RD Sharma Solutions Class 12 Maths Chapter 11 Ex 11.8

Chapter: Differentiation

Exercise: 11.8

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Q1.

Answer :

$$Let\ u = x^2 \quad and\ v = x^3$$

$$\Rightarrow \frac{du}{dx} = 2x\ and\ \frac{dv}{dx} = 3x^2$$

$$\therefore \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{2x}{3x^2} = \frac{2}{3x}$$

Q2.

Answer :

$$Let\ u = \log\left(1+x^2\right)\ and\ v = \tan^{-1} x$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{\left(1+x^2\right)}\frac{d}{dx}\left(1+x^2\right) = \frac{2x}{\left(1+x^2\right)}\ and\ \frac{dv}{dx} = \frac{1}{1+x^2}$$

$$\therefore \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{2x}{1+x^2} \times \frac{1+x^2}{1} = 2x$$

Q3.

Answer :

$$Let\, u = \left(\log x\right)^x$$

Taking log on both sides,

$$\log u = \log\left(\log x\right)^x$$

$$\Rightarrow \log u = x\ \log\left(\log x\right)$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}\{\log(\log x)\} + \log(\log x)\frac{d}{dx}(x)$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx} = x\left(\frac{1}{\log x}\right)\frac{d}{dx}(\log x) + \log\log x(1)$$

$$\Rightarrow \frac{du}{dx} = u\left[\frac{x}{\log x}\left(\frac{1}{x}\right) + \log\ \log x\right]$$

$$\Rightarrow \frac{du}{dx} = (\log x)^x\left[\frac{1}{\log x} + \log\ \log x\right] \quad ..(i)$$

Again, let $v = \log x$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{x} \quad ...(ii)$$

Dividing equation (i) *by* (ii), *we get*

$$\frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{(\log x)^x\left[\frac{1}{\log x} + \log\ \log x\right]}{\frac{1}{x}}$$

$$\Rightarrow \frac{du}{dv} = \frac{(\log x)^x\left[\frac{1 + \log x(\log\ \log x)}{\log x}\right]}{\frac{1}{x}}$$

$$\Rightarrow \frac{du}{dv} = x(\log x)^{x-1}(1 + \log x \times \log\ \log x)$$

Q4.

Answer :

(i) *Let,* $u = \sin^{-1}\sqrt{1-x^2}$

Put $x = \cos\theta$

$\Rightarrow \quad u = \sin^{-1}\sqrt{1-\cos^2\theta}$

$\Rightarrow \quad u = \sin^{-1}(\sin\theta) \qquad \ldots(i)$

And, $v = \cos^{-1}x \qquad \ldots(ii)$

Now, $x \in (0,1)$

$\Rightarrow \quad \cos\theta \in (0,1)$

$\Rightarrow \quad \theta \in \left(0, \frac{\pi}{2}\right)$

So, from equation (i),

$u = \theta \qquad \left[\text{Since, } \sin^{-1}(\sin\theta) = \theta \text{ if } \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right]$

$\Rightarrow u = \cos^{-1}x \qquad [\text{Since, } \cos\theta = x]$

Differentiating it with respect to x,

$$\frac{du}{dx} = \frac{-1}{\sqrt{1-x^2}} \qquad \ldots(iii)$$

from equation (ii),

$v = \cos^{-1}x$

Differentiating it with respect to x,

$$\frac{dv}{dx} = \frac{-1}{\sqrt{1-x^2}} \qquad \ldots(iv)$$

Dividing equation (iii) *by* (iv),

$$\frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{-1}{\sqrt{1-x^2}} \times \frac{\sqrt{1-x^2}}{-1} \therefore \frac{du}{dx} = 1$$

(ii) $Let, \quad u = \sin^{-1}\sqrt{1-x^2}$

$Put \quad x = \cos\theta$

$\Rightarrow \quad u = \sin^{-1}\sqrt{1-\cos^2\theta}$

$\Rightarrow \quad u = \sin^{-1}(\sin\theta) \qquad ...(i)$

$And, \; v = \cos^{-1}x \qquad ...(ii)$

$Now, \quad x \in (-1, 0)$

$\Rightarrow \quad \cos\theta \in (-1, 0)$

$\Rightarrow \quad \theta \in \left(\frac{\pi}{2}, \pi\right)$

$So, \; from \; equation \; (i),$

$u = \pi - \theta \qquad \left[Since, \; \sin^{-1}(\sin\theta) = \pi - \theta \; if \; \theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)\right]$

$\Rightarrow u = \pi - \cos^{-1}x \qquad [Since, \; x = \cos\theta]$

Differentiating it with respect to x,

$$\frac{du}{dx} = 0 - \frac{-1}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} \qquad ...(iii)$$

$from \; equation \; (ii),$

$v = \cos^{-1}x$

Differentiating it with respect to x,

$$\frac{dv}{dx} = \frac{-1}{\sqrt{1-x^2}} \qquad ...(iv)$$

$Dividing \; equation \; (iii) \; by \; (iv),$

$$\frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{1}{\sqrt{1-x^2}} \times \frac{\sqrt{1-x^2}}{-1}$$

$$\therefore \frac{du}{dx} = -1$$

Q5.

Answer :

(i) *Let,* $u=\sin^{-1}\left(4x\sqrt{1-4x^2}\right)$

put $2x=\cos\theta$

$\Rightarrow \quad u=\sin^{-1}\left(2\times\cos\theta\sqrt{1-\cos^2\theta}\right)$

$\Rightarrow \quad u=\sin^{-1}\left(2\cos\theta\sin\theta\right)$

$\Rightarrow \quad u=\sin^{-1}\left(\sin 2\theta\right) \qquad \ldots(i)$

Let, $v=\sqrt{1-4x^2} \qquad \ldots(ii)$

Here,

$$x\in\left(-\frac{1}{2\sqrt{2}},\frac{1}{2\sqrt{2}}\right)$$

$$\Rightarrow 2x\in\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \theta\in\left(\frac{\pi}{4},\frac{3\pi}{4}\right)$$

So, from equation (i),

$u=\pi-2\theta \qquad \left[\text{Since, } \sin^{-1}(\sin\theta)=\pi-\theta, \text{if } \theta\in\left(\frac{\pi}{2},\pi\right)\right]$

$\Rightarrow u=\pi-2\cos^{-1}(2x) \qquad \left[\text{Since, } 2x=\cos\theta\right]$

Differentiating it with respect to x,

$$\frac{du}{dx}=0-2\left(\frac{-1}{\sqrt{1-(2x)^2}}\right)\frac{d}{dx}(2x)$$

$$\Rightarrow \frac{du}{dx}=\frac{2}{\sqrt{1-4x^2}}(2)$$

$$\Rightarrow \frac{du}{dx}=\frac{4}{\sqrt{1-4x^2}} \qquad ...(iii)$$

from equation (ii)

$$\frac{dv}{dx}=\frac{-4x}{\sqrt{1-4x^2}}$$

$$but,\ x\in\left(-\frac{1}{2},-\frac{1}{2\sqrt{2}}\right)$$

$$\frac{dv}{dx}=\frac{-4(-x)}{\sqrt{1-4(-x)^2}}$$

$$\Rightarrow \frac{dv}{dx}=\frac{4x}{\sqrt{1-4x^2}} \qquad ...(iv)$$

Diferentiating equation (ii) *with respect to* x,

$$\frac{dv}{dx}=\frac{1}{2\sqrt{1-4x^2}}\frac{d}{dx}(1-4x^2)$$

$$\Rightarrow \frac{dv}{dx}=\frac{1}{2\sqrt{1-4x^2}}(-8x)$$

$$\Rightarrow \frac{dv}{dx}=\frac{-4x}{\sqrt{1-4x^2}} \qquad ...(v)$$

Dividing equation (iii) *by* (v)

$$\frac{\frac{du}{dx}}{\frac{dv}{dx}}=\frac{4}{\sqrt{1-4x^2}}\times\frac{\sqrt{1-4x^2}}{-4x}$$

$$\therefore \frac{du}{dv}=-\frac{1}{x}$$

(ii) $Let,\quad u=\sin^{-1}\left(4x\sqrt{1-4x^2}\right)$

$put\ \ 2x=\cos\theta$

$$u=\sin^{-1}\left(2\times\cos\theta\sqrt{1-\cos^2\theta}\right)$$

$$\Rightarrow\quad u=\sin^{-1}\left(2\cos\theta\sin\theta\right)$$

$$\Rightarrow\quad u=\sin^{-1}\left(\sin 2\theta\right)\qquad ...(i)$$

$$Let,\quad v=\sqrt{1-4x^2}\qquad ...(ii)$$

$Here,$

$$x\in\left(\frac{1}{2\sqrt{2}},\frac{1}{2}\right)$$

$$\Rightarrow\ 2x\in\left(\frac{1}{\sqrt{2}},1\right)$$

$$\Rightarrow\cos\theta\in\left(\frac{1}{\sqrt{2}},1\right)$$

$$\Rightarrow\theta\in\left(0,\frac{\pi}{4}\right)$$

$So,\ from\ equation\ (i),$

$$u=2\theta\qquad\left[Since,\ \sin^{-1}\left(\sin\theta\right)=\theta,if\ \ \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right]$$

$$\Rightarrow\ u=2\cos^{-1}\left(2x\right)\qquad\left[Since,\ 2x=\cos\theta\right]$$

Differentiate it with respect to x,

$$\frac{du}{dx}=2\left(\frac{-1}{\sqrt{1-(2x)^2}}\right)\frac{d}{dx}(2x)$$

$$\frac{du}{dx}=\frac{-2}{\sqrt{1-4x^2}}(2)$$

$$\frac{du}{dx}=\frac{-4}{\sqrt{1-4x^2}} \qquad ...(iii)$$

Diferentiating equation (ii) *with respect to x,*

$$\frac{dv}{dx}=\frac{1}{2\sqrt{1-4x^2}}\frac{d}{dx}(1-4x^2)$$

$$\Rightarrow \frac{dv}{dx}=\frac{1}{2\sqrt{1-4x^2}}(-8x)$$

$$\Rightarrow \frac{dv}{dx}=\frac{-4x}{\sqrt{1-4x^2}} \qquad ...(iv)$$

Dividing equation (iii) *by* (iv)

$$\frac{\frac{du}{dx}}{\frac{dv}{dx}}=\frac{-4}{\sqrt{1-4x^2}}\times\frac{\sqrt{1-4x^2}}{-4x}$$

$$\therefore \frac{du}{dv}=\frac{1}{x}$$

$(iii)\quad Let,\quad u=\sin^{-1}\left(4x\sqrt{1-4x^2}\right)$

$put,\ 2x=\cos\theta$

$\Rightarrow\quad u=\sin^{-1}\left(2\times\cos\theta\sqrt{1-\cos^2\theta}\right)$

$\Rightarrow\quad u=\sin^{-1}(2\cos\theta\sin\theta)$

$\Rightarrow\quad u=\sin^{-1}(\sin 2\theta)\qquad ...(i)$

$Let,\quad v=\sqrt{1-4x^2}\qquad ...(ii)$

Here,

$x\in\left(-\frac{1}{2},-\frac{1}{2\sqrt{2}}\right)$

$\Rightarrow\ 2x\in\left(-1,-\frac{1}{\sqrt{2}}\right)$

$\Rightarrow\theta\in\left(\frac{3\pi}{4},\pi\right)$

So, from equation (i),

$u=\pi-2\theta\qquad\left[Since,\ \sin^{-1}(\sin\theta)=\pi-\theta, if\ \ \theta\in\left(\frac{\pi}{2},\frac{3\pi}{2}\right)\right]$

$\Rightarrow\ u=\pi-2\cos^{-1}(2x)\qquad[Since,\ 2x=\cos\theta]$

Differentiate it with respect to x,

$$\frac{du}{dx} = 0 - 2\left(\frac{-1}{\sqrt{1-(2x)^2}}\right)\frac{d}{dx}(2x)$$

$$\Rightarrow \frac{du}{dx} = \frac{2}{\sqrt{1-4x^2}}(2)$$

$$\Rightarrow \frac{du}{dx} = \frac{4}{\sqrt{1-4x^2}} \qquad ...(iii)$$

from equation (ii),

$$\frac{dv}{dx} = \frac{-4x}{\sqrt{1-4x^2}}$$

$$but,\ x \in \left(-\frac{1}{2}, -\frac{1}{2\sqrt{2}}\right)$$

$$\therefore \frac{dv}{dx} = \frac{-4(-x)}{\sqrt{1-4(-x)^2}}$$

$$\Rightarrow \frac{dv}{dx} = \frac{4x}{\sqrt{1-4x^2}} \qquad ...(iv)$$

Dividing equation (iii) *by* (iv)

$$\frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{4}{\sqrt{1-4x^2}} \times \frac{\sqrt{1-4x^2}}{4x}$$

$$\therefore \frac{du}{dv} = \frac{1}{x}$$

Q6.

Answer :

Let, $u = \tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$

put $x = \tan\theta$

$\Rightarrow \quad u = \tan^{-1}\left(\dfrac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$

$\Rightarrow \quad u = \tan^{-1}\left(\dfrac{sec\theta-1}{\tan\theta}\right)$

$\Rightarrow \quad u = \tan^{-1}\left(\dfrac{1-\cos\theta}{\sin\theta}\right)$

$\Rightarrow \quad u = \tan^{-1}\left(\dfrac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$

$\Rightarrow \quad u = \tan^{-1}\left(\tan\dfrac{\theta}{2}\right) \qquad \ldots(i)$

And,

$v = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right)$

$\Rightarrow \quad v = \sin^{-1}\left(\dfrac{2\tan\theta}{1+\tan^2\theta}\right)$

$\Rightarrow \quad v = \sin^{-1}(\sin 2\theta) \qquad \ldots(ii)$

Here,

$-1 < x < 1$

$\Rightarrow -1 < \tan\theta < 1$

$\Rightarrow -\dfrac{\pi}{4} < \theta < \dfrac{\pi}{4} \qquad \ldots(A)$

So, from equation (i)

$u = \dfrac{\theta}{2} \qquad \left[Since,\ \tan^{-1}(\tan\theta) = \theta,\ if\ \theta \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)\right]$

$\Rightarrow u = \dfrac{1}{2}\tan^{-1}x \qquad [since,\ x = \tan\theta]$

$\dfrac{du}{dx} = \dfrac{1}{2}\left(\dfrac{1}{1+x^2}\right)$

$\Rightarrow \dfrac{du}{dx} = \dfrac{1}{2(1+x^2)} \qquad \ldots(i)$

Now, from equation (ii) *and* (A),

$v = 2\theta \qquad \left[Since,\ \sin^{-1}(\sin\theta) = \theta,\ if\ \theta \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]\right]$

$\Rightarrow v = 2\tan^{-1}x \qquad [Since,\ x = \tan\theta]$

$$\frac{dv}{dx}=2\left(\frac{1}{1+x^2}\right) \qquad ...(iv)$$

dividing equation (iii) by (iv),

$$\frac{\frac{du}{dx}}{\frac{dv}{dx}}=\frac{1}{2\left(1+x^2\right)}\times\frac{1+x^2}{2}$$

$$\therefore \frac{du}{dv}=\frac{1}{4}$$

Q7.

Answer :

(i) Let, $\quad u = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

Put $\quad x = \sin\theta$

$\Rightarrow \quad u = \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$

$\Rightarrow \quad u = \sin^{-1}\left(2\sin\theta\cos\theta\right)$

And,

Let $\quad v = sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$

$\Rightarrow \quad v = sec^{-1}\left(\frac{1}{\sqrt{1-\sin^2\theta}}\right)$

$\Rightarrow \quad v = sec^{-1}\left(\frac{1}{\cos\theta}\right)$

$\Rightarrow \quad v = sec^{-1}\left(sec\theta\right)$

$\Rightarrow \quad v = \cos^{-1}\left(\frac{1}{\frac{1}{\cos\theta}}\right)$

$\Rightarrow \quad v = \cos^{-1}\left(\cos\theta\right) \qquad ...(ii)$

Here,

$x \in \left(0, \frac{1}{\sqrt{2}}\right)$

$\Rightarrow \quad \sin\theta \in \left(0, \frac{1}{\sqrt{2}}\right)$

$\Rightarrow \quad \theta \in \left(0, \frac{\pi}{4}\right)$

So, from equation (i),

$u = 2\theta \qquad \left[Since,\ \sin^{-1}(\sin\theta) = \theta,\ if\ \ \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right]$

Let, $u = 2\sin^{-1}x \qquad \left[Since,\ x = \sin\theta\right]$

Differentiating it with respect to x,

$$\frac{du}{dx} = 2\left(\frac{1}{\sqrt{1-x^2}}\right)$$

$$\Rightarrow \frac{du}{dx} = \frac{2}{\sqrt{1-x^2}} \qquad ...(iii)$$

And, from equation (ii),

$$v = \theta \qquad \left[Since,\ \cos^{-1}(\cos\theta) = \theta,\ if\ \theta \in [0, \pi]\right]$$

$$\Rightarrow v = \sin^{-1} x \qquad [Since,\ x = \sin\theta]$$

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-x^2}} ...(iv)$$

dividing equation (iii) *by* (iv),

$$\frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{2}{\sqrt{1-x^2}} \times \frac{\sqrt{1-x^2}}{1}$$

$$\therefore \frac{du}{dv} = 2$$

(ii) *Let*, $u = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

Put $x = \sin\theta$

$\Rightarrow \quad u = \sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$

$\Rightarrow \quad u = \sin^{-1}\left(2\sin\theta\cos\theta\right)$

$\Rightarrow \quad u = \sin^{-1}\left(\sin 2\theta\right) \quad \ldots(i)$

And,

Let, $v = sec^{-1}\left(\dfrac{1}{\sqrt{1-x^2}}\right)$

$\Rightarrow \quad v = sec^{-1}\left(\dfrac{1}{\sqrt{1-\sin^2\theta}}\right)$

$\Rightarrow \quad v = sec^{-1}\left(\dfrac{1}{\cos\theta}\right)$

$\Rightarrow \quad v = sec^{-1}\left(sec\theta\right)$

$\Rightarrow \quad v = \cos^{-1}\left(\dfrac{1}{\dfrac{1}{\cos\theta}}\right) \quad \left[\text{Since, } sec^{-1}x = \cos^{-1}\left(\dfrac{1}{x}\right)\right]$

$\Rightarrow \quad v = \cos^{-1}\left(\cos\theta\right) \quad \ldots(ii)$

Here,

$x \in \left(\dfrac{1}{\sqrt{2}}, 1\right)$

$\Rightarrow \quad \sin\theta \in \left(\dfrac{1}{\sqrt{2}}, 1\right)$

$\Rightarrow \quad \theta \in \left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$

So, from equation (i),

$u = 2\theta \quad \left[\text{Since, } \sin^{-1}\left(\sin\theta\right) = \theta, \text{ if } \theta \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)\right]$

Let, $u = 2\sin^{-1}x \quad \left[\text{Since, } x = \sin\theta\right]$

$$\frac{du}{dx}=2\left(\frac{1}{\sqrt{1-x^2}}\right)$$

$$\Rightarrow \frac{du}{dx}=\frac{2}{\sqrt{1-x^2}} \qquad ...(iii)$$

And, from equation (ii),

$$v=\theta \qquad \left[Since,\ \cos^{-1}(\cos\theta)=\theta,\ if\ \ \theta\in[0,\pi]\right]$$

$$\Rightarrow v=\sin^{-1}x \qquad [Since,\ x=\sin\theta]$$

$$\frac{dv}{dx}=\frac{1}{\sqrt{1-x^2}} \qquad ...(iv)$$

dividing equation (iii) by (iv),

$$\frac{\frac{du}{dx}}{\frac{dv}{dx}}=\frac{2}{\sqrt{1-x^2}}\times\frac{\sqrt{1-x^2}}{1}$$

$$\therefore \frac{du}{dv}=2$$

Q8.

Answer :

$$Let,\quad u=(\cos x)^{\sin x}$$

Taking log on both sides,

$$\log u=\log(\cos x)^{\sin x}$$

$$\Rightarrow \log u=\sin x\ \log(\cos x)$$

Differentiating it with respect to x using chain rule,

$$\frac{1}{u}\frac{du}{dx}=\sin x\frac{d}{dx}(\log\ \cos x)+\log\ \cos x\frac{d}{dx}(\sin x) \qquad [using\ product\ rule]$$

$$\Rightarrow \frac{1}{u}\frac{du}{dx}=\sin x\left(\frac{1}{\cos x}\right)\frac{d}{dx}(\cos x)+\log\ \cos x(\cos x)$$

$$\Rightarrow \frac{du}{dx}=u\left[(\tan x)\times(-\sin x)+\log\ \log x(\cos x)\right]$$

$$\Rightarrow \frac{du}{dx}=(\cos x)^{\sin x}\left[\cos x\log\cos x-\sin x\ \tan x\right] \qquad ...(i)$$

$$Let,\quad v=(\sin x)^{\cos x}$$

Taking log on both sides,

$$\log v = \log(\sin x)^{\cos x}$$

$$\Rightarrow \log v = \cos x \log(\sin x)$$

Differentiating it with respect to x using chain rule,

$$\frac{1}{v}\frac{dv}{dx} = \cos x \frac{d}{dx}(\log \sin x) + logsinx \frac{d}{dx}(\cos x) \qquad [using\ product\ rule]$$

$$\Rightarrow \frac{1}{v}\frac{dv}{dx} = \cos x\left(\frac{1}{\sin x}\right)\frac{d}{dx}(\sin x) + \log \sin x(-\sin x)$$

$$\Rightarrow \frac{dv}{dx} = v\left[cotx(\cos x) - \sin x \log \sin x\right]$$

$$\Rightarrow \frac{dv}{dx} = (\sin x)^{\cos x}\left[cotx(\cos x) - \sin x \log \sin x\right]$$

$dividing\ equation(i)\ by\ (ii),$

$$\therefore \frac{du}{dv} = \frac{(\cos x)^{\sin x}[\cos x \log \cos x - \sin x \tan x]}{(\sin x)^{\cos x}\left[cotx(\cos x) - \sin x \log \sin x\right]}$$

Q9.

Answer :

Let, $u=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Put $x=\tan\theta$

$\Rightarrow \quad u=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$\Rightarrow \quad u=\sin^{-1}(\sin 2\theta) \qquad ...(i)$

Let $v=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$\Rightarrow \quad v=\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$\Rightarrow \quad v=\cos^{-1}(\cos 2\theta) \qquad ...(ii)$

Here, $0<x<1$

$\Rightarrow \quad 0<\tan\theta<1$

$\Rightarrow \quad 0<\theta<\frac{\pi}{4}$

So, from equation (i),

$u=2\theta \qquad \left[\text{Since, } \sin^{-1}(\sin\theta)=\theta, \text{ if } \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right]$

$\Rightarrow u=2\tan^{-1}x \qquad [\text{Since, } x=\tan\theta]$

$\frac{du}{dx}=\frac{2}{1+x^2} \qquad ...(iii)$

from equation (ii),

$v=2\theta \qquad \left[\text{Since, } \cos^{-1}(\cos\theta)=\theta, \text{ if } \theta\in[0,\pi]\right]$

$\Rightarrow v=2\tan^{-1}x \qquad [\text{Since, } x=\tan\theta]$

$\frac{dv}{dx}=\frac{2}{1+x^2} \qquad ...(iv)$

Dividing equation (iii) by (iv),

$$\frac{\frac{du}{dx}}{\frac{dv}{dx}}=\frac{2}{1+x^2}\times\frac{1+x^2}{2}$$

$\therefore \frac{du}{dv}=1$

Q10.

Answer :

$Let,\quad u=\tan^{-1}\left(\frac{1+ax}{1-ax}\right)$

$Put\quad ax=\tan\theta$

$$\Rightarrow\quad u=\tan^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$

$$\Rightarrow\quad u=\tan^{-1}\left(\frac{\tan\frac{\pi}{4}+\tan\theta}{1-\tan\frac{\pi}{4}\tan\theta}\right)$$

$$\Rightarrow\quad u=\tan^{-1}\left[\tan\left(\frac{\pi}{4}+\theta\right)\right]$$

$$\Rightarrow\quad u=\frac{\pi}{4}+\theta$$

$$\Rightarrow\quad u=\frac{\pi}{4}+\tan^{-1}(ax)\qquad [Since,\ \tan\theta=ax]$$

$$\frac{du}{dx}=0+\frac{1}{1+(ax)^2}\frac{d}{dx}(ax)$$

$$\Rightarrow\frac{du}{dx}=\frac{a}{1+a^2x^2}\qquad ...(i)$$

$Now,$

$Let,\quad v=\sqrt{1+a^2x^2}$

$$\frac{dv}{dx}=\frac{1}{2\sqrt{1+a^2x^2}}\frac{d}{dx}(1+a^2x^2)$$

$$\Rightarrow\frac{dv}{dx}=\frac{1}{2\sqrt{1+a^2x^2}}(2a^2x)$$

$$\Rightarrow\frac{dv}{dx}=\frac{a^2x}{\sqrt{1+a^2x^2}}\qquad ...(ii)$$

$Dividing\ equation\ (i)\ by\ (ii),$

$$\frac{\frac{du}{dx}}{\frac{dv}{dx}}=\frac{a}{1+a^2x^2}\times\frac{\sqrt{1+a^2x^2}}{a^2x}$$

$$\frac{du}{dv}=\frac{1}{ax\sqrt{1+a^2x^2}}$$

Q11.

Answer :

$$Let,\quad u=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$$

$$Put\quad x=\sin\theta$$

$$\Rightarrow\quad u=\sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$$

$$\Rightarrow\quad u=\sin^{-1}\left(2\sin\theta\cos\theta\right)$$

$$\Rightarrow\quad u=\sin^{-1}\left(\sin 2\theta\right)\qquad ...(i)$$

$$Let\ \ v=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

$$\Rightarrow\quad v=\tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right)$$

$$\Rightarrow\quad v=\tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$$

$$\Rightarrow\quad v=\tan^{-1}\left(\tan\theta\right)\qquad ...(ii)$$

$$Here,\quad -\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}$$

$$\Rightarrow\quad -\frac{1}{\sqrt{2}}<\sin\theta<\frac{1}{\sqrt{2}}$$

$$\Rightarrow\quad -\frac{\pi}{4}<\theta<\frac{\pi}{4}$$

$So,\ from\ equation\ (i),$

$$u=2\theta\qquad\left[Since,\ \sin^{-1}(\sin\theta)=\theta,\ if\ \ \theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right]$$

$$\Rightarrow u=2\sin^{-1}x\qquad\left[Since,\ x=\sin\theta\right]$$

Differentiating it with respect to x,

$$\frac{du}{dx}=\frac{2}{\sqrt{1-x^2}}\qquad ...(iii)$$

$from\ equation\ (ii),$

$$v=\theta\qquad\left[Since,\ \tan^{-1}(\tan\theta)=\theta,\ if\ \ \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right]$$

$$\Rightarrow v=\sin^{-1}x\qquad\left[Since,\ x=\sin\theta\right]$$

Differentiating it with respect to x,

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-x^2}} \qquad ...(iv)$$

Dividing equation (iii) *by* (iv),

$$\frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{2}{\sqrt{1-x^2}} \times \frac{\sqrt{1-x^2}}{1}$$

$$\therefore \frac{du}{dv} = 2$$

Q12.

Answer :

$$Let, \quad u = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

$$Put \quad x = \tan\theta$$

$$\Rightarrow \quad u = \tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$$

$$\Rightarrow \quad u = \tan^{-1}(\tan 2\theta) \qquad ...(i)$$

$$let, \quad v = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$\Rightarrow \quad v = \cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$$

$$\Rightarrow \quad v = \cos^{-1}(\cos 2\theta) \qquad ...(ii)$$

Here, $0 < x < 1$

$$\Rightarrow \quad 0 < \tan\theta < 1$$

$$\Rightarrow \quad 0 < \theta < \frac{\pi}{4}$$

So, from equation (i),

$$u = 2\theta \qquad \left[Since,\ \tan^{-1}(\tan\theta) = \theta,\ if\ \ \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right]$$

$$\Rightarrow u = 2\tan^{-1}x \qquad [Since,\ x = \tan\theta]$$

differentiating it with respect to x,

$$\frac{du}{dx}=\frac{2}{1+x^2} \qquad ...(iii)$$

From equation (ii),

$$v=\theta \qquad \left[Since,\ \cos^{-1}(\cos\theta)=\theta,\ if\ \ \theta\in[0,\pi]\right]$$

$$\Rightarrow v=2\tan^{-1}x \qquad [Since,\ x=\tan\theta]$$

$$\frac{dv}{dx}=\frac{2}{1+x^2} \qquad ...(iv)$$

Dividing equation (iii) *by* (iv),

$$\frac{\frac{du}{dx}}{\frac{dv}{dx}}=\frac{2}{1+x^2}\times\frac{1+x^2}{2}$$

$$\therefore \frac{du}{dv}=1$$

Q13.

Answer :

$$Let, \quad u=\tan^{-1}\left(\frac{x-1}{x+1}\right)$$

$$Put \quad x=\tan\theta$$

$$\Rightarrow \quad u=\tan^{-1}\left(\frac{\tan\theta-1}{\tan\theta+1}\right)$$

$$\Rightarrow \quad u=\tan^{-1}\left(\frac{\tan\theta-\tan\frac{\pi}{4}}{1+\tan\theta\tan\frac{\pi}{4}}\right)$$

$$\Rightarrow \quad u=\tan^{-1}\left[\tan\left(\theta-\frac{\pi}{4}\right)\right] \qquad ...(i)$$

$$Here,\ -\frac{1}{2}<x<\frac{1}{2}$$

$$\Rightarrow \quad -\frac{1}{2}<\tan\theta<\frac{1}{2}$$

$$\Rightarrow \quad -\tan^{-1}\left(\frac{1}{2}\right)<\theta<\tan^{-1}\left(\frac{1}{2}\right)$$

So, from equation (i),

$$u=\theta-\frac{\pi}{4} \qquad \left[Since,\ \tan^{-1}(\tan\theta)=\theta,\ if\ \ \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right]$$

$$\Rightarrow u=\tan^{-1}x-\frac{\pi}{4} \qquad [Since,\ x=\tan\theta]$$

differentiating it with respect to x,

$$\frac{du}{dx}=\frac{1}{1+x^2}-0$$

$$\Rightarrow \frac{du}{dx}=\frac{1}{1+x^2} \qquad ...(ii)$$

And,

Let, $v=\sin^{-1}\left(3x-4x^3\right)$

Put $x=\sin\theta$

$$\Rightarrow \quad v=\sin^{-1}\left(3\sin\theta-4\sin^3\theta\right)$$

$$\Rightarrow \quad v=\sin^{-1}\left(\sin 3\theta\right) \qquad ...(iii)$$

Now, $-\frac{1}{2}<x<\frac{1}{2}$

$$\Rightarrow \quad -\frac{1}{2}<\sin\theta<\frac{1}{2}$$

$$\Rightarrow \quad -\frac{1}{6}<\theta<\frac{\pi}{6}$$

So, from equation (iii),

$$v=3\theta \qquad \left[Since,\ \sin^{-1}(\sin\theta)=\theta,\ if\ \ \theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right]$$

$$\Rightarrow v=3\sin^{-1}x \qquad \left[Since,\ x=\sin\theta\right]$$

$$\frac{dv}{dx}=\frac{3}{\sqrt{1-x^2}} \qquad ...(iv)$$

Dividing equation (iii) *by* (iv),

$$\frac{\frac{du}{dx}}{\frac{dv}{dx}}=\frac{1}{1+x^2}\times\frac{\sqrt{1-x^2}}{3}$$

$$\therefore \frac{du}{dv}=\frac{\sqrt{1-x^2}}{3\left(1+x^2\right)}$$

Q14.

Answer :

$$Let,\quad u=\tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$$

$$\Rightarrow\quad u=\tan^{-1}\left[\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right]$$

$$\Rightarrow\quad u=\frac{\pi}{4}-\frac{x}{2}$$

Differentiating it with respect to x,

$$\frac{du}{dx}=0-\left(\frac{1}{2}\right)$$

$$\frac{du}{dx}=-\frac{1}{2}\qquad\ldots(i)$$

$$Let,\quad v=sec^{-1}x$$

Differentiating it with respect to x,

$$\frac{dv}{dx}=\frac{1}{x\sqrt{x^2-1}}\qquad\ldots(ii)$$

$$Dividing\ equation\ (i)\ by\ (ii),$$

$$\frac{\frac{du}{dx}}{\frac{dv}{dx}}=-\frac{1}{2}\times\frac{x\sqrt{x^2-1}}{1}$$

$$\frac{du}{dv}=\frac{-x\sqrt{x^2-1}}{2}$$

Q15.

Answer :

Let, $u=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Put $x=\tan\theta \Rightarrow \theta=\tan^{-1}x$,

$\Rightarrow \quad u=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$

$\Rightarrow \quad u=\sin^{-1}(\sin 2\theta) \qquad ...(i)$

Let, $v=\tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$\Rightarrow \quad v=\tan^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right)$

$\Rightarrow \quad v=\tan^{-1}(\tan 2\theta) \qquad ...(ii)$

Here, $-1<x<1$

$\Rightarrow \quad -1<\tan\theta<1$

$\Rightarrow \quad -\frac{\pi}{4}<\tan\theta<\frac{\pi}{4}$

So, from equation (i),

$u=2\theta \qquad \left[\text{Since, } \sin^{-1}(\sin\theta)=\theta, \text{ if } \theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right]$

$\Rightarrow u=2\tan^{-1}x$

Differentiating it with respect to x,

$\frac{du}{dx}=\frac{2}{1+x^2} \qquad ...(iii)$

from equation (ii),

$v=2\theta \qquad \left[\text{Since, } \tan^{-1}(\tan\theta)=\theta, \text{ if } \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right]$

$\Rightarrow v=2\tan^{-1}x$

Differentiating it with respect to x,

$\frac{dv}{dx}=\frac{2}{1+x^2} \qquad ...(iv)$

Dividing equation (iii) *by* (iv),

$$\frac{\frac{du}{dx}}{\frac{dv}{dx}}=\frac{2}{1+x^2}\times\frac{1+x^2}{2}$$

$$\therefore \frac{du}{dv}=1$$

Q16.

Answer :

$$Let, u = \cos^{-1}\left(4x^3 - 3x\right)$$
$$Put, x = \cos\theta$$
$$\Rightarrow \theta = \cos^{-1} x$$
$$Now, u = \cos^{-1}\left(4\cos^3\theta - 3\cos\theta\right)$$
$$\Rightarrow u = \cos^{-1}(\cos 3\theta)...(i)$$
$$Let, v = \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$$
$$\Rightarrow v = \tan^{-1}\left(\frac{\sqrt{1-\cos^2\theta}}{\cos\theta}\right)$$
$$\Rightarrow v = \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$$
$$\Rightarrow v = \tan^{-1}(\tan\theta)...(ii)$$
$$Here,$$
$$\frac{1}{2} < x < 1$$
$$\Rightarrow \frac{1}{2} < \cos\theta < 1$$
$$So, from\ equation(i),$$
$$u = 3\theta\left[Since, \cos^{-1}(\cos\theta) = \theta, if\ \theta \in [0, \pi]\right]$$
$$\Rightarrow u = 3\cos^{-1} x$$

Differentiating it with respect to x,

$$\frac{du}{dx} = \frac{-3}{\sqrt{1-x^2}} \qquad ...(iii)$$
$$From\ equation\ (ii),$$
$$v = \theta \qquad \left[Since,\ \tan^{-1}(\tan\theta) = \theta,\ if\ \ \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right]$$
$$\Rightarrow v = \cos^{-1} x$$

Differentiating it with respect to x,

$$\frac{dv}{dx}=\frac{-1}{\sqrt{1-x^2}} \qquad ...(iv)$$

Dividing equation (iii) *by* (iv),

$$\frac{\frac{du}{dx}}{\frac{dv}{dx}}=\left(\frac{-3}{\sqrt{1-x^2}}\right)\left(-\frac{\sqrt{1-x^2}}{1}\right)$$

$$\therefore \frac{du}{dv}=3$$

Q17.

Answer :

Let, $u=\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$

Put $x=\sin\theta$

$\Rightarrow \quad \theta=\sin^{-1}x$

$\Rightarrow \quad u=\tan^{-1}\left(\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}\right)$

$\Rightarrow \quad u=\tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)$

$\Rightarrow \quad u=\tan^{-1}(\tan\theta) \qquad ...(i)$

And

Let, $v=\sin^{-1}\left(2x\sqrt{1-x^2}\right)$

$v=\sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$

$v=\sin^{-1}(2\sin\theta\cos\theta)$

$v=\sin^{-1}(\sin 2\theta) \qquad ...(ii)$

Here,

$-\frac{1}{\sqrt{2}}<x<\frac{1}{\sqrt{2}}$

$\Rightarrow \quad -\frac{1}{\sqrt{2}}<\sin\theta<\frac{1}{\sqrt{2}}$

$\Rightarrow \quad -\frac{\pi}{4}<\theta<\frac{\pi}{4}$

So, from equation (i),

$u=\theta \qquad \left[\text{Since, } \tan^{-1}(\tan\theta)=\theta, \text{ if } \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right]$

$\Rightarrow u=\sin^{-1}x$

Differentiating it with respect to x,

$$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} \qquad ...(iii)$$

from equation (ii),

$$v = 2\theta \qquad \left[Since,\ \sin^{-1}(\sin\theta) = \theta,\ if\ \ \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right]$$

$$\Rightarrow\ v = 2\sin^{-1} x$$

Differentiating it with respect to x,

$$\frac{dv}{dx} = \frac{2}{\sqrt{1-x^2}} \qquad ...(iv)$$

Dividing equation (iii) *by* (iv),

$$\frac{\frac{du}{dx}}{\frac{dv}{dx}} = \left(\frac{1}{\sqrt{1-x^2}}\right)\left(\frac{\sqrt{1-x^2}}{2}\right)$$

$$\therefore \frac{du}{dv} = \frac{1}{2}$$

Q18.

Answer :

$$Let, \quad u=\sin^{-1}\left(\sqrt{1-x^2}\right)$$

$$Put \quad x=\cos\theta$$

$$\Rightarrow \quad \theta=\cos^{-1}x$$

$$We\ get, \quad u=\sin^{-1}(\sin\theta) \qquad ...(i)$$

$$Let, \quad v=cot^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

$$\Rightarrow \quad v=cot^{-1}\left(\frac{\cos\theta}{\sqrt{1-\cos^2\theta}}\right)$$

$$\Rightarrow \quad v=cot^{-1}\left(\frac{\cos\theta}{\sin\theta}\right)$$

$$\Rightarrow \quad v=cot^{-1}(cot\theta) \qquad ...(ii)$$

Here,

$$0<x<1$$

$$\Rightarrow \quad 0<\cos\theta<1$$

$$\Rightarrow \quad 0<\theta<\frac{\pi}{2}$$

$$So,\ from\ equation\ (i),$$

$$u=\theta \qquad \left[Since,\ \sin^{-1}(\sin\theta)=\theta,\ if\ \theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\right]$$

$$\Rightarrow u=\cos^{-1}x$$

Differentiating it with respect to x,

$$\frac{du}{dx}=\frac{-1}{\sqrt{1-x^2}} \qquad ...(iii)$$

$$From\ equation\ (ii),$$

$$v=\theta \qquad \left[Since,\ cot^{-1}(cot\theta)=\theta,\ if\ \ \theta\in(0,\pi)\right]$$

$$\Rightarrow v=\cos^{-1}x$$

Differentiating it with respect to x,

$$\frac{dv}{dx}=\frac{-1}{\sqrt{1-x^2}} \quad ...(iv)$$

Dividing equation (iii) by (iv),

$$\frac{\frac{du}{dx}}{\frac{dv}{dx}}=\left(\frac{-1}{\sqrt{1-x^2}}\right)\left(\frac{\sqrt{1-x^2}}{-1}\right)$$

$$\therefore \frac{du}{dv}=1$$

Q19.

Answer :

Let, $u=\sin^{-1}\left(2ax\sqrt{1-a^2x^2}\right)$

Put $ax=\sin\theta \Rightarrow \quad \theta=\sin^{-1}(ax)$

$\therefore \quad u=\sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$

$\Rightarrow \quad u=\sin^{-1}(2\sin\theta\cos\theta)$

$\Rightarrow \quad u=\sin^{-1}(\sin 2\theta) \quad ...(i)$

And

Let,

$v=\sqrt{1-a^2x^2}$

Differentiating it with respect to x,

$$\frac{dv}{dx}=\frac{1}{2\sqrt{1-a^2x^2}}\times\frac{d}{dx}\left(1-a^2x^2\right)$$

$$\Rightarrow \frac{dv}{dx}=\left(\frac{0-2a^2x}{2\sqrt{1-a^2x^2}}\right)$$

$$\Rightarrow \frac{dv}{dx}=\frac{-a^2x}{\sqrt{1-a^2x^2}} \quad ...(ii)$$

Differentiating it with respect to x,

$$\frac{du}{dx}=2\times\frac{1}{\sqrt{1-(ax)^2}}\frac{d}{dx}(ax)$$

$$\Rightarrow \frac{du}{dx}=\frac{2}{1-a^2x^2}(a)$$

$$\Rightarrow \frac{du}{dx}=\frac{2a}{1-a^2x^2} \quad ...(iii)$$

Dividing equation (iii) *by* (ii),

$$\frac{\frac{du}{dx}}{\frac{dv}{dx}}=\left(\frac{2a}{\sqrt{1-a^2x^2}}\right)\left(\frac{\sqrt{1-a^2x^2}}{-a^2x}\right)$$

$$\therefore \frac{du}{dv}=-\frac{2}{ax}$$

Q20.

Answer :

Let, $u=\tan^{-1}\left(\frac{1-x}{1+x}\right)$

Put $x=\tan\theta$

$\Rightarrow \quad \theta=\tan^{-1}x$

$\Rightarrow \quad u=\tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$

$\Rightarrow \quad u=\tan^{-1}\left[\tan\left(\frac{\pi}{4}-\theta\right)\right] \qquad ...(i)$

Here,

$-1<x<1$

$\Rightarrow \quad -1<\tan\theta<1$

$\Rightarrow \quad -\frac{\pi}{4}<\theta<\frac{\pi}{4}$

$\Rightarrow \quad \frac{\pi}{4}>-\theta>\frac{\pi}{4}$

$\Rightarrow \quad -\frac{\pi}{4}<-\theta<\frac{\pi}{4}$

$\Rightarrow \quad 0<\frac{\pi}{4}-\theta<\frac{\pi}{2}$

So, from equation (i),

$u=\frac{\pi}{4}-\theta \qquad \left[Since,\ \tan^{-1}(\tan\theta)=\theta,\ if\ \ \theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\right]$

$\Rightarrow u=\frac{\pi}{4}-\tan^{-1}x$

$\frac{du}{dx}=0-\left(\frac{1}{1+x^2}\right)$

$\Rightarrow \frac{du}{dx}=-\frac{1}{1+x^2} \qquad ...(ii)$

And let, $v=\sqrt{1-x^2}$

$\frac{dv}{dx}=\frac{1}{2\sqrt{1-x^2}}\times\frac{d}{dx}\left(1-x^2\right)$

$\Rightarrow \frac{dv}{dx}=\frac{1}{2\sqrt{1-x^2}}(-2x)$

$\Rightarrow \frac{dv}{dx}=\frac{-x}{\sqrt{1-x^2}} \qquad ...(iii)$

Dividing equation (ii) by (iii),

$\frac{\frac{du}{dx}}{\frac{dv}{dx}}=-\frac{1}{1+x^2}\times\frac{\sqrt{1-x^2}}{-x}$

$\therefore \frac{du}{dv}=\frac{\sqrt{1-x^2}}{x\left(1+x^2\right)}$