{\pi}{6}$

Solution

Correct option: (b)

$$\tan^{-1}\left(\frac{a}{b+c}\right)+\tan^{-1}\left(\frac{b}{c+a}\right)$$

$$=\tan^{-1}\left(\frac{\frac{a}{b+c}+\frac{b}{c+a}}{1-\frac{a}{b+c}\times\frac{b}{c+a}}\right)$$

$$=\tan^{-1}\left(\frac{ac+a^2+b^2+bc}{bc+ba+c^2+ca-ab}\right)$$

$$=\tan^{-1}\left(\frac{ac+c^2+bc}{bc+ba+c^2+ca-ab}\right) \quad \cdots\cdots \left(\begin{matrix}\because \Delta ABC \text{ is right angle at } C\\ c^2=a^2+b^2\end{matrix}\right)$$

$$=\tan^{-1}\left(\frac{ac+c^2+bc}{bc+c^2+ca}\right)$$

$$=\tan^{-1}(1)$$

$$=\frac{\pi}{4}$$

Q28

The value of $\sin\left(\frac{1}{4}\sin^{-1}\frac{\sqrt{63}}{8}\right)$ is

a. $\frac{1}{\sqrt{2}}$

b. $\frac{1}{\sqrt{3}}$

c. $\frac{1}{2\sqrt{2}}$

d. $\frac{1}{3\sqrt{3}}$

Solution

Correct option: (c)

$\sin\left(\frac{1}{4}\sin^{-1}\frac{\sqrt{63}}{8}\right)$

Let, $\sin^{-1}\frac{\sqrt{63}}{8} = x$

$\sin x = \frac{\sqrt{63}}{8}$

$\cos x = \sqrt{1-\sin^2 x}$

$\cos x = \sqrt{1-\frac{63}{64}}$

$\cos x = \frac{1}{8}$

Consider,

$\sin\left(\frac{1}{4}\sin^{-1}\frac{\sqrt{63}}{8}\right)$

$= \sin\left(\frac{1}{4}x\right)$

$= \sqrt{\frac{1-\cos\frac{x}{2}}{2}} \qquad \left(\because \sin x = \frac{1-\cos 2x}{2}\right)$

$= \sqrt{\frac{1-\sqrt{\frac{1+\cos x}{2}}}{2}} \qquad \left(\because \cos x = \frac{1+\cos 2x}{2}\right)$

$= \sqrt{\frac{1-\sqrt{\frac{1+\frac{1}{8}}{2}}}{2}}$

$= \sqrt{\frac{1-\frac{3}{4}}{2}}$

$= \sqrt{\frac{1}{8}}$

$= \frac{1}{2\sqrt{2}}$

Q29

$\cot\left(\frac{\pi}{4} - 2\cot^{-1}3\right) =$

a. 7
b. 6
c. 5
d. None of these

Solution

Correct option:(a)

$\cot\left(\frac{\pi}{4} - 2\cot^{-1}3\right)$

Put, $2\cot^{-1}3 = x$

$\Rightarrow \cot^{-1}3 = \frac{x}{2}$

$\Rightarrow \cot\frac{x}{2} = 3$

$\Rightarrow \tan\frac{x}{2} = \frac{1}{3}$

$\cot\left(\frac{\pi}{4} - 2\cot^{-1}3\right)$

$= \cot\left(\frac{\pi}{4} - x\right)$

We will find $\tan\left(\frac{\pi}{4} - x\right)$ then $\cot\left(\frac{\pi}{4} - x\right)$

$\tan\left(\frac{\pi}{4} - x\right)$

$= \frac{1 - \tan x}{1 + \tan x}$

$= \frac{1 - \frac{3}{4}}{1 + \frac{3}{4}} \qquad \left(\because \tan x = \frac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}}\right)$

$= \frac{1}{7}$

$\Rightarrow \tan\left(\frac{\pi}{4} - x\right) = \frac{1}{7}$

$\cot\left(\frac{\pi}{4} - x\right) = 7$

Q30

If $\tan^{-1}(\cot\theta) = 2\theta$, then θ =

a. $\pm\frac{\pi}{3}$
b. $\pm\frac{\pi}{4}$
c. $\pm\frac{\pi}{6}$
d. none of these

Solution

Correct option: (c)

$\tan^{-1}(\cot\theta) = 2\theta$

$\cot\theta = \tan 2\theta$

$\frac{\cos\theta}{\sin\theta} = \frac{\sin 2\theta}{\cos 2\theta}$

$\frac{\cos\theta}{\sin\theta} = \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$

$\cos^2\theta - \sin^2\theta = 2\sin^2\theta$

$\cos^2\theta = 3\sin^2\theta$

$\tan^2\theta = \frac{1}{3}$

$\tan\theta = \pm\frac{1}{\sqrt{3}}$

$\theta = \pm\frac{\pi}{6}$

Q31

If $\sin^{-1}\left(\frac{2a}{1+a^2}\right) + \cos^{-1}\left(\frac{1-a^2}{1+a^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, where $a, x \in (0, 1)$, then, the value of x is

a. 0

b. $\frac{a}{2}$

c. a

d. $\frac{2a}{1-a^2}$

Solution

Correct option: (d)

$\sin^{-1}\left(\frac{2a}{1+a^2}\right) + \cos^{-1}\left(\frac{1-a^2}{1+a^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

Let, $a = \tan\theta \Rightarrow \theta = \tan^{-1} a$

$\sin^{-1}(\sin 2\theta) + \cos^{-1}(\cos 2\theta) = 2\tan^{-1}(x)$

$2\theta + 2\theta = 2\tan^{-1}(x)$

$4\theta = 2\tan^{-1}(x)$

$2\tan^{-1} a = \tan^{-1}(x)$

$\tan^{-1}\left(\frac{2a}{1-a^2}\right) = \tan^{-1}(x)$

$x = \frac{2a}{1-a^2}$

Q32

The value of sin $(2(\tan^{-1}0.75))$ is equal to

a. 0.75
b. 1.5
c. 0.96
d. $\sin^{-1}1.5$

Solution

Correct option: (c)

$\sin\{2(\tan^{-1}0.75)\}$

Let, $\tan^{-1}0.75 = x$

$\Rightarrow \tan^{-1}\frac{3}{4} = x$

$\Rightarrow \tan x = \frac{3}{4}$

Using trigonometric identities,

$\sin x = \frac{3}{5},\ \cos x = \frac{4}{5}$

$\sin\{2(\tan^{-1}0.75)\}$

$= \sin(2x)$

$= 2\sin x\cos x$

$= 2\times\frac{3}{5}\times\frac{4}{5}$

$= \frac{24}{25} = 0.96$

Q33

If $x > 1$, then $2\tan^{-1}x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ is equal to

a. $4\tan^{-1}x$
b. 0
c. $\frac{\pi}{2}$
d. π

Solution

Correct option: (a)

$2\tan^{-1}x + \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Let, $x = \tan\theta \Rightarrow \theta = \tan^{-1}x$

$\Rightarrow 2\tan^{-1}x + \sin^{-1}(\sin 2\theta)$

$\Rightarrow 2\theta + 2\theta$

$\Rightarrow 4\theta$

$\Rightarrow 4\tan^{-1}x$

Q34

The domain of $\cos^{-1}(x^2 - 4)$ is

a. [3, 5]
b. [-1, 1]
c. $\left[-\sqrt{5}, -\sqrt{3}\right] \cup \left[\sqrt{3}, \sqrt{5}\right]$
d. $\left[-\sqrt{3}, -\sqrt{3}\right] \cap \left[-\sqrt{5}, \sqrt{3}\right]$

Solution

Correct option: (c)

Let, $\cos^{-1}\left(x^2 - 4\right) = y$

$\Rightarrow \cos y = x^2 - 4$

$\Rightarrow -1 \le x^2 - 4 \le 1$

$\Rightarrow 3 \le x^2 \le 5$

$\Rightarrow \pm\sqrt{3} \le x \le \pm\sqrt{5}$

$x \in \left[-\sqrt{5}, -\sqrt{3}\right] \cup \left[\sqrt{3}, \sqrt{5}\right]$

Q35

The value of $\tan\left(\cos^{-1}\frac{3}{5} + \tan^{-1}\frac{1}{4}\right)$ is

a. $\frac{19}{8}$
b. $\frac{8}{19}$
c. $\frac{19}{12}$
d. $\frac{3}{4}$

Solution

Correct option: (a)

$\tan\left(\cos^{-1}\frac{3}{5}+\tan^{-1}\frac{1}{4}\right)$

Let, $\cos^{-1}\frac{3}{5}=x$ and $\tan^{-1}\frac{1}{4}=y$

$\Rightarrow \frac{3}{5}=\cos x$ and $\frac{1}{4}=\tan y$

Using trigonometric identities,

$\tan x=\frac{4}{3}$

Consider,

$\tan\left(\cos^{-1}\frac{3}{5}+\tan^{-1}\frac{1}{4}\right)$

$=\tan(x+y)$

$=\frac{\tan x+\tan y}{1-\tan x\tan y}$

$=\frac{\frac{4}{3}+\frac{1}{4}}{1-\frac{4}{3}\times\frac{1}{4}}$

$=\frac{19}{8}$