Exercise 6.1

Q1

Write the minors and co-factors of each element of the first column of the following matrices and hence evaluate the determinant.

$A = \begin{bmatrix} 5 & 20 \\ 0 & -1 \end{bmatrix}$

Solution

Let M_{ij} and C_{ij} represents the minor and co-factor respectively of an element which is placed at the i^{th} row and j^{th} column.

Now,

$M_{11} = -1$ $\left[\text{In a } 2\times 2 \text{ matrix, the minor is obtained for a particular element, by deleting that row and column where the element is present.}\right]$

$M_{21} = 20$

$C_{11} = (-1)^{1+1} \times M_{11}$ $\left[\because C_{ij} = (-1)^{i+j} \times M_{ij}\right]$

$= (+1)(-1)$

$= -1$

$C_{21} = (-1)^{2+1} M_{21}$

$= (-1)^3 \times 20$

$= -20$

Also,

$|A| = 5(-1) - (0)\times(20)$ $\left[\text{If } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ then } |A| = a_{11}a_{22} - a_{21}a_{12}\right]$

$= -5$

Q2

Write the minors and co-factors of each element of the first column of the following matrices and hence evaluate the determinant.

$A = \begin{bmatrix} -1 & 4 \\ 2 & 3 \end{bmatrix}$

Solution

Let M_{ij} and C_{ij} represents the minor and co-factor respectively of an element which is present at the i^{th} row and j^{th} column.

Now,

$M_{11} = 3$ $\left[\begin{array}{l}\text{In a } 2\times 2 \text{ matrix, the minor of an element is obtained by}\\ \text{deleting that row and that column in which it is present.}\end{array}\right]$

$M_{21} = 4$

$C_{11} = (-1)^{1+1} \times M_{11}$ $\left[C_{ij} = (-1)^{i+j} \times M_{ij}\right]$

$$\begin{aligned} C_{21} &= (-1)^{2+1} \times M_{21} \\ &= (-1)^3 \times 4 \\ &= -4 \end{aligned}$$

Also,

$$\begin{aligned} |A| &= (-1)\times(3) - (2)\times(4) \\ &= -3 - 8 \\ &= -11 \end{aligned}$$

Q3

Write the minors and co-factors of each element of the first column of the following matrices and hence evaluate the determinant.

$$A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{bmatrix}$$

Solution

Let M_{ij} and C_{ij} represents the minor and co-factor respectively of an element which is placed at the i^{th} row and j^{th} column.

Now,

$$M_{11} = \begin{bmatrix} -1 & 2 \\ 5 & 2 \end{bmatrix}$$

$\left[\text{In a } 3\times3 \text{ matrix, } M_{ij} \text{ equals to the determinant of the } 2\times2 \text{ sub-matrix obtained by leaving the } i^{th} \text{ row and } j^{th} \text{ column of } A.\right]$

$$= (-1)\times(2)-(5)\times(2)$$
$$= -2-10$$
$$= -12$$

$$M_{21} = \begin{bmatrix} -3 & 2 \\ 5 & 2 \end{bmatrix} = (-3)\times(2)-(5)\times(2) = -6-10 = -16$$

$$M_{31} = \begin{bmatrix} -3 & 2 \\ -1 & 2 \end{bmatrix} = (-3)(2)-(-1)(2) = -6+2 = -4$$

$$C_{11} = (-1)^{1+1}M_{11} \qquad \left(C_{ij} = (-1)^{i+j}\times M_{ij}\right)$$
$$= (+)(-12) = -12$$
$$C_{21} = (-1)^{2+1}M_{21} = (-1)^3(-16) = 16$$
$$C_{31} = (-1)^{3+1}M_{31} = (-1)^4(-4) = -4$$

Also, expanding the determinant along the first column.

$$|A| = a_{11}\times\left((-1)^{1+1}\times M_{11}\right) + a_{21}\times\left((-1)^{2+1}\times M_{21}\right) + a_{31}\times\left((-1)^{3+1}\times M_{31}\right)$$
$$= a_{11}\times C_{11} + a_{21}\times C_{21} + a_{31}\times C_{31}$$
$$= 1\times(-12) + 4\times16 + 3\times(-4)$$
$$= -12+48-12 = 24$$

Q4

Write the minors and co-factors of each element of the first column of the following matrices and hence evaluate the determinant.

$$A = \begin{bmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{bmatrix}$$

Solution

Let M_{ij} and C_{ij} are respectively the minor and co-factor of the element a_{ij}.

Now,

$$M_{11} = \begin{bmatrix} b & ca \\ c & ab \end{bmatrix}$$
$$= ab^2 - ac^2$$

$$M_{21} = \begin{bmatrix} a & bc \\ c & ab \end{bmatrix}$$
$$= a^2b - c^2b$$

$$M_{31} = \begin{bmatrix} a & bc \\ b & ca \end{bmatrix}$$
$$= a^2c - b^2c$$

$$C_{11} = (-1)^{1+1} \times M_{11} = +(ab^2 - ac^2)$$
$$C_{21} = (-1)^{2+1} \times M_{21} = -(a^2b - c^2b)$$
$$C_{31} = (-1)^{3+1} \times M_{31} = +(a^2c - b^2c)$$

Also, expanding the determinant, along the first column.

$$|A| = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31}$$
$$= 1(ab^2 - ac^2) + 1(c^2b - a^2b) + 1 \times (a^2c - b^2c)$$
$$= ab^2 - ac^2 + c^2b - a^2b + a^2c - b^2c$$

Q5

Write the minors and co-factors of each element of the first column of the following matrices and hence evaluate the determinant.

$$A = \begin{bmatrix} 0 & 2 & 6 \\ 1 & 5 & 0 \\ 3 & 7 & 1 \end{bmatrix}$$

Solution

Let M_{ij} and C_{ij} are respectively the minor and co-factor of the element a_{ij}.

Now,

$$M_{11} = \begin{bmatrix} 5 & 0 \\ 7 & 1 \end{bmatrix} = 5 - 0 = 5$$

$$M_{21} = \begin{bmatrix} 2 & 6 \\ 7 & 1 \end{bmatrix} = 2 - 42 = -40$$

$$M_{31} = \begin{bmatrix} 2 & 6 \\ 5 & 0 \end{bmatrix} = 0 - 30 = -30$$

$$C_{11} = (-1)^{1+1} \times M_{11} = +5$$

$$C_{21} = (-1)^{2+1} \times M_{21} = (-)(-40) = 40$$

$$C_{31} = (-1)^{3+1} \times M_{31} = +(-30) = -30$$

Now, expanding the determinant along the first column.

$$\begin{aligned} |A| &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} \\ &= 0 \times 5 + 1 \times (40) + 3 \times (-30) \\ &= 40 - 90 \\ &= -50 \end{aligned}$$

Q6

Write the minors and co-factors of each element of the first column of the following matrices and hence evaluate the determinant.

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

Solution

Let M_{ij} and C_{ij} are respectively the minor and co-factor of the element a_{ij}.

Now,

$$M_{11} = \begin{bmatrix} b & f \\ f & c \end{bmatrix} = bc - f^2$$

$$M_{21} = \begin{bmatrix} h & g \\ f & c \end{bmatrix} = hc - gf$$

$$M_{31} = \begin{bmatrix} h & g \\ b & f \end{bmatrix} = hf - bg$$

$$Also\ C_{11} = (-1)^{1+1} M_{11} = bc - f^2$$

$$C_{21} = (-1)^{2+1} M_{21} = -(hc - gf)$$

$$C_{31} = (-1)^{3+1} M_{31} = hf - bg$$

Also, expanding along the first column.

$$\begin{aligned} |A| &= a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} \\ &= a(bc - f^2) + h(-)(hc - gf) + g(hf - bg) \\ &= abc - af^2 + hgf - h^2c + ghf - bg^2 \end{aligned}$$

Q7

Write the minors and cofactors of each element of the first column of the matrix $A = \begin{bmatrix} 2 & -1 & 0 & 1 \\ -3 & 0 & 1 & -2 \\ 1 & 1 & -1 & 1 \\ 2 & -1 & 5 & 0 \end{bmatrix}$ and evaluate the determinant.

Solution

We have,

$$\begin{bmatrix} 2 & -1 & 0 & 1 \\ -3 & 0 & 1 & -2 \\ 1 & 1 & -1 & 1 \\ 2 & -1 & 5 & 0 \end{bmatrix}$$

Here, $M_{11} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -1 & 1 \\ -1 & 5 & 0 \end{bmatrix} = -1(0+10) - 1(1-2) = -9$

$M_{21} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 1 \\ -1 & 5 & 0 \end{bmatrix} = 9$

$M_{31} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -2 \\ -1 & 5 & 0 \end{bmatrix} = -9$

$M_{41} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & - & 1 \end{bmatrix} = 0$

$\therefore$ $C_{11} = (-1)^{1+1} M_{11} = -9$

$C_{21} = (-1)^3 M_{21} = -9$

$C_{31} = (-4)^4 M_{31} = -9$

$C_{41} = (-1)^5 M_{41} = 0$

Hence,

$$\begin{bmatrix} 2 & -1 & 0 & 1 \\ -3 & 0 & 1 & -2 \\ 1 & 1 & -1 & 1 \\ 2 & -1 & 5 & 0 \end{bmatrix} = 2 \times C_{11} + (-3) C_{21} + 1 \times C_{31} + 2 \times C_{41} = -9[2 - 3 + 1] = 0$$

Q8

Evaluate the following determinant:

$$\begin{bmatrix} x & -7 \\ x & 5x+1 \end{bmatrix}$$

Solution

Let $A = \begin{vmatrix} x & -7 \\ x & 5x+1 \end{vmatrix}$

$|A| = x(5x+1) + 7 \times x$

$= 5x^2 + x + 7x$

$= 5x^2 + 8x$

Hence $|A| = 5x^2 + 8x$

Q9

Evaluate the following determinant:

$$\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$$

Solution

Let $A = \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$

$$\begin{aligned} |A| &= \cos\theta \times \cos\theta + \sin\theta \times \sin\theta \\ &= \cos^2\theta + \sin^2\theta \\ &= 1 \end{aligned}$$

Hence $|A| = 1$

Q10

Evaluate the following determinant:

$$\begin{vmatrix} \cos 15^\circ & \sin 15^\circ \\ \sin 75^\circ & \cos 75^\circ \end{vmatrix}$$

Solution

Let $A = \begin{vmatrix} \cos 15^\circ & \sin 15^\circ \\ \sin 75^\circ & \cos 75^\circ \end{vmatrix}$

$$\begin{aligned} |A| &= \cos 15^\circ \cos 75^\circ - \sin 15^\circ \sin 75^\circ \\ &= \cos(75+15) \qquad (\because \cos A \cos B - \sin A \sin B = \cos(A+B)) \\ &= \cos 90^\circ \\ &= 0 \end{aligned}$$

Hence $|A| = 0$

Q11

Evaluate the following determinant:

$$\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix}$$

Solution

Let $A = \begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix}$

$$\begin{aligned} |A| &= (a+ib)(a-ib)-(c+id)(-c+id) \\ &= (a^2+b^2)+(c+id)(c-id) \qquad \{\text{Taking } (-) \text{ sign common from } -c+id\} \\ & \qquad \{Also\ (a+ib)(a-ib)=a^2+b^2\} \end{aligned}$$

$= a^2+b^2+c^2+d^2$

Hence $|A| = a^2+b^2+c^2+d^2$

Q12

Evaluate $\begin{vmatrix} 2 & 3 & 7 \\ 13 & 17 & 5 \\ 15 & 20 & 12 \end{vmatrix}^2$

Solution

Since $|AB| = |A| \times |B|$

Hence $|A|^2 = |A| \times |A| \qquad ---(1)$

Now let $A = \begin{vmatrix} 2 & 3 & 7 \\ 13 & 17 & 5 \\ 15 & 20 & 12 \end{vmatrix}$

Expanding along the first column, we get

$$\begin{aligned} |A| &= 2\begin{vmatrix} 17 & 5 \\ 20 & 12 \end{vmatrix} - 3\begin{vmatrix} 13 & 5 \\ 15 & 12 \end{vmatrix} + 7\begin{vmatrix} 13 & 17 \\ 15 & 20 \end{vmatrix} \\ &= 2(204-100)-3(156-75)+7(260-255) \\ &= 2(104)-3(81)+7(5) \\ &= 208-243+35 \\ &= 243-243 \\ &= 0 \end{aligned}$$

Hence from eq. (1)

$|A|^2 = |A| \times |A| = 0 \times 0 = 0$

Q13

Show that $\begin{vmatrix} \sin 10^\circ & \cos 10^\circ \\ \sin 80^\circ & \cos 80^\circ \end{vmatrix} = 1$

Solution

Evaluating the given determinant

$\sin 10^\circ \times \cos 80^\circ + \cos 10^\circ \sin 80^\circ$

$= \sin(10^\circ + 80^\circ)$ $\quad [\because \sin A \cos B + \cos A \sin B = \sin(A+B)]$

$= \sin 90^\circ$

$= 1$

Hence proved

Q14

Evaluate $\begin{vmatrix} 2 & 3 & -5 \\ 7 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix}$ by two methords.

Solution

We will evaluate the given determinant

(i) Along the first row

(ii) Along the first column

(i) Along the first row

$$|A| = 2\begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} - 3\begin{vmatrix} 7 & -2 \\ -3 & 1 \end{vmatrix} - 5\begin{vmatrix} 7 & 1 \\ -3 & 4 \end{vmatrix}$$

$= 2(1+8) - 3(7-6) - 5(28+3)$

$= 2(9) - 3(1) - 5(31)$

$= 18 - 3 - 155 = -140$

(ii) Along the first column

$$|A| = 2\begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} - 7\begin{vmatrix} 3 & -5 \\ 4 & 1 \end{vmatrix} - 3\begin{vmatrix} 3 & -5 \\ 1 & -2 \end{vmatrix}$$

$= 2(1+8) - 7(3+20) - 3(-6+5)$

$= 18 - 7(23) - 3(-1)$

$= 18 - 161 + 3$

$= 21 - 161$

$= -140$

We can see, the answer is same with both the methods.

Q15

Evaluate $\Delta = \begin{vmatrix} 0 & \sin\alpha & -\cos\alpha \\ -\sin\alpha & 0 & \sin\beta \\ \cos\alpha & -\sin\beta & 0 \end{vmatrix}$

Solution

$$\Delta=\begin{vmatrix}0 & \sin\alpha & -\cos\alpha\\ -\sin\alpha & 0 & \sin\beta\\ \cos\alpha & -\sin\beta & 0\end{vmatrix}$$
$$=-\sin\alpha(-\sin\beta\cos\alpha)-\cos\alpha(\sin\alpha\sin\beta)$$
$$=0$$

Q16

Evaluate

$$\begin{vmatrix}\cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha\\ -\sin\beta & \cos\beta & 0\\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha\end{vmatrix}$$

Solution

$$\Delta=\begin{vmatrix}\cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha\\ -\sin\beta & \cos\beta & 0\\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha\end{vmatrix}$$

Expanding along C_3, we have:

$$\Delta=-\sin\alpha\left(-\sin\alpha\sin^2\beta-\cos^2\beta\sin\alpha\right)+\cos\alpha\left(\cos\alpha\cos^2\beta+\cos\alpha\sin^2\beta\right)$$
$$=\sin^2\alpha\left(\sin^2\beta+\cos^2\beta\right)+\cos^2\alpha\left(\cos^2\beta+\sin^2\beta\right)$$
$$=\sin^2\alpha(1)+\cos^2\alpha(1)$$
$$=1$$

Q17

If $A=\begin{bmatrix}2 & 5\\ 2 & 1\end{bmatrix}$ and $B=\begin{bmatrix}4 & -3\\ 2 & 5\end{bmatrix}$, verify that $|AB|=|A||B|$

Solution

Let $A = \begin{bmatrix} 2 & 5 \\ 2 & 1 \end{bmatrix}$

$\Rightarrow |A| = 2 - 10 = -8$

$B = \begin{bmatrix} 4 & -3 \\ 2 & 5 \end{bmatrix}$

$\Rightarrow |B| = 20 + 6 = 26$

Now $AB = \begin{bmatrix} 2 & 5 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ 2 & 5 \end{bmatrix}$

$= \begin{bmatrix} 2\times 4 + 5\times 2 & 2\times(-3) + 5\times 5 \\ 2\times 4 + 1\times 2 & 2\times(-3) + 1\times 5 \end{bmatrix}$

$= \begin{bmatrix} 8+10 & -6+25 \\ 8+2 & -6+5 \end{bmatrix}$

$= \begin{bmatrix} 18 & 19 \\ 10 & -1 \end{bmatrix}$

$\Rightarrow |AB| = 18\times(-1) - (10)(19)$

$= -18 - 190 = -208$

Now $|AB| = |A| \times |B|$

$-208 = (-8)\times(26)$

$-208 = -208$

Hence verified.

Q18

If $A = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{vmatrix}$, then show that $|3A| = 27|A|$.

Solution

Let $A = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{vmatrix}$

Evaluating the determinant along the first column

$$|A| = 1\begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} - 0\begin{vmatrix} 0 & 1 \\ 0 & 4 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}$$
$$= 1 \times (4 - 0) - 0 + 0$$
$$= 4$$

Again $3A = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{bmatrix}$ (every element of A will be multiplied by 3)

Now, evaluating this determinant

$$|3A| = 3\begin{vmatrix} 3 & 6 \\ 0 & 12 \end{vmatrix} - 0\begin{vmatrix} 0 & 3 \\ 0 & 12 \end{vmatrix} + 0\begin{vmatrix} 0 & 3 \\ 3 & 6 \end{vmatrix}$$
$$= 3(36 - 0) - 0 + 0$$
$$= 108$$

Now, according to the question

$|3A| = 27|A|$

$108 = 27(4)$ (Substituting values)

$108 = 108$

Hence proved

Q19

Find values of x, if

(i) $\begin{vmatrix} 2 & 4 \\ 2 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$ (ii) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$ (iii) $\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$ (iv) $\begin{vmatrix} 3x & 7 \\ 2 & 4 \end{vmatrix} = 10$

Solution

(i) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$

$\Rightarrow 2\times1-5\times4 = 2x\times x-6\times4$
$\Rightarrow 2-20 = 2x^2-24$
$\Rightarrow 2x^2 = 6$
$\Rightarrow x^2 = 3$
$\Rightarrow x = \pm\sqrt{3}$

(ii) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

$\Rightarrow 2\times5-3\times4 = x\times5-3\times2x$
$\Rightarrow 10-12 = 5x-6x$
$\Rightarrow -2 = -x$
$\Rightarrow x = 2$

(iii)

$\begin{vmatrix} 3 & x \\ x & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix}$
$3-x^2 = 3-8$
$x^2 = 8$
$x = \pm2\sqrt{2}$

(iv)

$\begin{vmatrix} 3x & 7 \\ 2 & 4 \end{vmatrix} = 10$
$12x-14 = 10$
$12x = 24$
$x = 2$

Q20

$\begin{vmatrix} x+1 & x-1 \\ x-3 & x+2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$, find the value of x.

Solution

$\begin{vmatrix} x+1 & x-1 \\ x-3 & x+2 \end{vmatrix} = \begin{vmatrix} 4 & -1 \\ 1 & 3 \end{vmatrix}$
$\Rightarrow (x+1)(x+2)-(x-1)(x-3) = 4\times3+1\times1$
$\Rightarrow x^2+3x+2-(x^2-4x+3) = 13$
$\Rightarrow x^2+3x+2-x^2+4x-3 = 13$
$\Rightarrow 7x-1 = 13$
$\Rightarrow 7x = 14$
$\Rightarrow x = \frac{14}{7}$
$\therefore x = 2$

Q21

Find the values of x, if

$$\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & 5 \\ 8 & 3 \end{vmatrix}$$

Solution

$$\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & 5 \\ 8 & 3 \end{vmatrix}$$

$$(2x)(x) - (5)(8) = 6 \times 3 - 8 \times 5$$

$$2x^2 = 18$$

$$x^2 = 9$$

$$x = \pm 3$$

Q22

Find the integral value of x, if $\begin{vmatrix} x^2 & x & 1 \\ 0 & 2 & 1 \\ 3 & 1 & 4 \end{vmatrix} = 28$

Solution

Let $A = \begin{vmatrix} x^2 & x & 1 \\ 0 & 2 & 1 \\ 3 & 1 & 4 \end{vmatrix}$

Expanding the given determinant along the first column

$$|A| = x^2 \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} - 0 \begin{vmatrix} x & 1 \\ 1 & 4 \end{vmatrix} + 3 \begin{vmatrix} x & 1 \\ 2 & 1 \end{vmatrix}$$

$$28 = x^2(8 - 1) - 0(4x - 1) + 3(x - 2)$$

$$28 = 7x^2 + 3x - 6$$

or

$$7x^2 + 3x - 6 = 28$$

$$7x^2 + 3x - 34 = 0$$

Solving using quadratic formula, we get $x = 2$.

Q23

For what value of x the matrix $A = \begin{vmatrix} 1+x & 7 \\ 3-x & 8 \end{vmatrix}$ is singular ?

Solution

A matrix A is said to be singular if $|A| = 0$

Now

$$\begin{vmatrix} 1+x & 7 \\ 3-x & 8 \end{vmatrix} = 0$$
$$8 + 8x - 21 + 7x = 0$$
$$15x = 13$$
$$x = \frac{13}{15}$$

Q24

For what value of x the matrix $A = \begin{vmatrix} x-1 & 1 & 1 \\ 1 & x-1 & 1 \\ 1 & 1 & x-1 \end{vmatrix}$ is singular?

Solution

A matrix A is called singular if $|A| = 0$

Now expanding along the first row $|A|$

$$= (x-1)\begin{vmatrix} x-1 & 1 \\ 1 & x-1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & x-1 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ x-1 & 1 \end{vmatrix}$$
$$= (x-1)\left[(x-1)^2 - 1\right] - 1[x-1-1] + 1[1-x+1]$$
$$= (x-1)\left(x^2 + 1 - 2x - 1\right) - 1(x-2) + 1(2-x)$$
$$= (x-1)\left(x^2 - 2x\right) - x + 2 + 2 - x$$
$$= (x-1) \times x \times (x-2) + (4-2x)$$
$$= (x-1) \times x \times (x-2) + 2(2-x)$$
$$= (x-1) \times x \times (x-2) - 2(x-2)$$
$$= (x-2)[x(x-1) - 2] \qquad (\text{Taking } (x-2) \text{ common})$$

Since A is a singular matrix, so $|A| = 0$

$i.e\ (x-2)\left(x^2 - x - 2\right) = 0$

either $(x-2) = 0$ or $x^2 - x - 2 = 0$

$x = 2$ or $x^2 - 2x + x - 2 = 0$

$$x(x-2) + 1(x-2) = 0$$
$$(x-2)(x+1) = 0$$
$$x = 2, -1$$

$x = 2$ or -1

Exercise 6.2

Q1

Evaluate the determinant:

$$\begin{vmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 31 & 11 & 38 \end{vmatrix}$$

Solution

$$\begin{vmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 31 & 11 & 38 \end{vmatrix} = 2\begin{vmatrix} 1 & 3 & 5 \\ 1 & 3 & 5 \\ 31 & 11 & 38 \end{vmatrix} = 0$$

Q2

Evaluate the determinant:

$$\begin{vmatrix} 67 & 19 & 21 \\ 39 & 13 & 14 \\ 81 & 24 & 26 \end{vmatrix}$$

Solution

Consider the determinant

$$\Delta = \begin{vmatrix} 67 & 19 & 21 \\ 39 & 13 & 14 \\ 81 & 24 & 26 \end{vmatrix}$$

Applying $C_1 \to C_1 - 4C_3$, we get,

$$\Delta = \begin{vmatrix} 4 & 19 & 21 \\ -3 & 13 & 14 \\ -3 & 24 & 26 \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 4 & 19 & 21 \\ -3 & 13 & 14 \\ -3 & 24 & 26 \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 32 & 35 \\ -3 & 13 & 14 \\ 0 & 11 & 12 \end{vmatrix} \quad [Applying\ R_3 \to R_3 - R_2\ and\ R_1 \to R_1 + R_2]$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 32 & 35 \\ 0 & 109 & 119 \\ 0 & 11 & 12 \end{vmatrix} \quad [Applying\ R_2 \to 3R_1 + R_2]$$

$$\Rightarrow \Delta = 1(109 \times 12 - 119 \times 11)$$

$$\Rightarrow \Delta = -1$$

Q3

Evaluate the determinant:

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Solution

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$
$$= a(bc - f^2) - h(hc - fg) + g(hf - bg)$$
$$= abc - af^2 - h^2c + hfg + ghf - bg^2$$

Q4

Evaluate the determinant:

$$\begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix}$$

Solution

$$\begin{vmatrix} 1 & -3 & 2 \\ 4 & -1 & 2 \\ 3 & 5 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & -3 & 1 \\ 4 & -1 & 1 \\ 3 & 5 & 1 \end{vmatrix} = 2\begin{vmatrix} 1 & -3 & 1 \\ 3 & 2 & 0 \\ 2 & 8 & 0 \end{vmatrix} = 2(24 - 4) = 40$$

Q5

Evaluate the determinant:

$$\begin{vmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \\ 9 & 16 & 25 \end{vmatrix}$$

Solution

Let Δ be the determinant.

$$\Delta = \begin{vmatrix} 1 & 4 & 9 \\ 4 & 9 & 16 \\ 9 & 16 & 25 \end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$, we get,

$$\Delta = \begin{vmatrix} 1 & 4 & 9-4 \\ 4 & 9 & 16-9 \\ 9 & 16 & 25-16 \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 4 & 5 \\ 4 & 9 & 7 \\ 9 & 16 & 9 \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 5 & 5 \\ 4 & 13 & 7 \\ 9 & 25 & 9 \end{vmatrix} \quad [\text{Applying } C_2 \to C_1 + C_2]$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 0 & 0 \\ 4 & -7 & -13 \\ 9 & -20 & -36 \end{vmatrix} \quad [\text{Applying } C_2 \to 5C_1 - C_2 \text{ and } C_3 \to 5C_1 - C_3]$$

$$\Rightarrow \Delta = 1(7 \times 36 - 13 \times 20) = 252 - 260 = -8$$

Q6

Evaluate the determinant:

$$\begin{vmatrix} 6 & -3 & 2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{vmatrix}$$

Solution

$$\begin{vmatrix} 6 & -3 & 2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{vmatrix}$$

Apply: $R_1 \to R_1 + (-3)R_2$ and $R_3 \to R_3 + 5R_2$

$$= \begin{vmatrix} 0 & 0 & -4 \\ 2 & -1 & 2 \\ 0 & 0 & 12 \end{vmatrix} = 0$$

Q7

Evaluate the determinant:

$$\begin{vmatrix} 1 & 3 & 9 & 27 \\ 3 & 9 & 27 & 1 \\ 9 & 27 & 1 & 3 \\ 27 & 1 & 3 & 9 \end{vmatrix}$$

Solution

$$\begin{vmatrix} 1 & 3 & 9 & 27 \\ 3 & 9 & 27 & 1 \\ 9 & 27 & 1 & 3 \\ 27 & 1 & 3 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 3^2 & 3^3 \\ 3 & 3^2 & 3^3 & 1 \\ 3^2 & 3^3 & 1 & 3 \\ 3^3 & 1 & 3 & 3^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1+3+3^2+3^3 & 3 & 3^2 & 3^3 \\ 1+3+3^2+3^3 & 3^2 & 3^3 & 1 \\ 1+3+3^2+3^3 & 3^3 & 1 & 3 \\ 1+3+3^2+3^3 & 1 & 3 & 3^2 \end{vmatrix}$$

$$= \left(1+3+3^2+3^3\right)\begin{vmatrix} 1 & 3 & 3^2 & 3^3 \\ 1 & 3^2 & 3^3 & 1 \\ 1 & 3^3 & 1 & 3 \\ 1 & 1 & 3 & 3^2 \end{vmatrix}$$

$$= \left(1+3+3^2+3^3\right)\begin{vmatrix} 1 & 3 & 3^2 & 3^3 \\ 0 & 3^2-3 & 3^3-3^2 & 1-3^3 \\ 0 & 3^3-3 & 1-3^2 & 3-3^3 \\ 0 & 1-3 & 3-3^2 & 3^2-3^3 \end{vmatrix}$$

$$= \left(1+3+3^2+3^3\right)\begin{vmatrix} 6 & 18 & -26 \\ 24 & -8 & -24 \\ -2 & -6 & -18 \end{vmatrix}$$

$$= \left(1+3+3^2+3^3\right)2^3\begin{vmatrix} 3 & -9 & 13 \\ 12 & 4 & 12 \\ -1 & 3 & 9 \end{vmatrix}$$

$$= \left(1+3+3^2+3^3\right)2^3\begin{vmatrix} 0 & 0 & 40 \\ 12 & 4 & 12 \\ -1 & 3 & 9 \end{vmatrix}$$

$$= \left(1+3+3^2+3^3\right)2^3 \times 40(36+4) = 512000$$

Q8

Evaluate the determinant $\begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$

Solution

$$Let\ \Delta = \begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 17 & 3 & 6 \end{vmatrix}$$

Applying $R_3 \to 17R_2 - R_3$, we get,

$$\Delta = \begin{vmatrix} 102 & 18 & 36 \\ 1 & 3 & 4 \\ 0 & 48 & 62 \end{vmatrix}$$

Applying $R_2 \to 102R_2 - R_1$, we get,

$$\Delta = \begin{vmatrix} 102 & 18 & 36 \\ 0 & 288 & 372 \\ 0 & 48 & 62 \end{vmatrix}$$

Thus,

$$\Delta = 102(288 \times 62 - 372 \times 48)$$
$$\Rightarrow \Delta = 0$$

Q9

Without expanding, show that the values of determinant is zero:

$$\begin{vmatrix} 8 & 2 & 7 \\ 12 & 3 & 5 \\ 16 & 4 & 3 \end{vmatrix}$$

Solution

$$\begin{vmatrix} 8 & 2 & 7 \\ 12 & 3 & 5 \\ 16 & 4 & 3 \end{vmatrix}$$

Apply: $R_3 \to R_3 - R_2$

$$= \begin{vmatrix} 8 & 2 & 7 \\ 12 & 3 & 5 \\ 4 & 1 & -2 \end{vmatrix}$$

Apply: $R_2 \to R_2 - R_1$

$$= \begin{vmatrix} 8 & 2 & 7 \\ 4 & 1 & -2 \\ 4 & 1 & -2 \end{vmatrix}$$

Since, $R_3 = R_2$, the value of the determinant is zero.

Q10

Without expanding, show that the value of determinant is zero:

$$\begin{vmatrix} 6 & -3 & 2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{vmatrix}$$

Solution

$$\begin{vmatrix} 6 & -3 & 2 \\ 2 & -1 & 2 \\ -10 & 5 & 2 \end{vmatrix}$$

Taking (-2) common from C_1, we get

$$= (-2)\begin{vmatrix} -3 & -3 & 2 \\ -1 & -1 & 2 \\ 5 & 5 & 2 \end{vmatrix}$$

$= 0$

$\because C_1$ and C_2 are identical.

Q11

Without expanding, show that the value of determinant is zero:

$$\begin{vmatrix} 2 & 3 & 7 \\ 13 & 17 & 5 \\ 15 & 20 & 12 \end{vmatrix}$$

Solution

$$\begin{vmatrix} 2 & 3 & 7 \\ 13 & 17 & 5 \\ 15 & 20 & 12 \end{vmatrix}$$

Use: $R_3 \to R_3 - R_2$

$$= \begin{vmatrix} 2 & 3 & 7 \\ 13 & 17 & 5 \\ 2 & 3 & 7 \end{vmatrix}$$

$= 0$

$\because R_3 = R_1$

Q12

Without expanding, show that the value of determinant is zero:

$$\begin{vmatrix} \frac{1}{a} & a^2 & bc \\ \frac{1}{b} & b^2 & ca \\ \frac{1}{c} & c^2 & ab \end{vmatrix}$$

Solution

$$\begin{vmatrix} \frac{1}{a} & a^2 & bc \\ \frac{1}{b} & b^2 & ca \\ \frac{1}{c} & c^2 & ab \end{vmatrix}$$

Multiply: R_1, R_2 and R_3 by a, b and c respectively, we get

$$= \frac{1}{abc}\begin{vmatrix} 1 & a^3 & abc \\ 1 & b^3 & bca \\ 1 & c^3 & cab \end{vmatrix}$$

Take abc common from C_3, we get,

$$= \begin{vmatrix} 1 & a^3 & 1 \\ 1 & b^3 & 1 \\ 1 & c^3 & 1 \end{vmatrix}$$

$= 0$

$\because \quad C_1 = C_3$

Q13

Without expanding, show that the value of determinant is zero:

$$\begin{vmatrix} a+b & 2a+b & 3a+b \\ 2a+b & 3a+b & 4a+b \\ 4a+b & 5a+b & 6a+b \end{vmatrix}$$

Solution

$$\begin{vmatrix} a+b & 2a+b & 3a+b \\ 2a+b & 3a+b & 4a+b \\ 4a+b & 5a+b & 6a+b \end{vmatrix}$$

Apply: $C_3 \to C_3 - C_2$

$$= \begin{vmatrix} a+b & 2a+b & a \\ 2a+b & 3a+b & a \\ 4a+b & 5a+b & a \end{vmatrix}$$

Apply: $C_2 \to C_2 - C_1$

$$= \begin{vmatrix} a+b & a & a \\ 2a+b & a & a \\ 4a+b & a & a \end{vmatrix}$$

$= 0$

$\because \quad C_3 = C_2$

Q14

Without expanding, show that the value of determinant is zero:

$$\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ac \\ 1 & c & c^2-ab \end{vmatrix}$$

Solution

$$\begin{vmatrix}1 & a & a^2-bc\\1 & b & b^2-ac\\1 & c & c^2-ab\end{vmatrix}$$

$$=\begin{vmatrix}1 & a & a^2\\1 & b & b^2\\1 & c & c^2\end{vmatrix}-\begin{vmatrix}1 & a & bc\\1 & b & ac\\1 & c & ab\end{vmatrix}$$

$$=\begin{vmatrix}1 & a & a^2\\0 & b-a & b^2-a^2\\0 & c-a & c^2-a^2\end{vmatrix}-\begin{vmatrix}1 & a & bc\\0 & b-a & (a-b)c\\0 & c-a & (a-c)b\end{vmatrix}$$

$$=(b-a)(c-a)\begin{vmatrix}1 & a & a^2\\0 & 1 & b+a\\0 & 1 & c+a\end{vmatrix}-(b-a)(c-a)\begin{vmatrix}1 & a & bc\\0 & 1 & -c\\0 & 1 & -b\end{vmatrix}$$

$$=(b-a)(c-a)(c+a-b-a)-(b-a)(c-a)(-b+c)$$

$$=(b-a)(c-a)(c-b)-(b-a)(c-a)(-b+c)$$

$$=0$$

Q15

Without expanding, show that the value of determinant is zero:

$$\begin{vmatrix}49 & 1 & 6\\39 & 7 & 4\\26 & 2 & 3\end{vmatrix}$$

Solution

$$\begin{vmatrix}49 & 1 & 6\\39 & 7 & 4\\26 & 2 & 3\end{vmatrix}$$

Apply: $C_1 \to C_1+(-8)C_3$

$$=\begin{vmatrix}1 & 1 & 6\\7 & 7 & 4\\2 & 2 & 3\end{vmatrix}=0$$

$\because \quad C_1 = C_2$

Q16

Without expanding, show that the values of determinants are zero:

$$\begin{vmatrix}0 & x & y\\-x & 0 & z\\-y & -z & 0\end{vmatrix}$$

Solution

$$\begin{vmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{vmatrix}$$

Multiply C_1, C_2 and C_3 by z, y, and x respectively

$$= \frac{1}{xyz}\begin{vmatrix} 0 & xy & yx \\ -xz & 0 & zx \\ -yz & -zy & 0 \end{vmatrix}$$

Take y, x, and z common from R_1, R_2 and R_3 respectively

$$= \begin{vmatrix} 0 & x & x \\ -z & 0 & z \\ -y & -y & 0 \end{vmatrix}$$

Apply: $C_2 \to C_2 - C_3$

$$= \begin{vmatrix} 0 & 0 & x \\ -z & -z & z \\ -y & -y & 0 \end{vmatrix}$$

$= 0$

$\because C_1 = C_2$

Q17

Without expanding, show that the value of determinant is zero:

$$\begin{vmatrix} 1 & 43 & 6 \\ 7 & 35 & 4 \\ 3 & 17 & 2 \end{vmatrix}$$

Solution

$$\begin{vmatrix} 1 & 43 & 6 \\ 7 & 35 & 4 \\ 3 & 17 & 2 \end{vmatrix}$$

Apply: $C_2 \to C_2 + (-7)C_3$

$$= \begin{vmatrix} 1 & 1 & 6 \\ 7 & 7 & 4 \\ 3 & 3 & 2 \end{vmatrix}$$

$= 0$

$\because \quad C_1 = C_2$

Q18

Without expanding, show that the value of determinant is zero:

$$\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$$

Solution

$$\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$$

Apply : $C3 \to C3 - C2, C4 \to C4 - C1$

$$= \begin{vmatrix} 1^2 & 2^2 & 3^2-2^2 & 4^2-1^2 \\ 2^2 & 3^2 & 4^2-3^2 & 5^2-2^2 \\ 3^2 & 4^2 & 5^2-4^2 & 6^2-3^2 \\ 4^2 & 5^2 & 6^2-5^2 & 7^2-4^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1^2 & 2^2 & 5 & 15 \\ 2^2 & 3^2 & 7 & 21 \\ 3^2 & 4^2 & 9 & 27 \\ 4^2 & 5^2 & 11 & 33 \end{vmatrix}$$

Take 3 common from C_4

$$= 3\begin{vmatrix} 1^2 & 2^2 & 5 & 5 \\ 2^2 & 3^2 & 7 & 7 \\ 3^2 & 4^2 & 9 & 9 \\ 4^2 & 5^2 & 11 & 11 \end{vmatrix}$$

$= 0$

$\because \quad C_3 = C_4$

Q19

Without expanding, show that the value of determinant is zero:

$$\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix}$$

Solution

$$\begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ 2a+2x & 2b+2y & 2c+2z \\ x+a & y+b & z+c \end{vmatrix}$$

$$= 2\begin{vmatrix} a & b & c \\ a+x & b+y & c+z \\ x+a & y+b & z+c \end{vmatrix}$$

$= 0$

Q20

Without expanding, show that the value of the following determinants is zero:

$$\begin{vmatrix} (2^x+2^{-x})^2 & (2^x-2^{-x})^2 & 1 \\ (3^x+3^{-x})^2 & (3^x-3^{-x})^2 & 1 \\ (4^x+4^{-x})^2 & (4^x-4^{-x})^2 & 1 \end{vmatrix}$$

Solution

$$\text{Let } \Delta = \begin{vmatrix} (2^x+2^{-x})^2 & (2^x-2^{-x})^2 & 1 \\ (3^x+3^{-x})^2 & (3^x-3^{-x})^2 & 1 \\ (4^x+4^{-x})^2 & (4^x-4^{-x})^2 & 1 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$

$$\Delta = \begin{vmatrix} (2^x+2^{-x})^2 & (2^x-2^{-x})^2-(2^x+2^{-x})^2 & 1 \\ (3^x+3^{-x})^2 & (3^x-3^{-x})^2-(3^x+3^{-x})^2 & 1 \\ (4^x+4^{-x})^2 & (4^x-4^{-x})^2-(4^x+4^{-x})^2 & 1 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} (2^x+2^{-x})^2 & -4 & 1 \\ (3^x+3^{-x})^2 & -4 & 1 \\ (4^x+4^{-x})^2 & -4 & 1 \end{vmatrix}$$

$$\Delta = (-4)\begin{vmatrix} (2^x+2^{-x})^2 & 1 & 1 \\ (3^x+3^{-x})^2 & 1 & 1 \\ (4^x+4^{-x})^2 & 1 & 1 \end{vmatrix}$$

$$\Delta = (-4)(0) \ldots\ldots\ldots\ldots\ldots\ldots \left[\because C_2 \text{ and } C_3 \text{ are identical}\right]$$

$$\Delta = 0$$

Q21

Evaluate the determinant $\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix}$

Solution

Consider the determinant $\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix}$

Let $\Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix}$

Applying $C_1 \to C_1\sin\delta$ *and* $C_2 \to C_2\cos\delta$, *we have,*

$$\Delta = \begin{vmatrix} \sin\alpha\sin\delta & \cos\alpha\cos\delta & \cos(\alpha+\delta) \\ \sin\beta\sin\delta & \cos\beta\cos\delta & \cos(\beta+\delta) \\ \sin\gamma\sin\delta & \cos\gamma\cos\delta & \cos(\gamma+\delta) \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, *we have*

$$\Delta = \begin{vmatrix} \sin\alpha\sin\delta & \cos\alpha\cos\delta - \sin\alpha\sin\delta & \cos(\alpha+\delta) \\ \sin\beta\sin\delta & \cos\beta\cos\delta - \sin\beta\sin\delta & \cos(\beta+\delta) \\ \sin\gamma\sin\delta & \cos\gamma\cos\delta - \sin\gamma\sin\delta & \cos(\gamma+\delta) \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} \sin\alpha\sin\delta & \cos(\alpha+\delta) & \cos(\alpha+\delta) \\ \sin\beta\sin\delta & \cos(\beta+\delta) & \cos(\beta+\delta) \\ \sin\gamma\sin\delta & \cos(\gamma+\delta) & \cos(\gamma+\delta) \end{vmatrix}$$

Applying $C_3 \to C_3 - C_2$, *we have,*

$$\Delta = \begin{vmatrix} \sin\alpha\sin\delta & \cos(\alpha+\delta) & 0 \\ \sin\beta\sin\delta & \cos(\beta+\delta) & 0 \\ \sin\gamma\sin\delta & \cos(\gamma+\delta) & 0 \end{vmatrix}$$

$\Rightarrow \Delta = 0$

Q22

Without expanding, show that the value of the following determinants is zero:

$$\begin{vmatrix} \sin^2 23^\circ & \sin^2 67^\circ & \cos 180^\circ \\ -\sin^2 67^\circ & -\sin^2 23^\circ & \cos^2 180^\circ \\ \cos 180^\circ & \sin^2 23^\circ & \sin^2 67^\circ \end{vmatrix}$$

Solution

$$\text{Let } \Delta = \begin{vmatrix} \sin^2 23° & \sin^2 67° & \cos 180° \\ -\sin^2 67° & -\sin^2 23° & \cos^2 180° \\ \cos 180° & \sin^2 23° & \sin^2 67° \end{vmatrix}$$

$$\Delta = \begin{vmatrix} \sin^2(90-67)° & \sin^2 67° & -1 \\ -\sin^2 67° & -\sin^2(90-67)° & 1 \\ -1 & \sin^2(90-67)° & \sin^2 67° \end{vmatrix}$$

$$\Delta = \begin{vmatrix} \cos^2 67° & \sin^2 67° & -1 \\ -\sin^2 67° & -\cos^2 67° & 1 \\ -1 & \cos^2 67° & \sin^2 67° \end{vmatrix}$$

Applying $C_2 \to C_2 + C_1$

$$\Delta = \begin{vmatrix} \cos^2 67° & 1 & -1 \\ -\sin^2 67° & -1 & 1 \\ -1 & -\sin^2 67° & \sin^2 67° \end{vmatrix}$$

Applying $C_2 \to C_2 + C_3$

$$\Delta = \begin{vmatrix} \cos^2 67° & 0 & -1 \\ -\sin^2 67° & 0 & 1 \\ -1 & 0 & \sin^2 67° \end{vmatrix}$$

$\Delta = 0$......................$[\because C_2$ is zero column$]$

Q23

Without expanding, show that the value of the following determinants is zero:

$$\begin{vmatrix} \cos(x+y) & -\sin(x+y) & \cos 2y \\ \sin x & \cos x & \sin y \\ -\cos x & \sin x & -\cos y \end{vmatrix}$$

Solution

Let $\Delta = \begin{vmatrix} \cos(x+y) & -\sin(x+y) & \cos 2y \\ \sin x & \cos x & \sin y \\ -\cos x & \sin x & -\cos y \end{vmatrix}$

Applying $R_3 \to R_3 \times (-\cos y)$

$$\Delta = \begin{vmatrix} \cos(x+y) & -\sin(x+y) & \cos 2y \\ \sin x & \cos x & \sin y \\ \cos x \cos y & -\sin x \cos y & \cos y \cos y \end{vmatrix}$$

Applying $R_3 \to R_3 - R_2 \times (-\sin y)$

$$\Delta = \begin{vmatrix} \cos(x+y) & -\sin(x+y) & \cos 2y \\ \sin x & \cos x & \sin y \\ \cos x \cos y & -\sin x \cos y & \cos y \cos y \end{vmatrix}$$

$$\Delta = \begin{vmatrix} \cos(x+y) & -\sin(x+y) & \cos 2y \\ \sin x & \cos x & \sin y \\ \cos x \cos y - \sin x \sin y & -\sin x \cos y - \cos x \sin y & \cos y \cos y - \sin y \sin y \end{vmatrix}$$

$$\Delta = \begin{vmatrix} \cos(x+y) & -\sin(x+y) & \cos 2y \\ \sin x & \cos x & \sin y \\ \cos(x+y) & -\sin(x+y) & \cos 2y \end{vmatrix}$$

$\Delta = 0$..........[$\because R_1$ and R_3 are identical]

Q24

Without expanding, show that the value of the following determinants is zero:

$$\begin{vmatrix} \sqrt{23}+\sqrt{3} & \sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{46} & 5 & \sqrt{10} \\ 3+\sqrt{115} & \sqrt{15} & 5 \end{vmatrix}$$

Solution

$$\text{Let } \Delta = \begin{vmatrix} \sqrt{23}+\sqrt{3} & \sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{46} & 5 & \sqrt{10} \\ 3+\sqrt{115} & \sqrt{15} & 5 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} \sqrt{3} & \sqrt{5} & \sqrt{5} \\ \sqrt{15} & 5 & \sqrt{10} \\ 3 & \sqrt{15} & 5 \end{vmatrix} + \begin{vmatrix} \sqrt{23} & \sqrt{5} & \sqrt{5} \\ \sqrt{46} & 5 & \sqrt{10} \\ \sqrt{115} & \sqrt{15} & 5 \end{vmatrix}$$

$$\Delta = \sqrt{3}\begin{vmatrix} 1 & \sqrt{5} & \sqrt{5} \\ \sqrt{5} & 5 & \sqrt{10} \\ \sqrt{3} & \sqrt{15} & 5 \end{vmatrix} + \sqrt{23}\begin{vmatrix} 1 & \sqrt{5} & \sqrt{5} \\ \sqrt{2} & 5 & \sqrt{10} \\ \sqrt{5} & \sqrt{15} & 5 \end{vmatrix}$$

$$\ldots\left[\begin{array}{l}\text{Taking } \sqrt{3} \text{ common from } C_1 \text{ of first determinant.} \\ \text{Taking } \sqrt{23} \text{ common from } C_1 \text{ of second determinant.}\end{array}\right]$$

$$\Delta = \sqrt{3}\sqrt{5}\begin{vmatrix} 1 & 1 & \sqrt{5} \\ \sqrt{5} & \sqrt{5} & \sqrt{10} \\ \sqrt{3} & \sqrt{3} & 5 \end{vmatrix} + \sqrt{23}\sqrt{5}\begin{vmatrix} 1 & \sqrt{5} & 1 \\ \sqrt{2} & 5 & \sqrt{2} \\ \sqrt{5} & \sqrt{15} & \sqrt{5} \end{vmatrix}$$

$$\ldots\left[\begin{array}{l}\text{Taking } \sqrt{5} \text{ common from } C_2 \text{ of first determinant.} \\ \text{Taking } \sqrt{5} \text{ common from } C_3 \text{ of second determinant.}\end{array}\right]$$

$$\Delta = \sqrt{3}\sqrt{5}(0) + \sqrt{23}\sqrt{5}(0)$$

$$\ldots\left[\begin{array}{l}\because C_1 \text{ and } C_2 \text{ of first determinant are identical.} \\ \because C_1 \text{ and } C_3 \text{ of second determinant are identical.}\end{array}\right]$$

$$\Delta = 0$$

Q25

Without expanding, show that the value of the following determinants is zero:

$$\begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin^2 B & \cot B & 1 \\ \sin^2 C & \cot C & 1 \end{vmatrix}, \text{ where } A, B, C \text{ are the angles of } \Delta ABC.$$

Solution

Let $\Delta = \begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin^2 B & \cot B & 1 \\ \sin^2 C & \cot C & 1 \end{vmatrix}$

Applying $C_3 \to C_3 - C_1$

$$\Delta = \begin{vmatrix} \sin^2 A & \cot A & 1 - \sin^2 A \\ \sin^2 B & \cot B & 1 - \sin^2 B \\ \sin^2 C & \cot C & 1 - \sin^2 C \end{vmatrix}$$

$$\Delta = \begin{vmatrix} \sin^2 A & \cot A & \cos^2 A \\ \sin^2 B & \cot B & \cos^2 B \\ \sin^2 C & \cot C & \cos^2 C \end{vmatrix}$$

$$\Delta = \begin{vmatrix} \frac{1-\cos 2A}{2} & \cot A & \frac{1+\cos 2A}{2} \\ \frac{1-\cos 2B}{2} & \cot B & \frac{1+\cos 2A}{2} \\ \frac{1-\cos 2C}{2} & \cot C & \frac{1+\cos 2C}{2} \end{vmatrix}$$

$$\Delta = \frac{1}{4}\begin{vmatrix} 1-\cos 2A & \cot A & 1+\cos 2A \\ 1-\cos 2B & \cot B & 1+\cos 2B \\ 1-\cos 2B & \cot C & 1+\cos 2C \end{vmatrix}$$

Applying $C_3 \to C_3 + C_1 - 2$

$$\Delta = \frac{1}{4}\begin{vmatrix} 1-\cos 2A & \cot A & 0 \\ 1-\cos 2B & \cot B & 0 \\ 1-\cos 2B & \cot C & 0 \end{vmatrix}$$

$= 0$

Q26

Evaluate the determinant:

$$\begin{vmatrix} a & b+c & a^2 \\ b & c+a & b^2 \\ c & a+b & c^2 \end{vmatrix}$$

Solution

$$\begin{vmatrix} a & b+c & a^2 \\ b & c+a & b^2 \\ c & a+b & c^2 \end{vmatrix}$$

Apply: $C_2 \to C_2 + C_1$.

$$= \begin{vmatrix} a & b+c+a & a^2 \\ b & c+a+b & b^2 \\ c & a+b+c & c^2 \end{vmatrix}$$

Take $(a+b+c)$ common from C_2

$$= (b+c+a)\begin{vmatrix} a & 1 & a^2 \\ b & 1 & b^2 \\ c & 1 & c^2 \end{vmatrix}$$

Apply: $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$= (b+c+a)\begin{vmatrix} a & 1 & a^2 \\ b-a & 0 & b^2-a^2 \\ c-a & 0 & c^2-a^2 \end{vmatrix}$$

$$= (b+c+a)(b-a)(c-a)\begin{vmatrix} a & 1 & a^2 \\ 1 & 0 & b+a \\ 1 & 0 & c+a \end{vmatrix}$$

$$= (b+c+a)(b-a)(c-a)(b-c)$$

Q27

Evaluate the determinant $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

Solution

Let $\Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ we get,

$$\Delta = \begin{vmatrix} 1 & a & bc \\ 0 & b-a & ca-bc \\ 0 & c-a & ab-ba \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 1 & a & bc \\ 0 & b-a & c(a-b) \\ 0 & c-a & b(a-c) \end{vmatrix}$$

Taking $(a-b)$ and $(a-c)$ common, we have

$$\Delta = (a-b)(a-c)\begin{vmatrix} 1 & a & bc \\ 0 & -1 & c \\ 0 & -1 & b \end{vmatrix}$$

$$\Rightarrow \Delta = (a-b)(c-a)(b-c)$$

Q28

Evaluate the determinant $\begin{vmatrix} x+\lambda & x & x \\ x & x+\lambda & x \\ x & x & x+\lambda \end{vmatrix}$

Solution

Let $\Delta = \begin{vmatrix} x+\lambda & x & x \\ x & x+\lambda & x \\ x & x & x+\lambda \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$ we get,

$\Delta = \begin{vmatrix} 3x+\lambda & x & x \\ 3x+\lambda & x+\lambda & x \\ 3x+\lambda & x & x+\lambda \end{vmatrix}$

Taking $(3x+\lambda)$ common, we have

$\Delta = (3x+\lambda)\begin{vmatrix} 1 & x & x \\ 1 & x+\lambda & x \\ 1 & x & x+\lambda \end{vmatrix}$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get,

$\Delta = (3x+\lambda)\begin{vmatrix} 1 & x & x \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix}$

$\to \Delta = \lambda^2(3x+\lambda)$

Q29

Evaluate the determinant $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$

Solution

$$\text{Let } \Delta = \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$ we get,

$$\Delta = \begin{vmatrix} a+b+c & b & c \\ a+b+c & a & b \\ a+b+c & c & a \end{vmatrix}$$

Taking $(a+b+c)$ common, we have

$$\Delta = (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & a & b \\ 1 & c & a \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, we get,

$$\Delta = (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & a-b & b-c \\ 0 & c-b & a-c \end{vmatrix}$$

$$\Rightarrow \Delta = (a+b+c)[(a-b)(a-c)-(b-c)(c-b)]$$

$$\Rightarrow \Delta = (a+b+c)[a^2 - ac - ab + bc + b^2 + c^2 - 2bc]$$

$$\Rightarrow \Delta = (a+b+c)[a^2 + b^2 + c^2 - ac - ab - bc]$$

Q30

Evaluate $\begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix}$

Solution

$$\begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix} = \begin{vmatrix} 2+x & 1 & 1 \\ 2+x & x & 1 \\ 2+x & 1 & x \end{vmatrix} = (2+x)\begin{vmatrix} 1 & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix}$$

$$= (2+x)\begin{vmatrix} 1 & 1 & 1 \\ 0 & x-1 & 0 \\ 0 & 0 & x-1 \end{vmatrix}$$

$$= (2+x)(x-1)^2$$

Q31

Evaluate the following:

$$\begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{vmatrix}$$

Solution

$$\begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{vmatrix}$$

$$= 0(0-y^3z^3) - xy^2(0-x^2yz^3) + xz^2(x^2y^3z-0)$$

$$= 0 + x^3y^3z^3 + x^3y^3z^3$$

$$= 2x^3y^3z^3$$

Q32

Evaluate the following:

$$\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix}$$

Solution

$$\text{Let } \Delta = \begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$ and $R_3 \to R_3 - R_2$

$$\Delta = \begin{vmatrix} a & -a & 0 \\ x & a+y & z \\ 0 & -a & a \end{vmatrix}$$

Applying $C_2 \to C_2 + C_1$

$$\Delta = \begin{vmatrix} a & 0 & 0 \\ x & a+x+y & z \\ 0 & -a & a \end{vmatrix}$$

$$\Delta = a[a(a+x+y)+az]+0+0$$

$$\Delta = a^2(a+x+y+z)$$

Q33

If $\Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}, \Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix}$, then prove that $\Delta + \Delta_1 = 0$.

Solution

$$\Delta + \Delta_1 = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix}$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} + \begin{vmatrix} 1 & yz & x \\ 1 & zx & y \\ 1 & xy & z \end{vmatrix} \ldots\ldots\ldots \left[\because |A| = |A^T|\right]$$

$$= \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} - \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$$

$\ldots\ldots$ [If any two rows (columns) of the determinant are interchanged then value of the determinant changes in sign.]

$$= \begin{vmatrix} 0 & 0 & x^2 - yz \\ 0 & 0 & y^2 - zx \\ 0 & 0 & z^2 - xy \end{vmatrix}$$

$= 0 \ldots\ldots\ldots [\because C_1$ and C_2 are identical$]$

Q34

Prove the identity:

$$\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3 + b^3 + c^3 - 3abc$$

Solution

$$\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3+b^3+c^3-3abc$$

$$\text{LHS} = \begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}$$

Apply: $C_1 \to C_1 + C_2 + C_3$.

$$= \begin{vmatrix} a+b+c & b & c \\ 0 & b-c & c-a \\ 2(a+b+c) & c+a & a+b \end{vmatrix}$$

Take $(a+b+c)$ common from C_1

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 2 & c+a & a+b \end{vmatrix}$$

Apply: $R_3 \to R_3 - 2R_1$

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 0 & c+a-2b & a+b-2c \end{vmatrix}$$

$$= (a+b+c)\left[(b-c)(a+b-2c)-(c-a)(c+a-2b)\right]$$

$$= a^3+b^3+c^3-3abc$$

$$= RHS$$

Q35

Prove the identity:

$$\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = 3abc - a^3 - b^3 - c^3$$

Solution

$$\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = 3abc - a^3 - b^3 - c^3$$

$$\text{LHS} = \begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix}$$

$$= \begin{vmatrix} b+c+a & -b & a \\ c+a+b & -c & b \\ a+b+c & -a & c \end{vmatrix}$$

$$= -(b+c+a)\begin{vmatrix} 1 & b & a \\ 1 & c & b \\ 1 & a & c \end{vmatrix}$$

$$= -(b+c+a)\begin{vmatrix} 1 & b & a \\ 0 & c-b & b-a \\ 0 & a-b & c-a \end{vmatrix}$$

$$= -(b+c+a)\left[(c-b)(c-a)-(b-a)(a-b)\right]$$

$$= 3abc - a^3 - b^3 - c^3$$

$$= RHS$$

Q36

Prove the identity:

$$\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Solution

$$\begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$\text{LHS} = \begin{vmatrix} a+b & b+c & c+a \\ b+c & c+a & a+b \\ c+a & a+b & b+c \end{vmatrix}$$

Apply: $C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} 2(a+b+c) & b+c & c+a \\ 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & a+b & b+c \end{vmatrix}$$

$$= 2\begin{vmatrix} a+b+c & b+c & c+a \\ a+b+c & c+a & a+b \\ a+b+c & a+b & b+c \end{vmatrix}$$

Apply: $C_2 \to C_2 - C_1, C_3 \to C_3 - C_1$

$$= 2\begin{vmatrix} a+b+c & -a & -b \\ a+b+c & -b & -c \\ a+b+c & -c & -a \end{vmatrix}$$

$$= 2\begin{vmatrix} a+b+c & a & b \\ a+b+c & b & c \\ a+b+c & c & a \end{vmatrix}$$

$$= 2\left(\begin{vmatrix} c & a & b \\ a & b & c \\ b & c & a \end{vmatrix} + \begin{vmatrix} a & a & b \\ b & b & c \\ c & c & a \end{vmatrix} + \begin{vmatrix} b & a & b \\ c & b & c \\ a & c & a \end{vmatrix} \right)$$

$$= 2\begin{vmatrix} c & a & b \\ a & b & c \\ b & c & a \end{vmatrix}$$

$$= 2\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$= RHS$

Q37

Prove the following identity:

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$$

Solution

We need to prove the following identity:

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$$

Applying $C_1 \to C_1 + C_2 + C_3$, we have,

$$L.H.S = \begin{vmatrix} 2a+2b+2c & a & b \\ 2a+2b+2c & b+c+2a & b \\ 2a+2b+2c & a & c+a+2b \end{vmatrix}$$

Taking the term $2a+2b+2$ as common, we have

$$L.H.S = (2a+2b+2c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

$$\Rightarrow L.H.S = 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$ we have

$$L.H.S = 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & a+b+c & 0 \\ 0 & 0 & a+b+c \end{vmatrix}$$

Thus, we have,

$$L.H.S = 2(a+b+c)\left[1 \times (a+b+c)^2\right]$$
$$= 2(a+b+c)(a+b+c)^2$$
$$= 2(a+b+c)^3$$

Q38

Prove the identity:

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$$

Solution

$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$$

$$LHS = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Apply: $R_1 \to R_1 + R_2 + R_3$.

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Take $(a+b+c)$ common from R_1

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Apply: $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -b-c-a & 0 \\ 2c & 0 & -c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & b+c+a & 0 \\ 2c & 0 & b+c+a \end{vmatrix}$$

$$= (a+b+c)^3$$

$$= RHS$$

Q39

Using properties of determinants, show that

$$\begin{vmatrix} 1 & b+c & b^2+c^2 \\ 1 & c+a & c^2+a^2 \\ 1 & a+b & a^2+b^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

Solution

$$LHS = \begin{vmatrix} 1 & b+c & b^2+c^2 \\ 1 & c+a & c^2+a^2 \\ 1 & a+b & a^2+b^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & b+c & b^2+c^2 \\ 0 & a-b & a^2-b^2 \\ 0 & a-c & a^2-c^2 \end{vmatrix}$$

$$= (a-b)(a-c)\begin{vmatrix} 1 & b+c & b^2+c^2 \\ 0 & 1 & a+b \\ 0 & 1 & a+c \end{vmatrix}$$

$$= (a-b)(a-c)\begin{vmatrix} 1 & b+c & b^2+c^2 \\ 0 & 1 & a+b \\ 0 & 0 & c-b \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)$$

$$= RHS$$

Q40

Show that

$$\begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix} = 9(a+b)b^2$$

Solution

$$LHS = \begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix}$$

$$= \begin{vmatrix} 3a+3b & 3a+3b & 3a+3b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix}$$

$$= (3a+3b)\begin{vmatrix} 1 & 1 & 1 \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix}$$

$$= 3(a+b)\begin{vmatrix} 0 & 1 & 0 \\ 2b & a & b \\ -b & a+2b & -2b \end{vmatrix}$$

$$= 3(a+b)b^2\begin{vmatrix} 0 & 1 & 0 \\ 2 & a & 1 \\ -1 & a+2b & -2 \end{vmatrix}$$

$$= 9(a+b)b^2$$

$$= RHS$$

Q41

Without expanding the determinants, show that

$$\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Solution

$$\begin{vmatrix}1 & a & bc\\1 & b & ca\\1 & c & ab\end{vmatrix}$$

Apply $R_1 \to R_1a,\ R_2 \to R_2b,\ R_3 \to R_3c$

$$= \frac{1}{abc}\begin{vmatrix}a & a^2 & abc\\b & b^2 & cab\\c & c^2 & abc\end{vmatrix}$$

$$= \frac{abc}{abc}\begin{vmatrix}a & a^2 & 1\\b & b^2 & 1\\c & c^2 & 1\end{vmatrix}$$

$$= -\begin{vmatrix}a & 1 & a^2\\b & 1 & b^2\\c & 1 & c^2\end{vmatrix}$$

$$= \begin{vmatrix}1 & a & a^2\\1 & b & b^2\\1 & c & c^2\end{vmatrix}$$

Q42

Prove that

$$\begin{vmatrix}z & x & y\\z^2 & x^2 & y^2\\z^4 & x^4 & y^4\end{vmatrix} = \begin{vmatrix}x & y & z\\x^2 & y^2 & z^2\\x^4 & y^4 & z^4\end{vmatrix} = \begin{vmatrix}x^2 & y^2 & z^2\\x^4 & y^4 & z^4\\x & y & z\end{vmatrix} = xyz(x-y)(y-z)(z-x)(x+y+z).$$

Solution

$$\begin{vmatrix} z & x & y \\ z^2 & x^2 & y^2 \\ z^4 & x^4 & y^4 \end{vmatrix} = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^4 & y^4 & z^4 \end{vmatrix} = \begin{vmatrix} x^2 & y^2 & z^2 \\ x^4 & y^4 & z^4 \\ x & y & z \end{vmatrix} = xyz(x-y)(y-z)(z-x)(x+y+z)$$

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^4 & y^4 & z^4 \end{vmatrix}$$

$$= xyz\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix}$$

$$= xyz\begin{vmatrix} 0 & 1 & 0 \\ x-y & y & z-y \\ x^3-y^3 & y^3 & z^3-y^3 \end{vmatrix}$$

$$= xyz(x-y)(z-y)\begin{vmatrix} 0 & 1 & 0 \\ 1 & y & 1 \\ x^2+y^2+xy & y^3 & z^2+y^2+zy \end{vmatrix}$$

$$= -xyz(x-y)(z-y)\left[z^2+y^2+zy-x^2-y^2-xy\right]$$

$$= -xyz(x-y)(z-y)\left[(z-x)(z+x)+y(z-x)\right]$$

$$= -xyz(x-y)(z-y)(z-x)\left[z+x+y\right]$$

$$= xyz(x-y)(y-z)(z-x)(x+y+z)$$

$$= RHS$$

Q43

Prove the identity:

$$\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)$$

Solution

$$\begin{vmatrix}(b+c)^2 & a^2 & bc\\(c+a)^2 & b^2 & ca\\(a+b)^2 & c^2 & ab\end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)$$

$$\text{LHS} = \begin{vmatrix}(b+c)^2 & a^2 & bc\\(c+a)^2 & b^2 & ca\\(a+b)^2 & c^2 & ab\end{vmatrix}$$

Apply: $C_1 \to C_1 + C_2 - 2C_3$

$$= \begin{vmatrix}(b+c)^2+a^2-2bc & a^2 & bc\\(c+a)^2+b^2-2ca & b^2 & ca\\(a+b)^2+c^2-2ab & c^2 & ab\end{vmatrix}$$

$$= \begin{vmatrix}a^2+b^2+c^2 & a^2 & bc\\a^2+b^2+c^2 & b^2 & ca\\a^2+b^2+c^2 & c^2 & ab\end{vmatrix}$$

Take $(a^2+b^2+c^2)$ common from C_1

$$= (a^2+b^2+c^2)\begin{vmatrix}1 & a^2 & bc\\1 & b^2 & ca\\1 & c^2 & ab\end{vmatrix}$$

$$= (a^2+b^2+c^2)\begin{vmatrix}1 & a^2 & bc\\0 & b^2-a^2 & ca-bc\\0 & c^2-a^2 & ab-bc\end{vmatrix}$$

$$= (a^2+b^2+c^2)(b-a)(c-a)\begin{vmatrix}1 & a^2 & bc\\0 & b+a & -c\\0 & c+a & -b\end{vmatrix}$$

$$= (a^2+b^2+c^2)(b-a)(c-a)[(b+a)(-b)-(-c)(c+a)]$$

$$= (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)$$

$$= RHS$$

Q44

Prove the identity:

$$\begin{vmatrix}(a+1)(a+2) & a+2 & 1\\(a+2)(a+3) & a+3 & 1\\(a+3)(a+4) & a+4 & 1\end{vmatrix} = -2$$

Solution

$$\begin{vmatrix} (a+1)(a+2) & a+2 & 1 \\ (a+2)(a+3) & a+3 & 1 \\ (a+3)(a+4) & a+4 & 1 \end{vmatrix} = -2$$

$$\text{LHS} = \begin{vmatrix} (a+1)(a+2) & a+2 & 1 \\ (a+2)(a+3) & a+3 & 1 \\ (a+3)(a+4) & a+4 & 1 \end{vmatrix}$$

Apply $R_3 \to R_3 - R_2$

$$= \begin{vmatrix} (a+1)(a+2) & a+2 & 1 \\ (a+2)(a+3) & a+3 & 1 \\ (a+3)2 & 1 & 0 \end{vmatrix}$$

Apply $R_2 \to R_2 - R_1$

$$= \begin{vmatrix} (a+1)(a+2) & a+2 & 1 \\ (a+2)2 & 1 & 0 \\ (a+3)2 & 1 & 0 \end{vmatrix}$$

$$= [(2a+4)(1) - (1)(2a+6)]$$

$$= -2$$

$$= \text{RHS}$$

Q45

Prove the identity:

$$\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)$$

Solution

$$\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)$$

$$\text{LHS} = \begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix}$$

Apply: $C_2 \to C_2 - 2C_1 - 2C_3$

$$= \begin{vmatrix} a^2 & a^2-(b-c)^2-2a^2-2bc & bc \\ b^2 & b^2-(c-a)^2-2b^2-2ca & ca \\ c^2 & c^2-(a-b)^2-2c^2-2ab & ab \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & -(b^2+c^2+a^2) & bc \\ b^2 & -(b^2+c^2+a^2) & ca \\ c^2 & -(b^2+c^2+a^2) & ab \end{vmatrix}$$

Take $-(a^2+b^2+c^2)$ common from C_2

$$= -(b^2+c^2+a^2)\begin{vmatrix} a^2 & 1 & bc \\ b^2 & 1 & ca \\ c^2 & 1 & ab \end{vmatrix}$$

$$= -(b^2+c^2+a^2)\begin{vmatrix} a^2 & 1 & bc \\ b^2-a^2 & 0 & ca-bc \\ c^2-a^2 & 0 & ab-bc \end{vmatrix}$$

$$= -(b^2+c^2+a^2)(a-b)(c-a)\begin{vmatrix} a^2 & 1 & bc \\ -(b+a) & 0 & c \\ c+a & 0 & -b \end{vmatrix}$$

$$= -(b^2+c^2+a^2)(a-b)(c-a)[(-(b+a))(-b)-(c)(c+a)]$$

$$= (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)$$

$$= RHS$$

Q46

Prove the identity:

$$\begin{vmatrix} 1 & a^2+bc & a^3 \\ 1 & b^2+ca & b^3 \\ 1 & c^2+ab & c^3 \end{vmatrix} = -(a-b)(b-c)(c-a)(a^2+b^2+c^2)$$

Solution

$$\begin{vmatrix} 1 & a^2+bc & a^3 \\ 1 & b^2+ca & b^3 \\ 1 & c^2+ab & c^3 \end{vmatrix} = -(a-b)(b-c)(c-a)(a^2+b^2+c^2)$$

$$\text{LHS} = \begin{vmatrix} 1 & a^2+bc & a^3 \\ 1 & b^2+ca & b^3 \\ 1 & c^2+ab & c^3 \end{vmatrix}$$

Apply: $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$= \begin{vmatrix} 1 & a^2+bc & a^3 \\ 0 & b^2+ca-a^2-bc & b^3-a^3 \\ 0 & c^2+abb^2+ca-a^2-bc & c^3-a^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a^2+bc & a^3 \\ 0 & (b^2-a^2)-c(b-a) & b^3-a^3 \\ 0 & (c^2-a^2)-b(c-a) & c^3-a^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a^2+bc & a^3 \\ 0 & (b-a)(b+a-c) & b^3-a^3 \\ 0 & (c-a)(c+a-b) & c^3-a^3 \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & a^2+bc & a^3 \\ 0 & (b+a-c) & b^2+a^2+ab \\ 0 & (c+a-b) & c^2+a^2+ac \end{vmatrix}$$

$$= (b-a)(c-a)\left[((b+a-c))(c^2+a^2+ac)-(b^2+a^2+ab)(c^2+a^2+ac)\right]$$

$$= -(a-b)(b-c)(c-a)(a^2+b^2+c^2)$$

$$= RHS$$

Q47

Prove the following identity:

$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$$

Solution

We need to prove the following identity:

$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$$

Taking the term a,b,c common from C_1, C_2 and C_3, respectively, we have,

$$L.H.S = abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, we have,

$$L.H.S = abc\begin{vmatrix} 2a+2c & c & a+c \\ 2a+2b & b & a \\ 2b+2c & b+c & c \end{vmatrix}$$

$$\Rightarrow L.H.S = 2abc\begin{vmatrix} a+c & c & a+c \\ a+b & b & a \\ b+c & b+c & c \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we have,

$$L.H.S = 2abc\begin{vmatrix} a+c & -a & 0 \\ a+b & -a & -b \\ b+c & 0 & -b \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, we have,

$$\Rightarrow L.H.S = 2abc\begin{vmatrix} c & -a & 0 \\ 0 & -a & -b \\ c & 0 & -b \end{vmatrix}$$

Taking c, a, and b from C_1, C_2 and C_3 respectively, we have,

$$L.H.S = 2a^2b^2c^2\begin{vmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 1 & 0 & -1 \end{vmatrix}$$

Applying $R_3 \to R_3 - R_1$, we have

$$L.H.S = 2a^2b^2c^2\begin{vmatrix} 1 & -1 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & -1 \end{vmatrix}$$

$$= 4a^2b^2c^2$$

Q48

Prove the following identity:

$$\begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix} = 16(3x+4)$$

Solution

We need to prove the following identity:

$$\begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix} = 16(3x+4)$$

Let us consider the L.H.S of the above equation.

$$\Delta = \begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$$

Applying $C_1 \rightarrow C_1 + C_2 + C_3$, we get,

$$\Delta = \begin{vmatrix} 3x+4 & x & x \\ 3x+4 & x+4 & x \\ 3x+4 & x & x+4 \end{vmatrix}$$

Taking the common term $3x+4$, we get,

$$\Delta = (3x+4)\begin{vmatrix} 1 & x & x \\ 1 & x+4 & x \\ 1 & x & x+4 \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we get,

$$\Delta = (3x+4)\begin{vmatrix} 1 & x & x \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{vmatrix}$$

$$\rightarrow \Delta = 16(3x+4)$$

Q49

Prove the following identity:

$$\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix} = 1$$

Solution

We need to prove the following identity:

$$\begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix} = 1$$

Let us consider the L.H.S of the above equation.

Applying $C_2 \to C_2 - pC_1$ and $C_3 \to C_3 - qC_1$, we get

$$\Delta = \begin{vmatrix} 1 & 1 & 1+p \\ 2 & 3 & 4+3p \\ 3 & 6 & 10+6p \end{vmatrix}$$

Applying $C_3 \to C_3 - pC_2$, we get

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix}$$

Applying $C_2 \to C_2 - pC_1$ and $C_3 \to C_3 - qC_1$, we get

$$\Rightarrow \Delta = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 2 \\ 3 & 3 & 7 \end{vmatrix}$$

$$\Rightarrow \Delta = 1[7-6] = 1$$

Q50

Prove that

$$\begin{vmatrix} a & b-c & c-b \\ a-c & b & c-a \\ a-b & b-a & c \end{vmatrix} = (a+b-c)(b+c-a)(c+a-b)$$

Solution

$$\begin{vmatrix} a & b-c & c-b \\ a-c & b & c-a \\ a-b & b-a & c \end{vmatrix}$$

$R_1 \to R_1 - R_2 - R_3$

$$= \begin{vmatrix} -a+c+b & -b-c+a & -c-b+a \\ a-c & b & c-a \\ a-b & b-a & c \end{vmatrix}$$

$$= (b+c-a)\begin{vmatrix} 1 & -1 & -1 \\ a-c & b & c-a \\ a-b & b-a & c \end{vmatrix}$$

$$= (b+c-a)\begin{vmatrix} 1 & 0 & 0 \\ a-c & b+a-c & 0 \\ a-b & 0 & c+a-b \end{vmatrix}$$

$$= (a+b-c)(b+c-a)(c+a-b)$$

$$= RHS$$

Q51

Prove that

$$\begin{vmatrix} a^2 & 2ab & b^2 \\ b^2 & a^2 & 2ab \\ 2ab & b^2 & a^2 \end{vmatrix} = \left(a^3+b^3\right)^2$$

Solution

$$LHS = \begin{vmatrix} a^2 & 2ab & b^2 \\ b^2 & a^2 & 2ab \\ 2ab & b^2 & a^2 \end{vmatrix}$$

$$= \begin{vmatrix} a^2+b^2+2ab & 2ab & b^2 \\ a^2+b^2+2ab & a^2 & 2ab \\ a^2+b^2+2ab & b^2 & a^2 \end{vmatrix}$$

$$= \left(a^2+b^2+2ab\right)\begin{vmatrix} 1 & 2ab & b^2 \\ 1 & a^2 & 2ab \\ 1 & b^2 & a^2 \end{vmatrix}$$

$$= \left(a^2+b^2+2ab\right)\begin{vmatrix} 1 & 2ab & b^2 \\ 0 & a^2-2ab & 2ab-b^2 \\ 0 & b^2-2ab & a^2-b^2 \end{vmatrix}$$

$$= \left(a^2+b^2+2ab\right)\begin{vmatrix} 1 & 2ab & b^2 \\ 0 & a^2-b^2 & 2ab-a^2 \\ 0 & b^2-2ab & a^2-b^2 \end{vmatrix}$$

$$= (a+b)^2\left[\left(a^2-b^2\right)\left(a^2-b^2\right)-\left(2ab-a^2\right)\left(b^2-2ab\right)\right]$$

$$= (a+b)^2\left(a^2+b^2-ab\right)^2$$

$$= \left(a^3+b^3\right)^2$$

$$= RHS$$

Q52

Prove the following identity:

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$$

Solution

We need to prove the following identity:

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1 + a^2 + b^2 + c^2$$

Let us consider the L.H.S of the above equation.

$$\Delta = \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$$

Applying $R_1 \to R_1(a)$, $R_2 \to R_2(b)$ *and* $R_3 \to R_3(c)$, *we get*

$$\Delta = \frac{1}{abc}\begin{vmatrix} a(a^2+1) & a^2b & a^2c \\ ab^2 & b(b^2+1) & b^2c \\ c^2a & c^2b & c(c^2+1) \end{vmatrix}$$

Taking a,b, and c common from C_1, C_2 *and* C_3, *respectively, we get,*

$$\Delta = \frac{abc}{abc}\begin{vmatrix} (a^2+1) & a^2 & a^2 \\ b^2 & (b^2+1) & b^2 \\ c^2 & c^2 & (c^2+1) \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, *we get,*

$$\Delta = \frac{abc}{abc}\begin{vmatrix} (a^2+b^2+c^2+1) & (a^2+b^2+c^2+1) & (a^2+b^2+c^2+1) \\ b^2 & (b^2+1) & b^2 \\ c^2 & c^2 & (c^2+1) \end{vmatrix}$$

Taking the term, $(a^2+b^2+c^2+1)$ *common from the above equation, we have,*

$$\Delta = (a^2+b^2+c^2+1)\begin{vmatrix} 1 & 1 & 1 \\ b^2 & (b^2+1) & b^2 \\ c^2 & c^2 & (c^2+1) \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, *we get,*

$$\Delta = (a^2+b^2+c^2+1)\begin{vmatrix} 1 & 0 & 1 \\ b^2 & 1 & 0 \\ c^2 & 0 & 1 \end{vmatrix}$$

$$\to \Delta = (a^2+b^2+c^2+1)$$

Q53

Prove the following identity:

$$\begin{vmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{vmatrix} = (a^3-1)^2$$

Solution

Let us consider the L.H.S of the given equation.

Let $\Delta = \begin{vmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, we have,

$$\Delta = \begin{vmatrix} 1+a+a^2 & a & a^2 \\ 1+a+a^2 & 1 & a \\ 1+a+a^2 & a^2 & 1 \end{vmatrix}$$

Taking the term $(1+a+a^2)$ common, we have,

$$\Delta = (1+a+a^2)\begin{vmatrix} 1 & a & a^2 \\ 1 & 1 & a \\ 1 & a^2 & 1 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have

$$\Delta = (1+a+a^2)\begin{vmatrix} 1 & a & a^2 \\ 0 & 1-a & a(1-a) \\ 0 & -a(1-a) & (1-a)(1+a) \end{vmatrix}$$

Taking the term $(1-a)$ common from R_2 and R_3, we have

$$\Rightarrow \Delta = (1+a+a^2)(1-a)^2\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & a \\ 0 & -a & (1+a) \end{vmatrix}$$

$$\Rightarrow \Delta = (1+a+a^2)(1-a)^2(1+a+a^2)$$

$$\Rightarrow \Delta = (1+a+a^2)^2(1-a)^2$$

$$\Rightarrow \Delta = [(1+a+a^2)(1-a)]^2$$

$$\Rightarrow \Delta = [(a^3-1)]^2$$

Q54

Prove the identity:

$$\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix} = 2(a+b)(b+c)(c+a)$$

Solution

$$\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix} = 2(a+b)(b+c)(c+a)$$

$$\text{LHS} = \begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix}$$

Apply: $C_1 \to C_1 + C_3$ and $C_2 \to C_2 + C_3$

$$= \begin{vmatrix} a+c & -(c+b) & -b \\ -(c+a) & b+c & -a \\ a+c & b+c & a+b+c \end{vmatrix}$$

$$= (c+a)(c+b) \begin{vmatrix} 1 & -1 & -b \\ -1 & 1 & -a \\ 1 & 1 & a+b+c \end{vmatrix}$$

$$= (c+a)(c+b) \begin{vmatrix} 1 & -1 & -b \\ 0 & 0 & -a-b \\ 0 & 2 & a+c \end{vmatrix}$$

$$= 2(a+b)(b+c)(c+a)$$

$$= RHS$$

Q55

Prove the following identity:

$$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$$

Solution

We need to prove the following identity:

$$\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc$$

Let us consider the L.H.S of the above equation.

$$\Delta = \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we have,

$$\Delta = \begin{vmatrix} 2(b+c) & 2(a+c) & 2(a+b) \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

Taking 2 common from the above equation, we have,

$$\Delta = 2\begin{vmatrix} (b+c) & (a+c) & (a+b) \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have,

$$\Delta = 2\begin{vmatrix} (b+c) & (a+c) & (a+b) \\ -c & 0 & -a \\ -b & -a & 0 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we have,

$$\Delta = 2\begin{vmatrix} 0 & c & b \\ -c & 0 & -a \\ -b & -a & 0 \end{vmatrix}$$

$\Rightarrow \Delta = 2(0 + 2abc + abc)$

$\Rightarrow \Delta = 4abc$

Q56

Prove the identity:

$$\begin{vmatrix} b^2+c^2 & ab & ac \\ ba & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2$$

Solution

$$\begin{vmatrix} b^2+c^2 & ab & ac \\ ba & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2$$

$$LHS = \begin{vmatrix} b^2+c^2 & ab & ac \\ ba & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix}$$

Multiply R_1, R_2 and R_3 by a, b and c respectively.

$$= \frac{1}{abc}\begin{vmatrix} ab^2+ac^2 & a^2b & a^2c \\ b^2a & bc^2+ba^2 & b^2c \\ c^2a & c^2b & ca^2+cb^2 \end{vmatrix}$$

Take a, b and c common from C_1, C_2 and C_3 respectively.

$$= \frac{abc}{abc}\begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$$

Now apply $R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} 2\left(b^2+c^2\right) & 2\left(c^2+a^2\right) & 2\left(a^2+b^2\right) \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$$

$$= 2\begin{vmatrix} \left(b^2+c^2\right) & \left(c^2+a^2\right) & \left(a^2+b^2\right) \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$$

$$= 2\begin{vmatrix} c^2 & 0 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$$

$$= 2\left[c^2\left\{\left(c^2+a^2\right)\left(a^2+b^2\right) - b^2c^2\right\} + a^2\left\{b^2c^2 - \left(c^2+a^2\right)c^2\right\}\right]$$

$$= 4a^2b^2c^2$$

$$= RHS$$

Q57

Prove the identity:

$$\begin{vmatrix} 0 & b^2a & c^2a \\ a^2b & 0 & c^2b \\ a^2c & b^2c & 0 \end{vmatrix} = 2a^3b^3c^3$$

Solution

$$\begin{vmatrix} 0 & b^2a & c^2a \\ a^2b & 0 & c^2b \\ a^2c & b^2c & 0 \end{vmatrix} = 2a^3b^3c^3$$

$$\text{LHS} = \begin{vmatrix} 0 & b^2a & c^2a \\ a^2b & 0 & c^2b \\ a^2c & b^2c & 0 \end{vmatrix}$$

$$= a^2b^2c^2 \begin{vmatrix} 0 & a & a \\ b & 0 & b \\ c & c & 0 \end{vmatrix}$$

$$= a^3b^3c^3 \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= a^3b^3c^3 \begin{vmatrix} 0 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= 2a^3b^3c^3$$

$$= RHS$$

Q58

Prove that

$$\begin{vmatrix} \frac{a^2+b^2}{c} & c & c \\ a & \frac{b^2+c^2}{a} & a \\ b & b & \frac{c^2+a^2}{b} \end{vmatrix} = 4abc$$

Solution

$$\begin{vmatrix} \frac{a^2+b^2}{c} & c & c \\ a & \frac{b^2+c^2}{a} & a \\ b & b & \frac{c^2+a^2}{b} \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} a^2+b^2 & c^2 & c^2 \\ a^2 & c^2+b^2 & a^2 \\ b^2 & b^2 & c^2+a^2 \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} 0 & -2b^2 & -2a^2 \\ a^2 & c^2+b^2 & a^2 \\ b^2 & b^2 & c^2+a^2 \end{vmatrix}$$

$$= \frac{-2}{abc}\begin{vmatrix} 0 & b^2 & a^2 \\ a^2 & c^2+b^2 & a^2 \\ b^2 & b^2 & c^2+a^2 \end{vmatrix}$$

$$= \frac{-2}{abc}\begin{vmatrix} 0 & b^2 & a^2 \\ a^2 & c^2+b^2 & a^2 \\ b^2 & 0 & c^2 \end{vmatrix}$$

$$= \frac{-2}{abc}\begin{vmatrix} 0 & b^2 & a^2 \\ a^2 & c^2 & 0 \\ b^2 & 0 & c^2 \end{vmatrix}$$

$$= \frac{-2}{abc}\left[\left(-a^2\right)\left(b^2c^2\right)+\left(b^2\right)\left(-a^2c^2\right)\right]$$

$$= \frac{-2}{abc}\left(-2a^2b^2c^2\right)$$

$= 4abc$

$= RHS$

Q59

Prove that

$$\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix} = (ab+bc+ca)^3$$

Solution

$$\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix}$$

Multiply R_1, R_2 and R_3 by a, b and c respectively

$$= \frac{1}{abc}\begin{vmatrix} -abc & ab^2+abc & ac^2+abc \\ a^2b+abc & -abc & bc^2+abc \\ a^2c+abc & b^2c+abc & -abc \end{vmatrix}$$

Take a, b and c common from C_1, C_2 and C_3 respectively.

$$= \frac{abc}{abc}\begin{vmatrix} -bc & ab+ac & ac+ab \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & -ab \end{vmatrix}$$

Apply: $R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} ab+bc+ca & ab+bc+ca & ab+bc+ca \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & -ab \end{vmatrix}$$

$$= (ab+bc+ca)\begin{vmatrix} 1 & 1 & 1 \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & -ab \end{vmatrix}$$

$$= (ab+bc+ca)\begin{vmatrix} 0 & 1 & 0 \\ ab+bc+ac & -ac & bc+ab+ac \\ 0 & bc+ac & -ab-bc-ac \end{vmatrix}$$

$$= (ab+bc+ca)^3\begin{vmatrix} 0 & 1 & 0 \\ 1 & -ac & 1 \\ 0 & bc+ac & -1 \end{vmatrix}$$

$= (ab+bc+ca)^3$

$= RHS$

Q60

Prove the following identity:

$$\begin{vmatrix} x+\lambda & 2x & 2x \\ 2x & x+\lambda & 2x \\ 2x & 2x & x+\lambda \end{vmatrix} = (5x+\lambda)(\lambda-x)^2$$

Solution

L.H.S.,

$$\begin{vmatrix} x+\lambda & 2x & 2x \\ 2x & x+\lambda & 2x \\ 2x & 2x & x+\lambda \end{vmatrix}$$

$$= \begin{vmatrix} x+\lambda & 2x & 2x \\ 2x & x+\lambda & 2x \\ 2x & 2x & x+\lambda \end{vmatrix} [C_1 \to C_1 - C_3, C_2 \to C_2 - C_3]$$

$$= \begin{vmatrix} \lambda - x & 0 & 2x \\ 0 & \lambda - x & 2x \\ x - \lambda & x - \lambda & x+\lambda \end{vmatrix}$$

$$= (\lambda - x)(\lambda - x) \begin{vmatrix} 1 & 0 & 2x \\ 0 & 1 & 2x \\ -1 & -1 & x+\lambda \end{vmatrix}$$

$$= (\lambda - x)^2 \begin{vmatrix} 1 & 0 & 2x \\ 0 & 1 & 2x \\ -1 & -1 & x+\lambda \end{vmatrix}$$

$$= (\lambda - x)^2[1(x+\lambda) + 2x + 2x(0+1)]$$

$$= (\lambda - x)^2[x+\lambda+2x+2x]$$

$$= (\lambda - x)^2[5x+\lambda]$$

= R.H.S

Hence Proved

Q61

Using properties of determinants prove that

$$\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$$

Solution

$$LHS = \begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

Apply $C_1 \to C_1 + C_2 + C_3$

$$= \begin{vmatrix} 5x+4 & 2x & 2x \\ 5x+4 & x+4 & 2x \\ 5x+4 & 2x & x+4 \end{vmatrix}$$

$$= (5x+4)\begin{vmatrix} 1 & 2x & 2x \\ 1 & x+4 & 2x \\ 1 & 2x & x+4 \end{vmatrix}$$

$$= (5x+4)\begin{vmatrix} 1 & 2x & 2x \\ 0 & -x+4 & 0 \\ 0 & 0 & -x+4 \end{vmatrix}$$

$$= (5x+4)(4-x)^2\begin{vmatrix} 1 & 2x & 2x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= (5x+4)(4-x)^2$$

$$= RHS$$

Q62

Prove the following identities:

$$\begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix} = 4xyz$$

Solution

$$\text{Let } \Delta = \begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix}$$

Applying $R_1 \to R_1 - R_2$

$$\Delta = \begin{vmatrix} y & -x & y-x \\ z & z+x & x \\ y & x & x+y \end{vmatrix}$$

Applying $R_1 \to R_1 - R_3$

$$\Delta = \begin{vmatrix} 0 & -2x & -2x \\ z & z+x & x \\ y & x & x+y \end{vmatrix}$$

$$\Delta = 2x[z(x+y) - xy] - 2x[zx - y(z+x)]$$

$$\Delta = 2x[zx + zy - xy - zx + yz + yx]$$

$$\Delta = 4xyz$$

Q63

Question 63

Prove the identity:

$$\begin{vmatrix} -a\left(b^2+c^2-a^2\right) & 2b^3 & 2c^3 \\ 2a^3 & -b\left(c^2+a^2-b^2\right) & 2c^3 \\ 2a^3 & 2b^3 & -c\left(a^2+b^2-c^2\right) \end{vmatrix} = abc\left(a^2+b^2+c^2\right)^3$$

Solution

$$\begin{vmatrix} -a\left(b^2+c^2-a^2\right) & 2b^3 & 2c^3 \\ 2a^3 & -b\left(c^2+a^2-b^2\right) & 2c^3 \\ 2a^3 & 2b^3 & -c\left(a^2+b^2-c^2\right) \end{vmatrix} = abc\left(a^2+b^2+c^2\right)^3$$

$$\text{LHS} = \begin{vmatrix} -a\left(b^2+c^2-a^2\right) & 2b^3 & 2c^3 \\ 2a^3 & -b\left(c^2+a^2-b^2\right) & 2c^3 \\ 2a^3 & 2b^3 & -c\left(a^2+b^2-c^2\right) \end{vmatrix}$$

Take a, b and c common from C_1, C_2 and C_3 respectively.

$$= abc\begin{vmatrix} -\left(b^2+c^2-a^2\right) & 2b^2 & 2c^2 \\ 2a^2 & -\left(c^2+a^2-b^2\right) & 2c^2 \\ 2a^2 & 2b^2 & -\left(a^2+b^2-c^2\right) \end{vmatrix}$$

Apply: $R_1 \to R_1 - R_3, R_2 \to R_2 - R_3$

$$= abc\begin{vmatrix} -\left(b^2+c^2-a^2\right)-2a^2 & 0 & 2c^2+\left(a^2+b^2-c^2\right) \\ 0 & -\left(c^2+a^2-b^2\right)-2b^2 & 2c^2+\left(a^2+b^2-c^2\right) \\ 2a^2 & 2b^2 & -\left(a^2+b^2-c^2\right) \end{vmatrix}$$

$$= abc\begin{vmatrix} -\left(b^2+c^2+a^2\right) & 0 & \left(a^2+b^2+c^2\right) \\ 0 & -\left(c^2+a^2+b^2\right) & \left(a^2+b^2+c^2\right) \\ 2a^2 & 2b^2 & -\left(a^2+b^2-c^2\right) \end{vmatrix}$$

$$= abc\left(b^2+c^2+a^2\right)^2\begin{vmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 2a^2 & 2b^2 & -\left(a^2+b^2-c^2\right) \end{vmatrix}$$

$$= abc\left(b^2+c^2+a^2\right)^2\begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 2a^2 & 2b^2 & -\left(a^2+b^2-c^2\right)+2a^2 \end{vmatrix}$$

$$= abc\left(b^2+c^2+a^2\right)^2\begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 2a^2 & 2b^2 & -b^2+c^2+a^2 \end{vmatrix}$$

$$= -abc\left(b^2+c^2+a^2\right)^2\left[(-1)\left(-b^2+c^2+a^2\right)-(1)\left(2b^2\right)\right]$$

$$abc\left(a^2+b^2+c^2\right)^3$$

$$= RHS$$

Q64

Prove that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix} = a^3 + 3a^2$

Solution

$$LHS = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix}$$

$$= \begin{vmatrix} 3+a & 3+a & 3+a \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix}$$

$$= (3+a)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix}$$

$$= (3+a)\begin{vmatrix} 1 & 1 & 1 \\ 0 & a & 0 \\ 0 & 1 & a \end{vmatrix}$$

$$= (3+a)a^2$$

$$= a^3 + 3a^2$$

$$= RHS$$

Q65

Prove the following identity:

$$\begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix} = (x+y+z)^3$$

Solution

L.H.S.,

$$\begin{vmatrix} 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix}$$

$$= \begin{vmatrix} x+y+z & x+y+z & x+y+z \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix} [R_1 = R_1 + R_2 + R_3]$$

$$= (x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix}$$

$$= (x+y+z)\begin{vmatrix} 1 & 0 & 0 \\ 2z & 0 & -x-y-z \\ x-y-z & x+y+z & x+y+z \end{vmatrix} [C_2 = C_2 - C_1, C_3 = C_3 - C_1]$$

$$= (x+y+z)[1\{0 + (x+y+z)(x+y+z)\}]$$

$$= (x+y+z)^3$$

= R.H.S.

Hence Proved

Q66

Show that $\begin{vmatrix} y+z & x & y \\ z+x & z & x \\ x+y & y & z \end{vmatrix} = (x+y+z)(x-z)^2$

Solution

$$\begin{vmatrix} y+z & x & y \\ z+x & z & x \\ x+y & y & z \end{vmatrix}$$

$$= \begin{vmatrix} 2(y+z+x) & y+z+x & y+z+x \\ z+x & z & x \\ x+y & y & z \end{vmatrix}$$

$$= (x+y+z)\begin{vmatrix} 2 & 1 & 1 \\ z+x & z & x \\ x+y & y & z \end{vmatrix}$$

$$= (x+y+z)\begin{vmatrix} 0 & 1 & 1 \\ z+x-z-x & z & x \\ x+y-y-z & y & z \end{vmatrix}$$

$$= (x+y+z)\begin{vmatrix} 0 & 1 & 1 \\ 0 & z & x \\ x-z & y & z \end{vmatrix}$$

$$= (x+y+z)(x-z)^2$$

$$= RHS$$

Q67

Prove the following identity:

$$\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix} = a^2(a+x+y+z)$$

Solution

L.H.S. =

$$\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix}$$

$$= \begin{vmatrix} a+x+y+z & y & z \\ a+x+y+z & a+y & z \\ a+x+y+z & x & a+z \end{vmatrix} [C_1 = C_1 + C_2 + C_3]$$

$$= (a+x+y+z)\begin{vmatrix} 1 & y & z \\ 1 & a+y & z \\ 1 & x & a+z \end{vmatrix}$$

$$= (a+x+y+z)\begin{vmatrix} 1 & y & z \\ 0 & a & 0 \\ 0 & x-y & a \end{vmatrix} [R_2 \to R_2 - R_1, R_3 \to R_3 - R_1]$$

$$= (a+x+y+z)[1(a^2 - 0)]$$

$$= a^2(a+x+y+z)$$

= R.H.S.

Hence Proved.

Q68

Prove the following identities:

$$\begin{vmatrix} a^3 & 2 & a \\ b^3 & 2 & b \\ c^3 & 2 & c \end{vmatrix} = 2(a-b)(b-c)(c-a)(a+b+c)$$

Solution

$$\text{Let } \Delta = \begin{vmatrix} a^3 & 2 & a \\ b^3 & 2 & b \\ c^3 & 2 & c \end{vmatrix}$$

$$\Delta = 2\begin{vmatrix} a^3 & 1 & a \\ b^3 & 1 & b \\ c^3 & 1 & c \end{vmatrix}$$

$$\Delta = 2\left\{a^3(c-b) - 1\left(b^3c - bc^3\right) + a\left(b^3 - c^3\right)\right\}$$

$$\Delta = 2\left\{a^3(c-b) - bc(b-c)(b+c) + a(b-c)\left(b^2 + bc + c^2\right)\right\}$$

$$\Delta = 2(b\text{-}c)\left\{-a^3 - bc(b+c) + a\left(b^2 + bc + c^2\right)\right\}$$

$$\Delta = 2(a\text{-}b)(b\text{-}c)(c-a)(a+b+c)$$

Q69

Without expanding, prove that

$$\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix} = \begin{vmatrix} y & b & p \\ x & a & q \\ z & c & r \end{vmatrix}$$

Solution

$$\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = -\begin{vmatrix} x & y & z \\ a & b & c \\ p & q & r \end{vmatrix} = (-1)^2\begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix} = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix}$$

$$= (-1)\begin{vmatrix} y & x & z \\ q & p & r \\ b & a & c \end{vmatrix}$$

$$= (-1)^2\begin{vmatrix} y & x & z \\ b & a & c \\ q & p & r \end{vmatrix}$$

Taking transpose, we get

$$= \begin{vmatrix} y & b & p \\ x & a & q \\ z & c & r \end{vmatrix}$$

Q70

Show that $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0$, *where a, b, c are in A.P.*

Solution

Consider the determinant $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$, *where a, b, c are in A.P.*

Let $\Delta = \begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$

Applying $C_1 \to C_1 + C_2 + C_3$, *we have,*

$$\Delta = \begin{vmatrix} 3x+1+2+a & x+2 & x+a \\ 3x+2+3+b & x+3 & x+b \\ 3x+3+4+c & x+4 & x+c \end{vmatrix}$$

$$\Rightarrow \Delta = \begin{vmatrix} 3x+3+a & x+2 & x+a \\ 3x+5+b & x+3 & x+b \\ 3x+7+c & x+4 & x+c \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ *and* $R_3 \to R_3 - R_2$, *we have,*

$$\Rightarrow \Delta = \begin{vmatrix} 3x+3+a & x+2 & x+a \\ 2+b-a & 1 & b-a \\ 2+c-b & 1 & c-b \end{vmatrix}$$

Since a,b and c are in arithmetic progression, we have
$b - a = c - b = k(say)$.
Thus,

$$\Delta = \begin{vmatrix} 3x+3+a & x+2 & x+a \\ 2+k & 1 & k \\ 2+k & 1 & k \end{vmatrix}$$

Since the second row and the third row are identical, we have
$\Delta = 0$

Q71

Show that $\begin{vmatrix} x-3 & x-4 & x-\alpha \\ x-2 & x-3 & x-\beta \\ x-1 & x-2 & x-\gamma \end{vmatrix} = 0$ where α, β, γ are in A.P.

Solution

Since, α, β, γ are in A.P, $2\beta = \alpha + \gamma$

$$LHS = \begin{vmatrix} x-3 & x-4 & x-\alpha \\ x-2 & x-3 & x-\beta \\ x-1 & x-2 & x-\gamma \end{vmatrix}$$

$R_2 \to R_2 - \frac{R_1}{2} - \frac{R_3}{2}$

$$= \begin{vmatrix} x-3 & x-4 & x-\alpha \\ (x-2) - \frac{x-3}{2} - \frac{x-1}{2} & (x-3) - \frac{x-4}{2} - \frac{x-2}{2} & (x-\beta) - \frac{x-\alpha}{2} - \frac{x-\gamma}{2} \\ x-1 & x-2 & x-\gamma \end{vmatrix}$$

$$= \begin{vmatrix} x-3 & x-4 & x-\alpha \\ 0 & 0 & 0 \\ x-1 & x-2 & x-\gamma \end{vmatrix} \quad [\because 2\beta = \alpha + \gamma]$$

$= 0$

Q72

If a, b and c are real numbers, and $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$,

Show that either $a + b + c = 0$ or $a = b = c$.

Solution

$$\Delta=\begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

Applying $R_1 \to R_1+R_2+R_3$, we have:

$$\Delta=\begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

$$=2(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

Applying $C_2 \to C_2-C_1$ and $C_3 \to C_3-C_1$, we have:

$$\Delta=2(a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ c+a & b-c & b-a \\ a+b & c-a & c-b \end{vmatrix}$$

Expanding along R_1, we have:

$$\Delta=2(a+b+c)(1)\left[(b-c)(c-b)-(b-a)(c-a)\right]$$
$$=2(a+b+c)\left[-b^2-c^2+2bc-bc+ba+ac-a^2\right]$$
$$=2(a+b+c)\left[ab+bc+ca-a^2-b^2-c^2\right]$$

It is given that $\Delta=0$.

$$(a+b+c)\left[ab+bc+ca-a^2-b^2-c^2\right]=0$$

$\Rightarrow$ Either $a+b+c=0$, or $ab+bc+ca-a^2-b^2-c^2=0$.

Now,

$$ab+bc+ca-a^2-b^2-c^2=0$$
$$\Rightarrow -2ab-2bc-2ca+2a^2+2b^2+2c^2=0$$
$$\Rightarrow (a-b)^2+(b-c)^2+(c-a)^2=0$$
$$\Rightarrow (a-b)^2=(b-c)^2=(c-a)^2=0 \quad \left[(a-b)^2,(b-c)^2,(c-a)^2 \text{ are non-negative}\right]$$
$$\Rightarrow (a-b)=(b-c)=(c-a)=0$$
$$\Rightarrow a=b=c$$

Hence, if $\Delta=0$, then either $a+b+c=0$ or $a=b=c$.

Q73

If $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix}=0$, find the value of

$$\frac{p}{p-a}+\frac{q}{q-b}+\frac{r}{r-c}, p\neq a, q=b, r\neq c.$$

Solution

$$\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} p-a & 0 & c-r \\ 0 & q-b & c-r \\ a & b & r \end{vmatrix} = 0 [R_1 = R_1 - R_3, R_2 = R_2 - R_3]$$

$$\Rightarrow (p-a)[r(q-b) - b(c-r)] + (c-r)[0 - a(q-b)] = 0$$

$$\Rightarrow (p-a)r(q-b) - (p-a)b(c-r) - (c-r)a(q-b) = 0$$

$$\Rightarrow \frac{(p-a)r(q-b)}{(p-a)(q-b)(r-c)} - \frac{(p-a)b(c-r)}{(p-a)(q-b)(r-c)} - \frac{(c-r)a(q-b)}{(p-a)(q-b)(r-c)} = 0$$

$$\Rightarrow \frac{r}{(r-c)} + \frac{b}{(q-b)} + \frac{a}{(p-a)} = 0$$

$$\Rightarrow \frac{r}{(r-c)} + \frac{b-q+q}{(q-b)} + \frac{a+p-p}{(p-a)} = 0$$

$$\Rightarrow \frac{r}{(r-c)} + \frac{q}{(q-b)} + \frac{(b-q)}{(q-b)} + \frac{(a-p)}{(p-a)} + \frac{p}{(p-a)} = 0$$

$$\Rightarrow \frac{r}{(r-c)} + \frac{q}{(q-b)} - 1 - 1 + \frac{p}{(p-a)} = 0$$

$$\Rightarrow \frac{r}{(r-c)} + \frac{q}{(q-b)} + \frac{p}{(p-a)} = 2$$

$$\therefore \frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} = 2$$

Exercise 6.3

Q1

Find the area of the triangle with vertices at the points $(3,8),(-4,2)$ and $(5,-1)$.

Solution

If the vertices of a triangle are $(x_1,y_1),(x_2,y_2)$ and (x_3,y_3)
then the area of the triangle is given by :

$$\Delta=\frac{1}{2}\begin{vmatrix}x_1 & y_1 & 1\\x_2 & y_2 & 1\\x_3 & y_3 & 1\end{vmatrix}$$

Substituting the values

$$\Delta=\frac{1}{2}\begin{vmatrix}3 & 8 & 1\\-4 & 2 & 1\\5 & -1 & 1\end{vmatrix}$$

expanding the determinant along R_1

$$=\frac{1}{2}\left[3\begin{vmatrix}2 & 1\\-1 & 1\end{vmatrix}-8\begin{vmatrix}-4 & 1\\5 & 1\end{vmatrix}+1\begin{vmatrix}-4 & 2\\5 & -1\end{vmatrix}\right]$$

$$=\frac{1}{2}\left[3(3)-8(-9)+1(-6)\right]$$

$$=\frac{1}{2}\left[9+72-6\right]=\frac{75}{2}\text{ sq. units}$$

The area of the Δ is $\frac{75}{2}$ sq. units

Q2

Find the area of the triangle with vertices at the points $(2,7),(1,1)$ and $(10,8)$

Solution

The area is given by:

$$\Delta=\frac{1}{2}\begin{vmatrix}2 & 7 & 1\\1 & 1 & 1\\10 & 8 & 1\end{vmatrix}$$

expanding along R_1

$$=\frac{1}{2}\left[2(-7)-7(-9)+1(-2)\right]$$

$$=\frac{1}{2}\left[-14+63-2\right]$$

$$=\frac{47}{2}\text{ sq. units}$$

The area of the Δ is $\frac{47}{2}$ sq. units

Q3

Find the area of the triangle with vertices at the points $(-1,-8)$, $(-2,-3)$ and $(3,2)$

Solution

The area is given by:

$$\Delta = \frac{1}{2}\begin{vmatrix} -1 & -8 & 1 \\ -2 & -3 & 1 \\ 3 & 2 & 1 \end{vmatrix}$$

$$= \frac{1}{2}[-1(-5)+8(-5)+1(5)]$$

$$= \frac{1}{2}[5-40+5] = \frac{-30}{2} = 15 \text{ sq. units}$$

$\because$ Area can not be negative, so answer will be 15 sq. units.

The area of the Δ is 15 sq. units.

Q4

find the area of the triangle with vertices at the points:
(0, 0), (6, 0), (4, 3)

Solution

The area is given by:

$$\Delta = \frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 6 & 0 & 1 \\ 4 & 3 & 1 \end{vmatrix}$$

Expanding along R_1

$$= \frac{1}{2}[0-0+1(18)] = 9 \text{ sq. units}$$

The area is 9 sq. units

Q5

Using determinants show that the points $(5,5)$, $(-5,1)$ and $(10,7)$ are collinear.

Solution

If 3 points are collinear, then the area of the triangle then form will be zero.

Hence

$$\frac{1}{2}\begin{vmatrix} 5 & 5 & 1 \\ -5 & 1 & 1 \\ 10 & 7 & 1 \end{vmatrix} = 0$$

Expanding along R_1

$= \frac{1}{2}\left[5(-6) - 5(-15) + 1(-35 - 10)\right]$

$= \frac{1}{2}\left[-35 + 75 - 45\right]$

$= \frac{1}{2}[0]$

$= 0$

Since the area of the triangle is zero, hence the points are collinear.

Q6

Using determinants show that the points $(3, -2)$, $(8, 8)$ and $(5, 2)$ are collinear.

Solution

If the points are collinear, then the area of the triangle will be zero.

So

$$\frac{1}{2}\begin{vmatrix} 3 & -2 & 1 \\ 8 & 8 & 1 \\ 5 & 2 & 1 \end{vmatrix} = 0$$

L.H.S

Expanding along R_1

$= \frac{1}{2}\left[3(6) + 2(3) + 1(-24)\right]$

$= \frac{1}{2}\left[18 + 6 - 24\right]$

$= \frac{1}{2}[0]$

$= 0$

Since the area of the triangle is zero, hence given points are collinear.

Q7

Using determinants show that the points $(2, 3)$, $(-1, -2)$ and $(5, 8)$ are collinear.

Solution

If given points are collinear, then the area of the triangle must be zero.

Hence

$$= \frac{1}{2}\begin{vmatrix} 2 & 3 & 1 \\ -1 & -2 & 1 \\ 5 & 8 & 1 \end{vmatrix}$$

$$= \frac{1}{2}[2(-10) - 3(-6) + 1(2)]$$

$$= \frac{1}{2}[-20 + 18 + 2]$$

$$= \frac{1}{2}[0]$$

$$= 0$$

Hence the given points are collinear.

Q8

Using determinants show that the points $(1, -1)$, $(2, 1)$ and $(4, 5)$ are collinear.

Solution

If 3 points are collinear, then the area of the triangle then form will be zero.

Hence

$$\frac{1}{2}\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ 4 & 5 & 1 \end{vmatrix} = 0$$

Expanding along R_1

$$= \frac{1}{2}[1(-4) + 1(-2) + 1(6)]$$

$$= 0$$

Since the area of the triangle is zero, hence the points are collinear.

Q9

If the points $(a, 0)$, $(0, b)$ and $(1, 1)$ are collinear, prove that $a + b = ab$

Solution

If the given points are collinear, the area of the triangle must be zero.

Hence

$$\frac{1}{2}\begin{vmatrix} a & 0 & 1 \\ 0 & b & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

Expanding along R_1

$$= \frac{1}{2}\left[a(b-1) - 0(0-1) + 1(-b)\right] = 0$$

or $ab - a - 0 - b = 0$

or $ab = a + b$

Hence proved

Q10

Using determinants prove that the points $(a,b)(a',b)(a-a',b-b')$ are collinear if $ab' = a'b$

Solution

If the given points are collinear, then the area of the triangle must be zero.

Hence

$$\frac{1}{2}\begin{vmatrix} a & b & 1 \\ a' & b' & 1 \\ a-a' & b-b' & 1 \end{vmatrix} = 0$$

or

$$\frac{1}{2}\left[a(b' - b + b') - b(a' - a + a') + 1(a'b - a'b' - ab' + a'b')\right] = 0$$

or $\frac{1}{2}\left[ab' - ab + ab' - a'b + ab - a'b + a'b - ab'\right] = 0$

or $ab' - a'b = 0$

$ab' = a'b$

Hence proved

Q11

Find the value of λ so that the points $(1,-5)$, $(-4,5)$ and $(\lambda,7)$ are collinear.

Solution

If the points are collinear, then the area of the triangle must be zero.

Hence

$$\begin{vmatrix} 1 & -5 & 1 \\ -4 & 5 & 1 \\ \lambda & 7 & 1 \end{vmatrix} = 0$$

Expanding along R_1

$$1(-2) + 5(-4 - \lambda) + 1(-28 - 5\lambda) = 0$$
$$-2 - 20 - 5\lambda - 28 - 5\lambda = 0$$
$$-50 - 10\lambda = 0$$
$$\lambda = -5$$

Hence $\lambda = -5$

Q13

Using determinants, find the area of the triangle whose vertices are $(1, 4)(2, 3)$ and $(-5, -3)$. Are the givenpoints collinear?

Solution

$$\text{Area} = \frac{1}{2}\begin{vmatrix} 1 & 4 & 1 \\ 2 & 3 & 1 \\ -5 & -3 & 1 \end{vmatrix}$$
$$= \frac{1}{2}\left[1(6) - 4(7) + 1(-6 + 15)\right]$$
$$= \frac{1}{2}[6 - 28 + 9]$$
$$= \frac{1}{2}[-13]$$
$$= \frac{13}{2} \text{ sq. units} \qquad [\because \text{Area can not be negative}]$$

Also, since the area of the triangle is non-zero.

Hence these points are non-collinear.

Q14

Using determinants, find the area of the triangle with vertices $(-3, 5)(3, -6)$ $(7, 2)$.

Solution

$$\text{Area} = \frac{1}{2}\begin{vmatrix} -3 & 5 & 1 \\ 3 & -6 & 1 \\ 7 & 2 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[-3(-8) - 5(-4) + 1(48)\right]$$

$$= \frac{1}{2}\left[24 + 20 + 48\right]$$

$$= 46 \text{ sq. units}$$

Hence the area is 46 sq. units.

Q15

Using determinants, find the value of k so that the points $(k, 2 - 2k)(-k + 1, 2k)$ and $(-4 - k, 6 - 2k)$ may be collinear.

Solution

If the given points are collinear, then the area of the triangle must be zero.

$$\text{so } \frac{1}{2}\begin{vmatrix} k & 2 - 2k & 1 \\ -k + 1 & 2k & 1 \\ -4 - k & 6 - 2k & 1 \end{vmatrix} = 0$$

expanding along R_1

$k(2k - 6 + 2k) - (2 - 2k)(-k + 1 + 4 + k) + 1(1 - k) \times (6 - 2k) - 2k(-4 - k) = 0$

$k(4k - 6) - (2 - 2k)(5) + 1\left[6 - 2k - 6k + 2k^2 + 8k + 2k^2\right] = 0$

$4k^2 - 6k - 10 + 10k + 6 + 4k^2 = 0$

$8k^2 + 4k - 4 = 0$

$8k^2 + 8k - 4k - 4 = 0$ (Middle term splitting)

$8k(k + 1) - 4(k + 1) = 0$

$(8k - 4)(k + 1) = 0$

If $8k - 4 = 0$ or if $k + 1 = 0$

$k = \frac{1}{2}$ $k = -1$

Hence $k = -1, \frac{1}{2}$

Q16

If the points $(x, -2), (5, 2), (8, 8)$ are collinear, find x using determinants.

Solution

Since the points are collinear, hence the area of the triangle must be zero.

$$\text{so } \frac{1}{2}\begin{vmatrix} x & -2 & 1 \\ 5 & 2 & 1 \\ 8 & 8 & 1 \end{vmatrix} = 0$$

$$\text{or } x(-6) + 2(-3) + 1(24) = 0$$

$$\begin{aligned} \text{or } -6x - 6 + 24 &= 0 \\ -6x + 18 &= 0 \\ x &= 3 \end{aligned}$$

Hence $x = 3$

Q17

If the points $(3,-2), (x,2), (8,8)$ are collinear, find x using determinants.

Solution

Since the points are collinear, hence the area of the triangle must be zero.

$$\frac{1}{2}\begin{vmatrix} 3 & -2 & 1 \\ x & 2 & 1 \\ 8 & 8 & 1 \end{vmatrix} = 0$$

$$\begin{aligned} 3(-6) + 2(x-8) + 1(8x-16) &= 0 \\ -18 + 2x - 16 + 8x - 16 &= 0 \\ 10x &= 50 \\ x &= 5 \end{aligned}$$

Hence $x = 5$

Q18

Using determinants, find the equation of the line joining the points $(1,2)$ and $(3,6)$

Solution

Let $A(x, y)$, $B(1, 2)$ and $C(3, 6)$ are 3 points in a line.

Since these points are collinear, hence area of the triangle must be zero.

$$\frac{1}{2}\begin{vmatrix} x & y & 1 \\ 1 & 2 & 1 \\ 3 & 6 & 1 \end{vmatrix} = 0$$

Expanding along R_1

$$x(-4) - y(-2) + 1(0) = 0$$
$$-4x + 2y = 0$$
$$\text{or } 2x - y = 0$$
$$\text{or } y = 2x$$

Hence the equation is $y = 2x$

Exercise 6.4

Q1

Solve the following systems of linear equations by Cramer's rule

$x - 2y = 4$

$-3x + 5y = -7$

Solution

$$\text{Let } D = \begin{vmatrix} 1 & -2 \\ -3 & 5 \end{vmatrix} = 5 - 6 = -1$$

$$D_1 = \begin{vmatrix} 4 & -2 \\ -7 & 5 \end{vmatrix} = 20 - 14 = 6$$

$$D_2 = \begin{vmatrix} 1 & 4 \\ -3 & -7 \end{vmatrix} = -7 + 12 = 5$$

$$\text{by definition } x = \frac{D_1}{D} = \frac{6}{-1} = -6$$

$$y = \frac{D_2}{D} = \frac{5}{-1} = -5$$

Hence $x = -6$

$y = -5$

Q2

Solve the following systems of linear equations by Cramer's rule

$2x - y = 1$

$7x - 2y = -7$

Solution

$$\text{Let } D = \begin{vmatrix} 2 & -1 \\ 7 & -2 \end{vmatrix} = -4 + 7 = 3$$

$$D_1 = \begin{vmatrix} 1 & -1 \\ -7 & -2 \end{vmatrix} = -9$$

$$D_2 = \begin{vmatrix} 2 & 1 \\ 7 & -7 \end{vmatrix} = -21$$

$$\text{Now, } x = \frac{D_1}{D} = \frac{-9}{3} = -3$$

$$y = \frac{+D_2}{D} = \frac{-21}{3} = -7$$

Hence $x = -3$

$y = -7$

Q3

Solve the following systems of linear equations by Cramer's rule
$2x - y = 17$
$3x + 5y = 6$

Solution

$$\text{Let } D = \begin{vmatrix} 2 & -1 \\ 3 & 5 \end{vmatrix} = 13$$
$$D_1 = \begin{vmatrix} 17 & -1 \\ 6 & 5 \end{vmatrix} = 91$$
$$D_2 = \begin{vmatrix} 2 & 17 \\ 3 & 6 \end{vmatrix} = -39$$

$$x = \frac{D_1}{D} = \frac{91}{13} = 7$$
$$y = \frac{D_2}{D} = \frac{-39}{13} = -3$$

Hence $x = 7$
$y = -3$

Q4

Solve the following systems of linear equations by Cramer's rule
$3x + y = 19$
$3x - y = 23$

Solution

$$\text{Let } D = \begin{vmatrix} 3 & 1 \\ 3 & -1 \end{vmatrix} = -6$$
$$D_1 = \begin{vmatrix} 19 & 1 \\ 23 & -1 \end{vmatrix} = -42$$
$$D_2 = \begin{vmatrix} 3 & 19 \\ 3 & 23 \end{vmatrix} = 12$$

$$x = \frac{D_1}{D} = \frac{-42}{-6} = 7$$
$$y = \frac{D_2}{D} = \frac{12}{-6} = -2$$

Hence $x = 7$
$y = -2$

Q5

Solve the following systems of linear equations by Cramer's rule
$2x - y = -2$
$3x + 4y = 3$

Solution

$$\text{Let } D = \begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix} = 11$$

$$D_1 = \begin{vmatrix} -2 & -1 \\ 3 & 4 \end{vmatrix} = -5$$

$$D_2 = \begin{vmatrix} 2 & -2 \\ 3 & 3 \end{vmatrix} = 12$$

$$x = \frac{D_1}{D} = \frac{-5}{11}$$

$$y = \frac{D_2}{D} = \frac{12}{11}$$

Q6

Solve the following systems of linear equations by Cramer's rule

$$3x + ay = 4$$
$$2x + ay = 2$$

Solution

$$\text{Let } D = \begin{vmatrix} 3 & a \\ 2 & a \end{vmatrix} = a$$

$$D_1 = \begin{vmatrix} 4 & a \\ 2 & a \end{vmatrix} = 2a$$

$$D_2 = \begin{vmatrix} 3 & 4 \\ 4 & 2 \end{vmatrix} = -2$$

$$x = \frac{D_1}{D} = \frac{2a}{a} = 2$$

$$y = \frac{D_2}{D} = \frac{-2}{a}$$

Q7

Solve the following systems of linear equation by Cramer's rule:
$2x + 3y = 10$
$x + 6y = 4$

Solution

$$\text{Let } D = \begin{vmatrix} 2 & 3 \\ 1 & 6 \end{vmatrix} = 9$$
$$D_1 = \begin{vmatrix} 10 & 3 \\ 4 & 6 \end{vmatrix} = 48$$
$$D_2 = \begin{vmatrix} 2 & 10 \\ 1 & 4 \end{vmatrix} = -2$$

$$x = \frac{D_1}{D} = \frac{48}{9} = \frac{16}{3}$$
$$y = \frac{D_2}{D} = \frac{-2}{9}$$

Q8

Solve the following systems of linear equations by Cramer's rule

$$5x + 7y = -2$$
$$4x + 6y = -3$$

Solution

$$\text{Let } D = \begin{vmatrix} 5 & 7 \\ 4 & 6 \end{vmatrix} = 2$$
$$D_1 = \begin{vmatrix} -2 & 7 \\ -3 & 6 \end{vmatrix} = 9$$
$$D_2 = \begin{vmatrix} 5 & -2 \\ 4 & -3 \end{vmatrix} = -7$$

$$x = \frac{D_1}{D} = \frac{9}{2}$$
$$y = \frac{D_2}{D} = \frac{-7}{2}$$

Q9

Solve the following systems of linear equations by Cramer's rule

$$9x + 5y = 10$$
$$3y - 2x = 8$$

Solution

$$\text{Let } D = \begin{vmatrix} 9 & 5 \\ -2 & 3 \end{vmatrix} = 37$$
$$D_1 = \begin{vmatrix} 10 & 5 \\ 8 & 3 \end{vmatrix} = -10$$
$$D_2 = \begin{vmatrix} 9 & 10 \\ -2 & 8 \end{vmatrix} = 92$$

$$x = \frac{D_1}{D} = \frac{-10}{37}$$
$$y = \frac{D_2}{D} = \frac{92}{37}$$

Q10

Solve the following systems of linear equations by Cramer's rule
x + 2y = 1
3x + y = 4

Solution

$$\text{Let } D = \begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5$$

$$D_1 = \begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} = -7$$

$$D_2 = \begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} = 1$$

$$x = \frac{D_1}{D} = \frac{7}{5}$$

$$y = \frac{D_2}{D} = \frac{-1}{5}$$

Exercise 6.5

Q1

Solve each of the following system of homogeneous linear equations.

$x + y - 2z = 0$

$2x + y - 3z = 0$

$5x + 4y - 9z = 0$

Solution

Solve each of the following system of homogeneous linear equations.

$x + y - 2z = 0$

$2x + y - 3z = 0$

$5x + 4y - 9z = 0$

Q2

Solve the following system of homogeneous linear equations:

$2x + 3y + 4z = 0$
$X + y + z = 0$
$2x + 5y - 2z = 0$

Solution

Solve the following system of homogeneous linear equations:

$2x + 3y + 4z = 0$
$X + y + z = 0$
$2x + 5y - 2z = 0$

Q3

Solve each of the following system of homogeneous linear equations.

$3x + y + z = 0$

$x - 4y + 3z = 0$

$2x + 5y - 2z = 0$

Solution

$$\text{Here } D = \begin{vmatrix} 3 & 1 & 1 \\ 1 & -4 & 3 \\ 2 & 5 & -2 \end{vmatrix}$$

$$= 3(8-15) - 1(-2-6) + 1(13)$$

$$= -21 + 8 + 13$$

$$= 0$$

So, the system has infinite solutions:

Let $z = k$,

so, $3x + y = -k$

$x - 4y = -3k$

Now,

$$x = \frac{D_1}{D} = \frac{\begin{vmatrix} -k & 1 \\ -3k & -4 \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 1 & -4 \end{vmatrix}} = \frac{7k}{-13}$$

$$y = \frac{D_2}{D} = \frac{\begin{vmatrix} 3 & -k \\ 1 & -3k \end{vmatrix}}{\begin{vmatrix} 3 & 1 \\ 1 & -4 \end{vmatrix}} = \frac{-8k}{-13}$$

$$x = \frac{-7k}{13}, y = \frac{8k}{13}, z = k$$

and there values satisfy eq.(3)

Hence $x = -7k, y = 8k, z = 13k$

Q4

Find the real values of λ for which the following system of linear equations has non - trivial solutions. Also, find the non - trivial solutions

$2\lambda x - 2y + 3z = 0$

$x + \lambda y + 2z = 0$

$2x + \lambda z = 0$

Solution

$$D = \begin{vmatrix} 2\lambda & -2 & 3 \\ 1 & \lambda & 2 \\ 2 & 0 & \lambda \end{vmatrix}$$
$$= 3\lambda^3 + 2\lambda - 8 - 6\lambda$$
$$= 2\lambda^3 - 4\lambda - 8$$

which is satisfied by $\lambda = 2$ $[\because$ for non-trivial solutions $\lambda = 2]$

Now Let $z = k$,

$$4x - 2y = -3k$$
$$x + 2y = -3k$$

$$x = \frac{D_1}{D} = \frac{\begin{vmatrix} -3k & -2 \\ -2k & 2 \end{vmatrix}}{\begin{vmatrix} 4 & -2 \\ 1 & 2 \end{vmatrix}} = \frac{-10k}{10} = -k$$

$$y = \frac{D_2}{D} = \frac{\begin{vmatrix} 4 & -3k \\ 1 & -2k \end{vmatrix}}{\begin{vmatrix} 4 & -2 \\ 1 & 2 \end{vmatrix}} = \frac{-5k}{10} = \frac{-k}{2}$$

Hence solution is $x = -k, y = \frac{-k}{2}, z = k$

Q5

If a, b, c are non-zero real numbers and if the system of equations

$(a-1)x = y + z$

$(b-1)y = z + x$

$(c-1)z = x + y$

has a non-trivial solution, then prove that $ab + bc + ca = abc$.

Solution

$$D = \begin{vmatrix} (a-1) & -1 & -1 \\ -1 & (b-1) & -1 \\ -1 & -1 & (c-1) \end{vmatrix}$$

Now for non-trivial solution, $D = 0$

$$0 = (a-1)[(b-1)(c-1)-1] + 1[-c + \cancel{1} - \cancel{1}] - [\cancel{1} + b - \cancel{1}]$$
$$0 = (a-1)[bc - b - c + \cancel{1} - \cancel{1}] - c - b$$
$$0 = abc - ab - ac + \cancel{b} + \cancel{c} - \cancel{c} - \cancel{b}$$
$$ab + bc + ac = abc$$

Hence proved

Exercise MCQ

Q1

If A and B are square matrices of order 2, then det (A + B) = 0 is possible only when

a. det (A) = 0 or det (B) = 0
b. det (A) + det (B) = 0
c. det (A) = 0 and det (B) = 0
d. A + B = O

Solution

Correct option: (d)

Deter min anat A denoted as $\left[a_{ij}\right]$ and det er min anat B as $\left[b_{ij}\right]$

$$\Rightarrow A+B=\left[a_{ij}\right]+\left[b_{ij}\right]$$
$$\Rightarrow A+B=\left[a_{ij}+b_{ij}\right]$$
$$\Rightarrow \det(A+B)=\det\left[a_{ij}+b_{ij}\right]$$
$$\Rightarrow \det(A+B)=0$$
$$\Rightarrow \det\left[a_{ij}+b_{ij}\right]=0$$
$$\Rightarrow a_{ij}+b_{ij}=0$$
$$\Rightarrow A+B=0$$

Q2

Which of the following is not correct?

a. $|A| = |A^T|$, where $A = [a_{ij}]_{3\times3}$
b. $|kA| = k^3\,|A|$, where $A = [a_{ij}]_{3\times3}$
c. If A is a skew-symmetric matrix of odd order, then $|A| = 0$
d. $\begin{vmatrix} a+b & c+d \\ e+f & g+h \end{vmatrix} = \begin{vmatrix} a & c \\ e & g \end{vmatrix} + \begin{vmatrix} b & d \\ f & h \end{vmatrix}$

Solution

Correct option: (d)

$$\begin{vmatrix} a+b & c+d \\ e+f & g+h \end{vmatrix}$$
$$= \begin{vmatrix} a+b & c \\ e+f & g \end{vmatrix} + \begin{vmatrix} a+b & d \\ e+f & h \end{vmatrix}$$
$$= \begin{vmatrix} a & c \\ e & g \end{vmatrix} + \begin{vmatrix} b & c \\ f & g \end{vmatrix} + \begin{vmatrix} a & d \\ e & h \end{vmatrix} + \begin{vmatrix} b & d \\ f & h \end{vmatrix}$$
$$\begin{vmatrix} a+b & c+d \\ e+f & g+h \end{vmatrix} \neq \begin{vmatrix} a & c \\ e & g \end{vmatrix} + \begin{vmatrix} b & d \\ f & h \end{vmatrix}$$

Q3

If $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and C_{ij} is cofactor of a_{ij} in A,
then value of $|A|$ is given by

a. $a_{11}C_{31}+a_{12}C_{32}+a_{13}C_{33}$
b. $a_{11}C_{11}+a_{12}C_{21}+a_{13}C_{31}$
c. $a_{21}C_{11}+a_{22}C_{12}+a_{23}C_{13}$
d. $a_{11}C_{11}+a_{21}C_{21}+a_{31}C_{31}$

Solution

Correct option: (d)
If A is a square matrix of order n then det(A) = $a_{11}C_{11}+a_{21}C_{21}+a_{31}C_{31}$

Q4

Which of the following is not correct in a given determinant of A, where A = $[a_{ij}]_{3\times3}$.

a. Order of minor is less than order of the det (A).
b. Minor of an element can never be equal to cofactor of the same element
c. Value of a determinant is obtained by multiplying elements of a row or column by corresponding cofactors
d. Order of minors and cofactors of elements of A is same

Solution

Correct option: (b)
Minor of an element can never be equal to cofactor of the same element.
$C_{ij}=(-1)^{i+j}M_{ij}$

Q5

Let $\begin{vmatrix} x & 2 & x \\ x^2 & x & 6 \\ x & x & 6 \end{vmatrix} = ax^4 + bx^3 + cx^2 + dx + e$. Then, the value of
$5a + 4b + 3c + 2d + e$ is equal

a. 0
b. -16
c. 16
d. 1
e. None of these

Solution

Correct option: (e)

$$\begin{vmatrix} x & 2 & x \\ x^2 & x & 6 \\ x & x & 6 \end{vmatrix}$$

$= x(6x-6x) - 2(6x^2-6x) + x(x^3-x^2)$

$= 0 - 12x^2 + 12x + x^4 - x^3$

$= x^4 - x^3 - 12x^2 + 12x$

Comparing with RHS $ax^4 + bx^3 + cx^2 + dx + e$.

$a = 1,\ b = -1,\ c = -12,\ d = 12,\ e = 0$

$\Rightarrow 5a + 4b + 3c + 2d + e = 5 - 4 - 36 + 24 = -11$

Q6

The value of the determinant $\begin{vmatrix} a^2 & a & 1 \\ \cos nx & \cos(n+1)x & \cos(n+2)x \\ \sin nx & \sin(n+1)x & \sin(n+2)x \end{vmatrix}$ is independent of

a. n
b. a
c. x
d. none of these

Solution

Correct option: (a)

$$\begin{vmatrix} a^2 & a & 1 \\ \cos nx & \cos(n+1)x & \cos(n+2)x \\ \sin nx & \sin(n+1)x & \sin(n+2)x \end{vmatrix}$$

Let, $nx = u,\ (n+1)x = v,\ (n+2)x = w$

$\Rightarrow \begin{vmatrix} a^2 & a & 1 \\ \cos u & \cos v & \cos w \\ \sin u & \sin v & \sin w \end{vmatrix}$

$\Rightarrow a^2 \sin(w-v) - a\sin(w-u) + \sin(v-u)$

$\Rightarrow a^2 \sin x - a\sin 2x + \sin x$

$\Rightarrow$ It is independent of n.

Q7

If $\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} 1 & bc & a \\ 1 & ca & b \\ 1 & ab & c \end{vmatrix}$, then

a. $\Delta_1 + \Delta_2 = 0$
b. $\Delta_1 + 2\Delta_2 = 0$
c. $\Delta_1 = \Delta_2$
d. none of these

Solution

Correct option: (a)

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = bc^2 - b^2c - (ac^2 - a^2c) + ab^2 - a^2b$$

$$= bc^2 - b^2c - ac^2 + a^2c + ab^2 - a^2b$$

$$\Delta_2 = \begin{vmatrix} 1 & bc & a \\ 1 & ca & b \\ 1 & ab & c \end{vmatrix} = c^2a - ab^2 - bc(c-b) + a(ab-ac)$$

$$= c^2a - ab^2 - bc^2 + b^2c + a^2b - a^2c$$

$$= -(bc^2 - b^2c - ac^2 + a^2c + ab^2 - a^2b)$$

$$\Rightarrow \Delta_1 = -\Delta_2$$

$$\Rightarrow \Delta_1 + \Delta_2 = 0$$

Q8

If $D_k = \begin{vmatrix} 1 & n & n \\ 2k & n^2+n+2 & n^2+n \\ 2k-1 & n^2 & n^2+n+2 \end{vmatrix}$ and $\sum_{k=1}^{n} D_k = 48$, then n equals

a. 4
b. 6
c. 8
d. none of these

Solution

Correct option: (a)

$$D_k = \begin{vmatrix} 1 & n & n \\ 2k & n^2+n+2 & n^2+n \\ 2k-1 & n^2 & n^2+n+2 \end{vmatrix}$$

Applying row transformation $R_2 \to R_2 - R_3$ we get

$$D_k = \begin{vmatrix} 1 & n & n \\ 1 & n+2 & -2 \\ 2k-1 & n^2 & n^2+n+2 \end{vmatrix}$$

Applying row transformation $R_1 \to R_1 - R_2$ we get

$$D_k = \begin{vmatrix} 0 & -2 & n+2 \\ 1 & n+2 & -2 \\ 2k-1 & n^2 & n^2+n+2 \end{vmatrix}$$

$$= 2\left(n^2+n+2+4k-2\right)+(n+2)\left(n^2-(n+2)(2k-1)\right)$$

$$= 2\left(n^2+n+4k\right)+(n+2)\left(n^2-2nk+n-4k+2\right)$$

$$= 2n^2+2n+8k+n^3-2n^2k+n^2-4nk+2n+2n^2-4nk+2n-8k+4$$

$$= n^3+5n^2-2n^2k+6n-8nk+4$$

$$\sum_{k=1}^{n} D_k = 48$$

$$\Rightarrow n(n^3+5n^2+6n+4)-2n^2\sum_{k=1}^{n} k-8n\sum_{k=1}^{n} k = 48$$

$$\Rightarrow n(n^3+5n^2+6n+4)-2n^2\frac{n(n+1)}{2}-8n\frac{n(n+1)}{2} = 48$$

$$\Rightarrow n^4+5n^3+6n^2+4n-n^4-n^3-4n^3-4n^2 = 48$$

$$\Rightarrow 2n^2+4n = 48$$

$$\Rightarrow n^2+2n-24 = 0$$

$$\Rightarrow (n+6)(n-4) = 0$$

$$\Rightarrow n = -6, 4$$

Q9

Let $\begin{vmatrix} x^2+3x & x-1 & x+3 \\ x+1 & -2x & x-4 \\ x-3 & x+4 & 3x \end{vmatrix} = ax^4+bx^3+cx^2+dx+e$

be an identity in x, where a, b, c, d, e, are independent of x. then the value of e is

a. 4
b. 0
c. 1
d. none of these

Solution

Correct option: (a)

$$\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix}$$

Applying row transformation $C_1 \to C_1 + C_2$ we get

$$\begin{vmatrix} b-a & a+b & a+c \\ a-b & -2b & b+c \\ 2c+b+a & c+b & -2c \end{vmatrix}$$

Applying column transformation $C_2 \to C_2 + C_3$ we get

$$\begin{vmatrix} b-a & 2a+b+c & a+c \\ a-b & c-b & b+c \\ 2c+b+a & b-c & -2c \end{vmatrix}$$

Applying row transformation $R_3 \to R_2 + R_3$ we get

$$\begin{vmatrix} b-a & 2a+b+c & a+c \\ a-b & c-b & b+c \\ 2(c+a) & 0 & b-c \end{vmatrix}$$

Applying row transformation $R_2 \to R_2 + R_1$ we get

$$\begin{vmatrix} b-a & 2a+b+c & a+c \\ 0 & 2(a+c) & a+b+2c \\ 2(c+a) & 0 & b-c \end{vmatrix}$$

Expanding along C_1 we get

$$= (b-a)(2(a+c)(b-c)) + 2(a+c)\left((2a+b+c)(a+b+2c) - 2(a+c)^2\right)$$

$$= 2(a+c)\left[(b-a)(b-c) + (2a+b+c)(a+b+2c) - 2(a+c)^2\right]$$

$$= 2(a+c)\left[b^2 - bc - ab + ac + 2a^2 + 2ab + 4ac + ab + b^2 + 2bc + ac + bc + 2c^2 - 2a^2 - 2c^2 - 4ac\right]$$

$$= 2(a+c)\left[2b^2 + 2ab + 2bc + 2ac\right]$$

$$= 4(a+c)\left[b^2 + ab + bc + ac\right]$$

$$= 4(a+c)\left[b(a+b) + c(a+b)\right]$$

$$= 4(a+c)(b+c)(a+b)$$

So another factor is 4

Q10

Using the factor theorem it is found that a + b, b + c and c + a are three factors of the determinant $\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix}$. The other factor in the value of the determinant is

a. 4
b. 2
c. a + b + c
d. none of these

Solution

Correct option: (a)

$$\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix}$$

Applying row transformation $C_1 \to C_1 + C_2$ we get

$$\begin{vmatrix} b-a & a+c & a+c \\ a-b & -2b & b+c \\ 2c+b+a & c+b & -2c \end{vmatrix}$$

Applying column transformation $C_2 \to C_2 + C_3$ we get

$$\begin{vmatrix} b-a & 2a+b+c & a+c \\ a-b & c-b & b+c \\ 2c+b+a & b-c & -2c \end{vmatrix}$$

Applying row transformation $R_3 \to R_2 + R_3$ we get

$$\begin{vmatrix} b-a & 2a+b+c & a+c \\ a-b & c-b & b+c \\ 2(c+a) & 0 & b-c \end{vmatrix}$$

Applying row transformation $R_2 \to R_2 + R_1$ we get

$$\begin{vmatrix} b-a & 2a+b+c & a+c \\ 0 & 2(a+c) & a+b+2c \\ 2(c+a) & 0 & b-c \end{vmatrix}$$

Expanding along C_1 we get

$$= (b-a)(2(a+c)(b-c)) + 2(a+c)\left((2a+b+c)(a+b+2c) - 2(a+c)^2\right)$$

$$= 2(a+c)\left[(b-a)(b-c) + (2a+b+c)(a+b+2c) - 2(a+c)^2\right]$$

$$= 2(a+c)\left[b^2 - bc - ab + ac + 2a^2 + 2ab + 4ac + ab + b^2 + 2bc + ac + bc + 2c^2 - 2a^2 - 2c^2 - 4ac\right]$$

$$= 2(a+c)\left[2b^2 + 2ab + 2bc + 2ac\right]$$

$$= 4(a+c)\left[b^2 + ab + bc + ac\right]$$

$$= 4(a+c)\left[b(a+b) + c(a+b)\right]$$

$$= 4(a+c)(b+c)(a+b)$$

So another factor is 4

Q11

If a, b, c are distinct, then the value of x satisfying

$$\begin{vmatrix} 0 & x^2-a & x^3-b \\ x^2+a & 0 & x^2+c \\ x^4+b & x-c & 0 \end{vmatrix}$$

a. c
b. a
c. b
d. 0

Solution

Correct option: (d)

$$\begin{vmatrix} 0 & x^2-a & x^3-b \\ x^2+a & 0 & x^2+c \\ x^4+b & x-c & 0 \end{vmatrix}$$

If we put $x = 0$ in the above determinant,

$$\Rightarrow \begin{vmatrix} 0 & -a & -b \\ a & 0 & c \\ b & -c & 0 \end{vmatrix} = A$$

$$A^T = \begin{vmatrix} 0 & a & b \\ -a & 0 & -c \\ -b & c & 0 \end{vmatrix} = -\begin{vmatrix} 0 & -a & -b \\ a & 0 & c \\ b & -c & 0 \end{vmatrix} = -A$$

Matrix is skew − symmetric.

Also, Power of the matrix is odd.

Hence, value of x is 0.

Q12

If the determinant $\begin{vmatrix} a & b & 2a\alpha+3b \\ b & c & 2b\alpha+3c \\ 2a\alpha+3b & 2b\alpha+3c & 0 \end{vmatrix} = 0$, then

a. A, b, c are in H.P.
b. α is a root of $4ax^2 + 12bx + 9c = 0$ or, a, b, c are in G.P.
c. a, b, c are in G.P. only
d. a, b, c are in A.P.

Solution

Correct option: (b)

$$\begin{vmatrix} a & b & 2a\alpha+3b \\ b & c & 2b\alpha+3c \\ 2a\alpha+3b & 2b\alpha+3c & 0 \end{vmatrix} = 0$$

$C_1 \to C_1 - C_2$

$$\begin{vmatrix} a-b & b & 2a\alpha+3b \\ b-c & c & 2b\alpha+3c \\ 2\alpha(a-b)+3b-3c & 2b\alpha+3c & 0 \end{vmatrix} = 0$$

$R_3 \to R_3 - \alpha R_1$

$$\begin{vmatrix} \alpha(a-b) & \alpha b+3c & 2a\alpha+3b \\ b-c & c & 2b\alpha+3c \\ 2\alpha(a-b)+3b-3c & 2b\alpha+3c & 0 \end{vmatrix} = 0$$

$C_1 \to C_1 + C_2$

$$\begin{vmatrix} \alpha a-3c & \alpha b+3c & 2a\alpha+3b \\ b & c & 2b\alpha+3c \\ 2\alpha a+3b & 2b\alpha+3c & 0 \end{vmatrix} = 0$$

$$-\left(4a\alpha^2+12b\alpha+9c\right)\left(ac-b^2\right) = 0$$

$$4a\alpha^2+12b\alpha+9c = 0 \text{ or } ac-b^2 = 0$$

$$ac-b^2 = 0 \Rightarrow ac = b^2$$

$\Rightarrow$ a, b, c are in GP.

Q13

If ω is a non-real cube root of unity and n is not a multiple of 3, then $\Delta = \begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^{2n} & 1 & \omega^n \\ \omega^n & \omega^{2n} & 1 \end{vmatrix}$ is equal to

a. 0
b. ω
c. ω^2
d. 1

Solution

Correct option: (a)

$$\Delta = \begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^{2n} & 1 & \omega^n \\ \omega^n & \omega^{2n} & 1 \end{vmatrix}$$

$C_1 \to C_1 + C_2 + C_3$

$$\Delta = \begin{vmatrix} 1+\omega^n+\omega^{2n} & \omega^n & \omega^{2n} \\ 1+\omega^n+\omega^{2n} & 1 & \omega^n \\ 1+\omega^n+\omega^{2n} & \omega^{2n} & 1 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} 0 & \omega^n & \omega^{2n} \\ 0 & 1 & \omega^n \\ 0 & \omega^{2n} & 1 \end{vmatrix} = 0$$

Q14

If $A_r = \begin{vmatrix} 1 & r & 2^r \\ 2 & n & n^2 \\ n & \frac{n(n+1)}{2} & 2^{n+1} \end{vmatrix}$, then the value of $\sum_{r=1}^{n} A_r$ is

a. n
b. 2n
c. -2n
d. n^2

Solution

Correct option: (c)

$$\text{If } A_r = \begin{vmatrix} 1 & r & 2^r \\ 2 & n & n^2 \\ n & \frac{n(n+1)}{2} & 2^{n+1} \end{vmatrix}$$

$$\sum_{r=1}^{n} A_r = \begin{vmatrix} n & \frac{n(n+1)}{2} & 2^{n+1}-2 \\ 2 & n & n^2 \\ n & \frac{n(n+1)}{2} & 2^{n+1} \end{vmatrix}$$

$R_1 \to R_1 - R_3$

$$\sum_{r=1}^{n} A_r = \begin{vmatrix} 0 & 0 & -2 \\ 2 & n & n^2 \\ n & \frac{n(n+1)}{2} & 2^{n+1} \end{vmatrix}$$

$$\sum_{r=1}^{n} A_r = -2\left[n(n+1) - n^2\right]$$

$$\sum_{r=1}^{n} A_r = -2\left[n^2 + n - n^2\right]$$

$$\sum_{r=1}^{n} A_r = -2n$$

Q15

If $a > 0$ and discriminant of $ax^2 + 2bx + c$ is negative,

$$\text{then } \Delta = \begin{vmatrix} a & b & ax+b \\ b & c & bx+c \\ ax+b & bx+c & 0 \end{vmatrix} \text{ is}$$

a. positive
b. $(ac - b^2)(ax^2 + 2bx + c)$
c. Negative
d. 0

Solution

Correct option: (c)

$\Rightarrow a > 0$ and discriminant of $ax^2 + 2bx + c$ is negative,

$(2b)^2 - 4ac < 0$

$4b^2 - 4ac < 0$

$b^2 - ac < 0$ But $a > 0$ ....(i)

$$\Delta = \begin{vmatrix} a & b & ax+b \\ b & c & bx+c \\ ax+b & bx+c & 0 \end{vmatrix}$$

$R_1 \to xR_1$

$$\Delta = \begin{vmatrix} xa & xb & ax^2+bx \\ b & c & bx+c \\ ax+b & bx+c & 0 \end{vmatrix}$$

$R_1 \to R_1 + R_2$

$$\Delta = \begin{vmatrix} xa+b & xb+c & ax^2+bx \\ b & c & bx+c \\ ax+b & bx+c & 0 \end{vmatrix}$$

$R_1 \to R_1 - R_3$

$$\Delta = \begin{vmatrix} 0 & 0 & ax^2+bx \\ b & c & bx+c \\ ax+b & bx+c & 0 \end{vmatrix}$$

$\Delta = (ax^2+bx)[b(bx+c) - c(ax+b)]$

$\Delta = (ax^2+bx)(b^2x + bc - acx - bc)$

$\Delta = (ax^2+bx)(b^2x - acx)$

$\Delta = (ax^2+bx) \times (b^2 - ac)$

As $(b^2 - ac) < 0 \Rightarrow \Delta = (ax^2+bx) \times (b^2 - ac) < 0$

Q16

The value $\begin{vmatrix} 5^2 & 5^3 & 5^4 \\ 5^3 & 5^4 & 5^5 \\ 5^4 & 5^5 & 5^6 \end{vmatrix}$ is

a. 5^2
b. 0
c. 5^{13}
d. 5^9

Solution

Correct option: (b)

$$\begin{vmatrix} 5^2 & 5^3 & 5^4 \\ 5^3 & 5^4 & 5^5 \\ 5^4 & 5^5 & 5^6 \end{vmatrix}$$

Taking 5^2 and 5^3 common from R_1 and R_2 respectively.

$$5^2 \times 5^3 \begin{vmatrix} 1 & 5 & 5^2 \\ 1 & 5 & 5^2 \\ 5^4 & 5^5 & 5^6 \end{vmatrix} = 0$$

As R_1 and R_2 are same.

Q17

$$\begin{vmatrix} \log_3 512 & \log_4 3 \\ \log_3 8 & \log_4 9 \end{vmatrix} \times \begin{vmatrix} \log_2 3 & \log_8 3 \\ \log_3 4 & \log_3 4 \end{vmatrix} =$$

a. 7
b. 10
c. 13
d. 17

Solution

Correct option: (b)

$$\begin{vmatrix} \log_3 512 & \log_4 3 \\ \log_3 8 & \log_4 9 \end{vmatrix} \times \begin{vmatrix} \log_2 3 & \log_8 3 \\ \log_3 4 & \log_3 4 \end{vmatrix}$$

$$= \begin{vmatrix} \log_3 2^9 & \log_4 3 \\ \log_3 2^3 & \log_4 3^2 \end{vmatrix} \times \begin{vmatrix} \log_2 3 & \log_8 3 \\ \log_3 2^2 & \log_3 2^2 \end{vmatrix}$$

$$= \begin{vmatrix} 9\log_3 2 & \log_4 3 \\ 3\log_3 2 & 2\log_4 3 \end{vmatrix} \times \begin{vmatrix} \log_2 3 & \log_8 3 \\ 2\log_3 2 & 2\log_3 2 \end{vmatrix}$$

$$= \left[(9\log_3 2 \times 2\log_4 3) - (3\log_3 2 \times \log_4 3)\right]$$

$$\times\left[\log_2 3 \times 2\log_3 2 - \log_8 3 \times 2\log_3 2\right]$$

$$= \left[\left(9 \times \frac{\log 2}{\log 3} \times 2 \times \frac{\log 3}{\log 4}\right) - 3 \times \frac{\log 2}{\log 3} \times \frac{\log 3}{\log 4}\right]$$

$$\times\left[\frac{\log 3}{\log 2} \times 2 \times \frac{\log 2}{\log 3} - \frac{\log 3}{\log 8} \times 2 \times \frac{\log 2}{\log 3}\right]$$

$$= \left(18 \times \frac{\log 2}{2\log 2} - 3 \times \frac{\log 2}{2\log 2}\right)\left(2 - 2 \times \frac{1}{3\log 2} \times \log 2\right)$$

$$= \left(9 - \frac{3}{2}\right)\left(2 - \frac{2}{3}\right)$$

$$= 10$$

Q18

If a, b, c are in A.P., then the determinant $\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$

a. 0
b. 1
c. x
d. 2x

Solution

Correct option:(a)

$$\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$

$R_1 + R_3 - R_2$

$$\begin{vmatrix} x+3 & x+4 & x+2(a+c-b) \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$

$R_1 - R_2$

$$\begin{vmatrix} 0 & 0 & 2(a+c-2b) \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$

Given that a, b, c are in A.P.

Hence, $2b = a + c$

$\Rightarrow a + c - 2b = 0$

$$\Rightarrow \begin{vmatrix} 0 & 0 & 0 \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix} = 0$$

Exercise 6VSAQ

Q1

If A and B are square matrices of order 2, then det (A + B) = 0 is possible only when

a. det (A) = 0 or det (B) = 0
b. det (A) + det (B) = 0
c. det (A) = 0 and det (B) = 0
d. A + B = O

Solution

Correct option: (d)

Determinanat A denoted as $[a_{ij}]$ and determinanat B as $[b_{ij}]$

$\Rightarrow A + B = [a_{ij}] + [b_{ij}]$

$\Rightarrow A + B = [a_{ij} + b_{ij}]$

$\Rightarrow \det(A + B) = \det[a_{ij} + b_{ij}]$

$\Rightarrow \det(A + B) = 0$

$\Rightarrow \det[a_{ij} + b_{ij}] = 0$

$\Rightarrow a_{ij} + b_{ij} = 0$

$\Rightarrow A + B = 0$

Q2

Which of the following is not correct?

a. $|A| = |A^T|$, where $A = [a_{ij}]_{3\times3}$
b. $|kA| = k^3 |A|$, where $A = [a_{ij}]_{3\times3}$
c. If A is a skew-symmetric matrix of odd order, then |A| = 0
d. $\begin{vmatrix} a+b & c+d \\ e+f & g+h \end{vmatrix} = \begin{vmatrix} a & c \\ e & g \end{vmatrix} + \begin{vmatrix} b & d \\ f & h \end{vmatrix}$

Solution

Correct option: (d)

$$\begin{vmatrix} a+b & c+d \\ e+f & g+h \end{vmatrix}$$

$$= \begin{vmatrix} a+b & c \\ e+f & g \end{vmatrix} + \begin{vmatrix} a+b & d \\ e+f & h \end{vmatrix}$$

$$= \begin{vmatrix} a & c \\ e & g \end{vmatrix} + \begin{vmatrix} b & c \\ f & g \end{vmatrix} + \begin{vmatrix} a & d \\ e & h \end{vmatrix} + \begin{vmatrix} b & d \\ f & h \end{vmatrix}$$

$$\begin{vmatrix} a+b & c+d \\ e+f & g+h \end{vmatrix} \neq \begin{vmatrix} a & c \\ e & g \end{vmatrix} + \begin{vmatrix} b & d \\ f & h \end{vmatrix}$$

Q3

If A = $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ and C_{ij} is cofactor of a_{ij} in A, then value of $|A|$ is given by

a. $a_{11}C_{31}+a_{12}C_{32}+a_{13}C_{33}$
b. $a_{11}C_{11}+a_{12}C_{21}+a_{13}C_{31}$
c. $a_{21}C_{11}+a_{22}C_{12}+a_{23}C_{13}$
d. $a_{11}C_{11}+a_{21}C_{21}+a_{31}C_{31}$

Solution

Correct option: (d)
If A is a square matrix of order n then det(A) = $a_{11}C_{11}+a_{21}C_{21}+a_{31}C_{31}$

Q4

Which of the following is not correct in a given determinant of A, where A = $[a_{ij}]_{3\times3}$.

a. Order of minor is less than order of the det (A).
b. Minor of an element can never be equal to cofactor of the same element
c. Value of a determinant is obtained by multiplying elements of a row or column by corresponding cofactors
d. Order of minors and cofactors of elements of A is same

Solution

Correct option: (b)
Minor of an element can never be equal to cofactor of the same element.
$C_{ij}=(-1)^{i+j}M_{ij}$

Q5

Let $\begin{vmatrix} x & 2 & x \\ x^2 & x & 6 \\ x & x & 6 \end{vmatrix} = ax^4 + bx^3 + cx^2 + dx + e$. Then, the value of $5a + 4b + 3c + 2d + e$ is equal

a. 0
b. -16
c. 16
d. 1
e. None of these

Solution

Correct option: (e)

$$\begin{vmatrix} x & 2 & x \\ x^2 & x & 6 \\ x & x & 6 \end{vmatrix}$$

$= x(6x-6x) - 2(6x^2-6x) + x(x^3-x^2)$

$= 0 - 12x^2 + 12x + x^4 - x^3$

$= x^4 - x^3 - 12x^2 + 12x$

Comparing with RHS $ax^4 + bx^3 + cx^2 + dx + e$,

$a = 1,\ b = -1,\ c = -12,\ d = 12,\ e = 0$

$\Rightarrow 5a + 4b + 3c + 2d + e = 5 - 4 - 36 + 24 = -11$

Q6

The value of the determinant $\begin{vmatrix} a^2 & a & 1 \\ \cos nx & \cos(n+1)x & \cos(n+2)x \\ \sin nx & \sin(n+1)x & \sin(n+2)x \end{vmatrix}$ is independent of

a. n
b. a
c. x
d. none of these

Solution

Correct option: (a)

$$\begin{vmatrix} a^2 & a & 1 \\ \cos nx & \cos(n+1)x & \cos(n+2)x \\ \sin nx & \sin(n+1)x & \sin(n+2)x \end{vmatrix}$$

Let, $nx = u,\ (n+1)x = v,\ (n+2)x = w$

$$\Rightarrow \begin{vmatrix} a^2 & a & 1 \\ \cos u & \cos v & \cos w \\ \sin u & \sin v & \sin w \end{vmatrix}$$

$\Rightarrow a^2 \sin(w-v) - a\sin(w-u) + \sin(v-u)$

$\Rightarrow a^2 \sin x - a\sin 2x + \sin x$

$\Rightarrow$ It is independent of n.

Q7

If $\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$, $\Delta_2 = \begin{vmatrix} 1 & bc & a \\ 1 & ca & b \\ 1 & ab & c \end{vmatrix}$, then

a. $\Delta_1 + \Delta_2 = 0$
b. $\Delta_1 + 2\Delta_2 = 0$
c. $\Delta_1 = \Delta_2$
d. none of these

Solution

Correct option: (a)

$$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = bc^2 - b^2c - (ac^2 - a^2c) + ab^2 - a^2b$$

$$= bc^2 - b^2c - ac^2 + a^2c + ab^2 - a^2b$$

$$\Delta_2 = \begin{vmatrix} 1 & bc & a \\ 1 & ca & b \\ 1 & ab & c \end{vmatrix} = c^2a - ab^2 - bc(c-b) + a(ab - ac)$$

$$= c^2a - ab^2 - bc^2 + b^2c + a^2b - a^2c$$

$$= -(bc^2 - b^2c - ac^2 + a^2c + ab^2 - a^2b)$$

$\Rightarrow \Delta_1 = -\Delta_2$

$\Rightarrow \Delta_1 + \Delta_2 = 0$

Q8

If $D_k = \begin{vmatrix} 1 & n & n \\ 2k & n^2+n+2 & n^2+n \\ 2k-1 & n^2 & n^2+n+2 \end{vmatrix}$ and $\sum_{k=1}^{n} D_k = 48$, then n equals

a. 4
b. 6
c. 8
d. none of these

Solution

Correct option: (a)

$$D_k = \begin{vmatrix} 1 & n & n \\ 2k & n^2+n+2 & n^2+n \\ 2k-1 & n^2 & n^2+n+2 \end{vmatrix}$$

Applying row transformation $R_2 \to R_2 - R_3$ we get

$$D_k = \begin{vmatrix} 1 & n & n \\ 1 & n+2 & -2 \\ 2k-1 & n^2 & n^2+n+2 \end{vmatrix}$$

Applying row transformation $R_1 \to R_1 - R_2$ we get

$$D_k = \begin{vmatrix} 0 & -2 & n+2 \\ 1 & n+2 & -2 \\ 2k-1 & n^2 & n^2+n+2 \end{vmatrix}$$

$$= 2(n^2+n+2+4k-2) + (n+2)(n^2-(n+2)(2k-1))$$

$$= 2(n^2+n+4k) + (n+2)(n^2-2nk+n-4k+2)$$

$$= 2n^2+2n+8k+n^3-2n^2k+n^2-4nk+2n+2n^2-4nk+2n-8k+4$$

$$= n^3+5n^2-2n^2k+6n-8nk+4$$

$$\sum_{k=1}^{n} D_k = 48$$

$$\Rightarrow n(n^3+5n^2+6n+4) - 2n^2\sum_{k=1}^{n} k - 8n\sum_{k=1}^{n} k = 48$$

$$\Rightarrow n(n^3+5n^2+6n+4) - 2n^2\frac{n(n+1)}{2} - 8n\frac{n(n+1)}{2} = 48$$

$$\Rightarrow n^4+5n^3+6n^2+4n-n^4-n^3-4n^3-4n^2 = 48$$

$$\Rightarrow 2n^2+4n = 48$$

$$\Rightarrow n^2+2n-24 = 0$$

$$\Rightarrow (n+6)(n-4) = 0$$

$$\Rightarrow n = -6, 4$$

Q9

$$\text{Let} \begin{vmatrix} x^2+3x & x-1 & x+3 \\ x+1 & -2x & x-4 \\ x-3 & x+4 & 3x \end{vmatrix} = ax^4+bx^3+cx^2+dx+e$$

be an identity in x, where a, b, c, d, e, are independent of x. then the value of e is

a. 4
b. 0
c. 1
d. none of these

Solution

Correct option: (a)

$$\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix}$$

Applying row transformation $C_1 \to C_1 + C_2$ we get

$$\begin{vmatrix} b-a & a+b & a+c \\ a-b & -2b & b+c \\ 2c+b+a & c+b & -2c \end{vmatrix}$$

Applying column transformation $C_2 \to C_2 + C_3$ we get

$$\begin{vmatrix} b-a & 2a+b+c & a+c \\ a-b & c-b & b+c \\ 2c+b+a & b-c & -2c \end{vmatrix}$$

Applying row transformation $R_3 \to R_2 + R_3$ we get

$$\begin{vmatrix} b-a & 2a+b+c & a+c \\ a-b & c-b & b+c \\ 2(c+a) & 0 & b-c \end{vmatrix}$$

Applying row transformation $R_2 \to R_2 + R_1$ we get

$$\begin{vmatrix} b-a & 2a+b+c & a+c \\ 0 & 2(a+c) & a+b+2c \\ 2(c+a) & 0 & b-c \end{vmatrix}$$

Expanding along C_1 we get

$$= (b-a)(2(a+c)(b-c)) + 2(a+c)\left((2a+b+c)(a+b+2c) - 2(a+c)^2\right)$$

$$= 2(a+c)\left[(b-a)(b-c) + (2a+b+c)(a+b+2c) - 2(a+c)^2\right]$$

$$= 2(a+c)\left[b^2 - bc - ab + ac + 2a^2 + 2ab + 4ac + ab + b^2 + 2bc + ac + bc + 2c^2 - 2a^2 - 2c^2 - 4ac\right]$$

$$= 2(a+c)\left[2b^2 + 2ab + 2bc + 2ac\right]$$

$$= 4(a+c)\left[b^2 + ab + bc + ac\right]$$

$$= 4(a+c)\left[b(a+b) + c(a+b)\right]$$

$$= 4(a+c)(b+c)(a+b)$$

So another factor is 4

Q10

Using the factor theorem it is found that a + b, b + c and c + a are three factors of the determinant $\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix}$. The other factor in the value of the determinant is

a. 4
b. 2
c. a + b + c
d. none of these

Solution

Correct option: (a)

$$\begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix}$$

Applying row transformation $C_1 \to C_1 + C_2$ we get

$$\begin{vmatrix} b-a & a+c & a+c \\ a-b & -2b & b+c \\ 2c+b+a & c+b & -2c \end{vmatrix}$$

Applying column transformation $C_2 \to C_2 + C_3$ we get

$$\begin{vmatrix} b-a & 2a+b+c & a+c \\ a-b & c-b & b+c \\ 2c+b+a & b-c & -2c \end{vmatrix}$$

Applying row transformation $R_3 \to R_2 + R_3$ we get

$$\begin{vmatrix} b-a & 2a+b+c & a+c \\ a-b & c-b & b+c \\ 2(c+a) & 0 & b-c \end{vmatrix}$$

Applying row transformation $R_2 \to R_2 + R_1$ we get

$$\begin{vmatrix} b-a & 2a+b+c & a+c \\ 0 & 2(a+c) & a+b+2c \\ 2(c+a) & 0 & b-c \end{vmatrix}$$

Expanding along C_1 we get

$$= (b-a)(2(a+c)(b-c)) + 2(a+c)\left((2a+b+c)(a+b+2c) - 2(a+c)^2\right)$$

$$= 2(a+c)\left[(b-a)(b-c) + (2a+b+c)(a+b+2c) - 2(a+c)^2\right]$$

$$= 2(a+c)\left[b^2 - bc - ab + ac + 2a^2 + 2ab + 4ac + ab + b^2 + 2bc + ac + bc + 2c^2 - 2a^2 - 2c^2 - 4ac\right]$$

$$= 2(a+c)\left[2b^2 + 2ab + 2bc + 2ac\right]$$

$$= 4(a+c)\left[b^2 + ab + bc + ac\right]$$

$$= 4(a+c)\left[b(a+b) + c(a+b)\right]$$

$$= 4(a+c)(b+c)(a+b)$$

So another factor is 4

Q11

If a, b, c are distinct, then the value of x satisfying

$$\begin{vmatrix} 0 & x^2-a & x^3-b \\ x^2+a & 0 & x^2+c \\ x^4+b & x-c & 0 \end{vmatrix}$$

a. c
b. a
c. b
d. 0

Solution

Correct option: (d)

$$\begin{vmatrix} 0 & x^2-a & x^3-b \\ x^2+a & 0 & x^2+c \\ x^4+b & x-c & 0 \end{vmatrix}$$

If we put $x = 0$ in the above determinant,

$$\Rightarrow \begin{vmatrix} 0 & -a & -b \\ a & 0 & c \\ b & -c & 0 \end{vmatrix} = A$$

$$A^T = \begin{vmatrix} 0 & a & b \\ -a & 0 & -c \\ -b & c & 0 \end{vmatrix} = -\begin{vmatrix} 0 & -a & -b \\ a & 0 & c \\ b & -c & 0 \end{vmatrix} = -A$$

Matrix is skew-symmetric.

Also, Power of the matrix is odd.

Hence, value of x is 0.

Q12

If the determinant $\begin{vmatrix} a & b & 2a\alpha+3b \\ b & c & 2b\alpha+3c \\ 2a\alpha+3b & 2b\alpha+3c & 0 \end{vmatrix} = 0$, then

a. A, b, c are in H.P.
b. α is a root of $4ax^2 + 12bx + 9c = 0$ or, a, b, c are in G.P.
c. a, b, c are in G.P. only
d. a, b, c are in A.P.

Solution

Correct option: (b)

$$\begin{vmatrix} a & b & 2a\alpha+3b \\ b & c & 2b\alpha+3c \\ 2a\alpha+3b & 2b\alpha+3c & 0 \end{vmatrix} = 0$$

$C_1 \to C_1 - C_2$

$$\begin{vmatrix} a-b & b & 2a\alpha+3b \\ b-c & c & 2b\alpha+3c \\ 2\alpha(a-b)+3b-3c & 2b\alpha+3c & 0 \end{vmatrix} = 0$$

$R_3 \to R_3 - \alpha R_1$

$$\begin{vmatrix} \alpha(a-b) & \alpha b+3c & 2a\alpha+3b \\ b-c & c & 2b\alpha+3c \\ 2\alpha(a-b)+3b-3c & 2b\alpha+3c & 0 \end{vmatrix} = 0$$

$C_1 \to C_1 + C_2$

$$\begin{vmatrix} \alpha a-3c & \alpha b+3c & 2a\alpha+3b \\ b & c & 2b\alpha+3c \\ 2\alpha a+3b & 2b\alpha+3c & 0 \end{vmatrix} = 0$$

$$-\left(4a\alpha^2+12b\alpha+9c\right)\left(ac-b^2\right) = 0$$

$$4a\alpha^2+12b\alpha+9c = 0 \text{ or } ac-b^2 = 0$$

$$ac-b^2 = 0 \Rightarrow ac = b^2$$

$\Rightarrow a, b, c$ are in GP.

Q13

If ω is a non-real cube root of unity and n is not a multiple of 3, then $\Delta = \begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^{2n} & 1 & \omega^n \\ \omega^n & \omega^{2n} & 1 \end{vmatrix}$ is equal to

a. 0
b. ω
c. ω^2
d. 1

Solution

Correct option: (a)

$$\Delta = \begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^{2n} & 1 & \omega^n \\ \omega^n & \omega^{2n} & 1 \end{vmatrix}$$

$$C_1 \to C_1 + C_2 + C_3$$

$$\Delta = \begin{vmatrix} 1+\omega^n+\omega^{2n} & \omega^n & \omega^{2n} \\ 1+\omega^n+\omega^{2n} & 1 & \omega^n \\ 1+\omega^n+\omega^{2n} & \omega^{2n} & 1 \end{vmatrix}$$

$$\Delta = \begin{vmatrix} 0 & \omega^n & \omega^{2n} \\ 0 & 1 & \omega^n \\ 0 & \omega^{2n} & 1 \end{vmatrix} = 0$$

Q14

If $A_r = \begin{vmatrix} 1 & r & 2^r \\ 2 & n & n^2 \\ n & \frac{n(n+1)}{2} & 2^{n+1} \end{vmatrix}$, then the value of $\sum_{r=1}^{n} A_r$ is

a. n
b. 2n
c. -2n
d. n^2

Solution

Correct option: (c)

$$\text{If } A_r = \begin{vmatrix} 1 & r & 2^r \\ 2 & n & n^2 \\ n & \frac{n(n+1)}{2} & 2^{n+1} \end{vmatrix}$$

$$\sum_{r=1}^{n} A_r = \begin{vmatrix} n & \frac{n(n+1)}{2} & 2^{n+1}-2 \\ 2 & n & n^2 \\ n & \frac{n(n+1)}{2} & 2^{n+1} \end{vmatrix}$$

$R_1 \to R_1 - R_3$

$$\sum_{r=1}^{n} A_r = \begin{vmatrix} 0 & 0 & -2 \\ 2 & n & n^2 \\ n & \frac{n(n+1)}{2} & 2^{n+1} \end{vmatrix}$$

$$\sum_{r=1}^{n} A_r = -2\left[n(n+1) - n^2\right]$$

$$\sum_{r=1}^{n} A_r = -2\left[n^2 + n - n^2\right]$$

$$\sum_{r=1}^{n} A_r = -2n$$

Q15

If $a > 0$ and discriminant of $ax^2 + 2bx + c$ is negative,

$$\text{then } \Delta = \begin{vmatrix} a & b & ax+b \\ b & c & bx+c \\ ax+b & bx+c & 0 \end{vmatrix} \text{ is}$$

a. positive
b. $(ac - b^2)(ax^2 + 2bx + c)$
c. Negative
d. 0

Solution

Correct option: (c)

$\Rightarrow a > 0$ and discriminant of $ax^2 + 2bx + c$ is negative,

$(2b)^2 - 4ac < 0$

$4b^2 - 4ac < 0$

$b^2 - ac < 0$ But $a > 0$ ....(i)

$$\Delta = \begin{vmatrix} a & b & ax+b \\ b & c & bx+c \\ ax+b & bx+c & 0 \end{vmatrix}$$

$R_1 \to xR_1$

$$\Delta = \begin{vmatrix} xa & xb & ax^2+bx \\ b & c & bx+c \\ ax+b & bx+c & 0 \end{vmatrix}$$

$R_1 \to R_1 + R_2$

$$\Delta = \begin{vmatrix} xa+b & xb+c & ax^2+bx \\ b & c & bx+c \\ ax+b & bx+c & 0 \end{vmatrix}$$

$R_1 \to R_1 - R_3$

$$\Delta = \begin{vmatrix} 0 & 0 & ax^2+bx \\ b & c & bx+c \\ ax+b & bx+c & 0 \end{vmatrix}$$

$\Delta = (ax^2 + bx)[b(bx+c) - c(ax+b)]$

$\Delta = (ax^2 + bx)(b^2x + bc - acx - bc)$

$\Delta = (ax^2 + bx)(b^2x - acx)$

$\Delta = (ax^2 + bx) \times (b^2 - ac)$

As $(b^2 - ac) < 0 \Rightarrow \Delta = (ax^2 + bx) \times (b^2 - ac) < 0$

Q16

The value $\begin{vmatrix} 5^2 & 5^3 & 5^4 \\ 5^3 & 5^4 & 5^5 \\ 5^4 & 5^5 & 5^6 \end{vmatrix}$ is

a. 5^2
b. 0
c. 5^{13}
d. 5^9

Solution

Correct option: (b)

$$\begin{vmatrix} 5^2 & 5^3 & 5^4 \\ 5^3 & 5^4 & 5^5 \\ 5^4 & 5^5 & 5^6 \end{vmatrix}$$

Taking 5^2 and 5^3 common from R_1 and R_2 respectively.

$$5^2 \times 5^3 \begin{vmatrix} 1 & 5 & 5^2 \\ 1 & 5 & 5^2 \\ 5^4 & 5^5 & 5^6 \end{vmatrix} = 0$$

As R_1 and R_2 are same.

Q17

$$\begin{vmatrix} \log_3 512 & \log_4 3 \\ \log_3 8 & \log_4 9 \end{vmatrix} \times \begin{vmatrix} \log_2 3 & \log_8 3 \\ \log_3 4 & \log_3 4 \end{vmatrix} =$$

a. 7
b. 10
c. 13
d. 17

Solution

Correct option: (b)

$$\begin{vmatrix} \log_3 512 & \log_4 3 \\ \log_3 8 & \log_4 9 \end{vmatrix} \times \begin{vmatrix} \log_2 3 & \log_8 3 \\ \log_3 4 & \log_3 4 \end{vmatrix}$$

$$= \begin{vmatrix} \log_3 2^9 & \log_4 3 \\ \log_3 2^3 & \log_4 3^2 \end{vmatrix} \times \begin{vmatrix} \log_2 3 & \log_8 3 \\ \log_3 2^2 & \log_3 2^2 \end{vmatrix}$$

$$= \begin{vmatrix} 9\log_3 2 & \log_4 3 \\ 3\log_3 2 & 2\log_4 3 \end{vmatrix} \times \begin{vmatrix} \log_2 3 & \log_8 3 \\ 2\log_3 2 & 2\log_3 2 \end{vmatrix}$$

$$= \left[(9\log_3 2 \times 2\log_4 3) - (3\log_3 2 \times \log_4 3)\right]$$
$$\times \left[\log_2 3 \times 2\log_3 2 - \log_8 3 \times 2\log_3 2\right]$$

$$= \left[\left(9 \times \frac{\log 2}{\log 3} \times 2 \times \frac{\log 3}{\log 4}\right) - 3 \times \frac{\log 2}{\log 3} \times \frac{\log 3}{\log 4}\right]$$
$$\times \left[\frac{\log 3}{\log 2} \times 2 \times \frac{\log 2}{\log 3} - \frac{\log 3}{\log 8} \times 2 \times \frac{\log 2}{\log 3}\right]$$

$$= \left(18 \times \frac{\log 2}{2\log 2} - 3 \times \frac{\log 2}{2\log 2}\right)\left(2 - 2 \times \frac{1}{3\log 2} \times \log 2\right)$$

$$= \left(9 - \frac{3}{2}\right)\left(2 - \frac{2}{3}\right)$$

$$= 10$$

Q18

If a, b, c are in A.P., then the determinant $\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$

a. 0
b. 1
c. x
d. 2x

Solution

Correct option:(a)

$$\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$

$R_1 + R_3 - R_2$

$$\begin{vmatrix} x+3 & x+4 & x+2(a+c-b) \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$

$R_1 - R_2$

$$\begin{vmatrix} 0 & 0 & 2(a+c-2b) \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$

Given that a, b, c are in A.P.

Hence, $2b = a + c$

$\Rightarrow a + c - 2b = 0$

$$\Rightarrow \begin{vmatrix} 0 & 0 & 0 \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix} = 0$$

Q19

If $A + B + C = \pi$, then the value of

$\begin{vmatrix} \sin(A+B+C) & \sin(A+C) & \cos C \\ -\sin B & 0 & \tan A \\ \cos(A+B) & \tan(B+C) & 0 \end{vmatrix}$ is equal to

a. 0
b. 1
c. 2 sin B tan A cos C
d. none of these

Solution

Correct option: (a)

$A+B+C=\pi$

$\Rightarrow \sin(A+C)=\sin B,\ \cos(A+B)=-\cos C,\ \tan(B+C)=-\tan C$

$$\begin{vmatrix} \sin(A+B+C) & \sin(A+C) & \cos C \\ -\sin B & 0 & \tan A \\ \cos(A+B) & \tan(B+C) & 0 \end{vmatrix}$$

$$=\begin{vmatrix} \sin\pi & \sin B & \cos C \\ -\sin B & 0 & \tan A \\ -\cos C & -\tan C & 0 \end{vmatrix}$$

$$=\begin{vmatrix} 0 & \sin B & \cos C \\ -\sin B & 0 & \tan A \\ -\cos C & -\tan C & 0 \end{vmatrix}$$

Deter minanat is skew − symmetric

$$\text{Hence, } \begin{vmatrix} 0 & \sin B & \cos C \\ -\sin B & 0 & \tan A \\ -\cos C & -\tan C & 0 \end{vmatrix}=0$$

Q20

The number of distinct real roots of $\begin{vmatrix} \operatorname{cosec} x & \sec x & \sec x \\ \sec x & \operatorname{cosec} x & \sec x \\ \sec x & \sec x & \operatorname{cosec} x \end{vmatrix}=0$ lies in the interval $-\frac{\pi}{4}\le x\le\frac{\pi}{4}$ is

a. 1
b. 2
c. 3
d. 0

Solution

Correct option: (b)

$$\begin{vmatrix} \operatorname{cosec} x & \sec x & \sec x \\ \sec x & \operatorname{cosec} x & \sec x \\ \sec x & \sec x & \operatorname{cosec} x \end{vmatrix} = 0$$

$C_1 \to C_1 - C_3,\ C_2 \to C_2 - C_3$

$$\begin{vmatrix} \operatorname{cosec} x - \sec x & 0 & \sec x \\ 0 & \operatorname{cosec} x - \sec x & \sec x \\ \sec x - \operatorname{cosec} x & \sec x - \operatorname{cosec} x & \operatorname{cosec} x \end{vmatrix} = 0$$

$$(\operatorname{cosec} x - \sec x)^2 \begin{vmatrix} 1 & 0 & \sec x \\ 0 & 1 & \sec x \\ -1 & -1 & \operatorname{cosec} x \end{vmatrix} = 0$$

$$(\operatorname{cosec} x - \sec x)^2 (\operatorname{cosec} x + \sec x + \sec x) = 0$$

$$(\operatorname{cosec} x - \sec x)^2 (\operatorname{cosec} x + 2\sec x) = 0$$

$$(\operatorname{cosec} x - \sec x)^2 = 0 \text{ or } \operatorname{cosec} x + 2\sec x = 0$$

$$\operatorname{cosec} x - \sec x = 0 \text{ or } \operatorname{cosec} x = -2\sec x$$

$$\sin x - \cos x = 0 \text{ or } \sin x = \frac{\cos x}{-2}$$

$$\tan x = 1 \text{ or } \tan x = \frac{-1}{2}$$

There are 2 solutions.

Q21

Let $A = \begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$, where $0 \le \theta \le 2\pi$. Then,

a. Det (A) = 0
b. Det (A) $\in (2, \infty)$
c. Det (A) $\in (2, 4)$
d. Det (A) $\in [2, 4]$

Solution

Correct option: (d)

$$A = \begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$$

$$|A| = 1 + \sin^2\theta - \sin\theta(-\sin\theta + \sin\theta) + \sin^2\theta + 1$$

$$|A| = 2 + 2\sin^2\theta$$

$$|A| = 2(1 + \sin^2\theta)$$

Given that $0 \le \theta \le 2\pi$

for $\theta = 0$

$\Rightarrow |A| = 2$

for $\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots\ldots$

$\Rightarrow |A| = 2(1 + 1) = 4$

Answer is $[2, 4]$