

Exercise 13.1

Question 1: Evaluate the Given limit: $\lim_{x \rightarrow 3} x + 3$

Solution 1: $\lim_{x \rightarrow 3} x + 3 = 3 + 3 = 6$

Question 2: Evaluate the Given limit: $\lim_{x \rightarrow \pi} \left(x - \frac{22}{7} \right)$

Solution 2: $\lim_{x \rightarrow \pi} \left(x - \frac{22}{7} \right) = \left(\pi - \frac{22}{7} \right)$

Question 3: Evaluate the Given limit: $\lim_{r \rightarrow 1} \pi r^2$

Solution 3: $\lim_{r \rightarrow 1} \pi r^2 = \pi(1^2) = \pi$

Question 4: Evaluate the Given limit: $\lim_{x \rightarrow 1} \frac{4x+3}{x-2}$

Solution 4: $\lim_{x \rightarrow 1} \frac{4x+3}{x-2} = \frac{4(1)+3}{1-2} = \frac{7}{-1} = -7$

Question 5: Evaluate the Given limit: $\lim_{x \rightarrow -1} \frac{x^{10} + x^5 + 1}{x - 1}$

Solution 5: $\lim_{x \rightarrow -1} \frac{x^{10} + x^5 + 1}{x - 1} = \frac{(-1)^{10} + (-1)^5 + 1}{-1 - 1} = \frac{1 - 1 + 1}{-2} = -\frac{1}{2}$

Question 6: Evaluate the Given limit: $\lim_{x \rightarrow 0} \frac{(x+1)^5 - 1}{x}$

Solution 6: $\lim_{x \rightarrow 0} \frac{(x+1)^5 - 1}{x}$

Put $x + 1 = y$ so that $y \rightarrow 1$ as $x \rightarrow 0$.

$$\text{Accordingly, } \lim_{x \rightarrow 0} \frac{(x+1)^5 - 1}{x} = \lim_{x \rightarrow 1} \frac{(y)^5 - 1}{y - 1}$$

$$= 5 \cdot 1^{5-1} \quad \left[\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1} \right]$$

$$= 5$$

$$\therefore \lim_{x \rightarrow 0} \frac{(x+1)^5 - 1}{x} = 5$$

Question 7: Evaluate the Given limit: $\lim_{x \rightarrow 2} \frac{3x^2 - x - 10}{x^2 - 4}$

Solution 7: At $x = 2$, the value of the given rational function takes the form $\frac{0}{0}$

$$\therefore \lim_{x \rightarrow 2} \frac{3x^2 - x - 10}{x^2 - 4} = \lim_{x \rightarrow 2} \frac{(x-2)(3x+5)}{(x-2)(x+2)}$$

$$= \lim_{x \rightarrow 2} \frac{3x+5}{x+2}$$

$$= \frac{3(2)+5}{2+2}$$

$$= \frac{11}{4}$$

Question 8: Evaluate the Given limit: $\lim_{x \rightarrow 3} \frac{x^4 - 81}{2x^2 - 5x - 3}$

Solution 8: At $x = 3$, the value of the given rational function takes the form $\frac{0}{0}$

$$\therefore \lim_{x \rightarrow 3} \frac{x^4 - 81}{2x^2 - 5x - 3} = \lim_{x \rightarrow 3} \frac{(x-3)(x+3)(x^2+9)}{(x-3)(2x+1)}$$

$$= \lim_{x \rightarrow 3} \frac{(x+3)(x^2+9)}{(2x+1)}$$

$$= \frac{(3+3)(3^2+9)}{2(3)+1}$$

$$= \frac{6 \times 18}{7}$$

$$= \frac{108}{7}$$

Question 9: Evaluate the Given limit: $\lim_{x \rightarrow 0} \frac{ax+b}{cx+1}$.

Solution 9:

$$\lim_{x \rightarrow 0} \frac{ax+b}{cx+1} = \frac{a(0)+b}{c(0)+1} = b$$

Question 10: Evaluate the Given limit: $\lim_{z \rightarrow 1} \frac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1}$

Solution 10: $\lim_{z \rightarrow 1} \frac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1}$

At $z = 1$, the value of the given function takes the form $\frac{0}{0}$

Put $z^{\frac{1}{6}} = x$ so that $z \rightarrow 1$ as $x \rightarrow 1$.

Accordingly, $\lim_{z \rightarrow 1} \frac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1} = \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$

$$= \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$$

$$= 2 \cdot 1^{2-1} \left[\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1} \right]$$

$$= 2$$

$$\lim_{z \rightarrow 1} \frac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1} = 2$$

Question 11: Evaluate the Given limit: $\lim_{x \rightarrow 1} \frac{ax^2 + bx + c}{cx^2 + bx + a}$, $a + b + c \neq 0$

Solution 11: $\lim_{x \rightarrow 1} \frac{ax^2 + bx + c}{cx^2 + bx + a} = \frac{a(1)^2 + b(1) + c}{c(1)^2 + b(1) + a}$

$$= \frac{a+b+c}{a+b+c}$$

$$= 1 \quad [a + b + c \neq 0]$$

Question 12: Evaluate the given limit: $\lim_{x \rightarrow -2} \frac{\frac{1}{x} + \frac{1}{2}}{x + 2}$

Solution 12: $\lim_{x \rightarrow -2} \frac{\frac{1}{x} + \frac{1}{2}}{x + 2}$

At $x = -2$, the value of the given function takes the form $\frac{0}{0}$

Now, $\lim_{x \rightarrow -2} \frac{\frac{1}{x} + \frac{1}{2}}{x + 2} = \lim_{x \rightarrow -2} \frac{\left(\frac{2+x}{2x}\right)}{x + 2}$

$$= \lim_{x \rightarrow -2} \frac{1}{2x}$$

$$= \frac{1}{2(-2)} = \frac{-1}{4}$$

Question 13: Evaluate the Given limit: $\lim_{x \rightarrow 0} \frac{\sin ax}{bx}$

Solution 13: $\lim_{x \rightarrow 0} \frac{\sin ax}{bx}$

At $x = 0$, the value of the given function takes the form $\frac{0}{0}$

Now, $\lim_{x \rightarrow 0} \frac{\sin ax}{bx} = \lim_{x \rightarrow 0} \frac{\sin ax}{ax} \times \frac{ax}{bx}$

$$\begin{aligned}
 &= \lim_{x \rightarrow 0} \left(\frac{\sin ax}{ax} \right) \times \frac{a}{b} \\
 &= \frac{a}{b} \lim_{ax \rightarrow 0} \left(\frac{\sin ax}{ax} \right) \quad [x \rightarrow 0 \Rightarrow ax \rightarrow 0] \\
 &= \frac{a}{b} \times 1 \quad \left[\lim_{x \rightarrow 0} \left(\frac{\sin y}{y} \right) \right] \\
 &= \frac{a}{b}
 \end{aligned}$$

Question 14: Evaluate the given limit: $\lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx}, a, b \neq 0$

Solution 14: $\lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx}, a, b \neq 0$

At $x = 0$, the value of the given function takes the form $\frac{0}{0}$

$$\begin{aligned}
 \text{Now, } \lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx} &= \lim_{x \rightarrow 0} \frac{\left(\frac{\sin ax}{ax} \right) \times ax}{\left(\frac{\sin bx}{bx} \right) \times bx} \\
 &= \frac{a}{b} \times \frac{\lim_{ax \rightarrow 0} \left(\frac{\sin ax}{ax} \right)}{\lim_{bx \rightarrow 0} \left(\frac{\sin bx}{bx} \right)} \quad \left[\begin{array}{l} x \rightarrow 0 \Rightarrow ax \rightarrow 0 \\ \text{and } x \rightarrow 0 \Rightarrow bx \rightarrow 0 \end{array} \right] \\
 &= \frac{a}{b} \times \frac{1}{1} \quad \left[\lim_{x \rightarrow 0} \left(\frac{\sin y}{y} \right) = 1 \right] \\
 &= \frac{a}{b}
 \end{aligned}$$

Question 15: Evaluate the Given limit: $\lim_{x \rightarrow \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)}$

Solution 15: $\lim_{x \rightarrow \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)}$

It is seen that $x \rightarrow \pi \Rightarrow (\pi - x) \rightarrow 0$

$$\begin{aligned}
 \therefore \lim_{x \rightarrow \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)} &= \frac{1}{\pi} \lim_{(\pi - x) \rightarrow 0} \frac{\sin(\pi - x)}{(\pi - x)} \\
 &= \frac{1}{\pi} \times 1 \quad \left[\lim_{y \rightarrow 0} \frac{\sin y}{y} = 1 \right] \\
 &= \frac{1}{\pi}
 \end{aligned}$$

Question 16: Evaluate the given limit: $\lim_{x \rightarrow 0} \frac{\cos x}{\pi - x}$

Solution 16: $\lim_{x \rightarrow 0} \frac{\cos x}{\pi - x} = \frac{\cos 0}{\pi - 0} = \frac{1}{\pi}$

Question 17: Evaluate the Given limit: $\lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\cos x - 1}$

Solution 17: $\lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\cos x - 1}$

At $x = 0$, the value of the given function takes the form $\frac{0}{0}$

Now, $\lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\cos x - 1} = \lim_{x \rightarrow 0} \frac{1 - 2\sin^2 x - 1}{1 - 2\sin^2 \frac{x}{2} - 1} \quad \left[\cos x = 1 - 2\sin^2 \frac{x}{2} \right]$

$$= \lim_{x \rightarrow 0} \frac{\sin^2 x}{\sin^2 \frac{x}{2}} = \lim_{x \rightarrow 0} \frac{\left(\frac{\sin^2 x}{x^2} \right) \times x^2}{\left(\frac{\sin^2 \frac{x}{2}}{\left(\frac{x}{2} \right)^2} \right) \times \frac{x^2}{4}}$$

$$= 4 \frac{\lim_{x \rightarrow 0} \left(\frac{\sin^2 x}{x^2} \right)}{\lim_{x \rightarrow 0} \left(\frac{\sin^2 \frac{x}{2}}{\left(\frac{x}{2} \right)^2} \right)}$$

$$\begin{aligned}
 &= 4 \frac{\left(\lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2} \right)^2}{\left(\lim_{x \rightarrow 0} \frac{\sin \frac{x}{2}}{\left(\frac{x}{2} \right)^2} \right)^2} \quad \left[x \rightarrow 0 \Rightarrow \frac{x}{2} \rightarrow 0 \right] \\
 &= 4 \frac{1^2}{1^2} \quad \left[\lim_{y \rightarrow 0} \frac{\sin y}{y} = 1 \right] \\
 &= 4
 \end{aligned}$$

Question 18: Evaluate the given limit: $\lim_{x \rightarrow 0} \frac{ax + x \cos x}{b \sin x}$

Solution 18: $\lim_{x \rightarrow 0} \frac{ax + x \cos x}{b \sin x}$

At $x = 0$, the value of the given function takes the form $\frac{0}{0}$

$$\text{Now, } \lim_{x \rightarrow 0} \frac{ax + x \cos x}{b \sin x} = \frac{1}{b} \lim_{x \rightarrow 0} \frac{x(a + \cos x)}{\sin x}$$

$$= \frac{1}{b} \lim_{x \rightarrow 0} \left(\frac{x}{\sin x} \right) \times \lim_{x \rightarrow 0} (a + \cos x)$$

$$= \frac{1}{b} \left(\frac{1}{\lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)} \right) \times \lim_{x \rightarrow 0} (a + \cos x)$$

$$= \frac{1}{b} \times (a + \cos 0) \quad \left[\lim_{y \rightarrow 0} \frac{\sin y}{y} = 1 \right]$$

$$= \frac{a+1}{b}$$

Question 19: Evaluate the given limit: $\lim_{x \rightarrow 0} x \sec x$

Solution 19: $\lim_{x \rightarrow 0} x \sec x = \lim_{x \rightarrow 0} \frac{x}{\cos x} = \frac{0}{\cos 0} = \frac{0}{1} = 0$

Question 20: Evaluate the given limit: $\lim_{x \rightarrow 0} \frac{\sin ax + bx}{ax + \sin bx}$ $a, b, a + b \neq 0$

Solution 20: At $x = 0$, the value of the given function takes the form $\frac{0}{0}$

$$\begin{aligned}
 &\text{Now, } \lim_{x \rightarrow 0} \frac{\sin ax + bx}{ax + \sin bx} \\
 &= \lim_{x \rightarrow 0} \frac{\left(\frac{\sin ax}{ax} \right) ax + bx}{ax + bx \left(\frac{\sin bx}{bx} \right)} \\
 &= \frac{\left(\lim_{x \rightarrow 0} \frac{\sin ax}{ax} \right) \times \lim_{x \rightarrow 0} (ax) + \lim_{x \rightarrow 0} (bx)}{\lim_{x \rightarrow 0} ax + \lim_{x \rightarrow 0} bx \left(\lim_{x \rightarrow 0} \frac{\sin bx}{bx} \right)} \quad [\text{As } x \rightarrow 0 \Rightarrow ax \rightarrow 0 \text{ and } bx \rightarrow 0] \\
 &= \frac{\lim_{x \rightarrow 0} (ax) + \lim_{x \rightarrow 0} bx}{\lim_{x \rightarrow 0} ax + \lim_{x \rightarrow 0} bx} \quad \left[\lim_{y \rightarrow 0} \frac{\sin y}{y} = 1 \right] \\
 &= \frac{\lim_{x \rightarrow 0} (ax + bx)}{\lim_{x \rightarrow 0} (ax + bx)} \\
 &= \lim_{x \rightarrow 0} (1) \\
 &= 1
 \end{aligned}$$

Question 21: Evaluate the given limit: $\lim_{x \rightarrow 0} (\operatorname{cosec} x - \cot x)$

Solution 21: At $x = 0$, the value of the given function takes the form $\infty - \infty$

Now, $\lim_{x \rightarrow 0} (\operatorname{cosec} x - \cot x)$

$$\begin{aligned}
 &= \lim_{x \rightarrow 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} \right) \\
 &= \lim_{x \rightarrow 0} \left(\frac{1 - \cos x}{\sin x} \right) \\
 &= \lim_{x \rightarrow 0} \frac{\left(\frac{1 - \cos x}{x} \right)}{\left(\frac{\sin x}{x} \right)}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{\lim_{x \rightarrow 0} \frac{1 - \cos x}{x}}{\lim_{x \rightarrow 0} \frac{\sin x}{x}} \\
 &= \frac{0}{1} \quad \left[\lim_{y \rightarrow 0} \frac{1 - \cos x}{x} = 0 \text{ and } \lim_{y \rightarrow 0} \frac{\sin x}{x} = 1 \right] \\
 &= 0
 \end{aligned}$$

Question 22: Evaluate the given limit: $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan 2x}{x - \frac{\pi}{2}}$

Solution 22: $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan 2x}{x - \frac{\pi}{2}}$

At $x = \frac{\pi}{2}$, the value of the given function takes the form $\frac{0}{0}$

Now, put So that $x - \frac{\pi}{2} = y$ so that $x \rightarrow \frac{\pi}{2}$, $y \rightarrow 0$

$$\begin{aligned}
 \therefore \lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan 2x}{x - \frac{\pi}{2}} &= \lim_{y \rightarrow 0} \frac{\tan 2\left(y + \frac{\pi}{2}\right)}{y} \\
 &= \lim_{y \rightarrow 0} \frac{\tan(\pi + 2y)}{y} \\
 &= \lim_{y \rightarrow 0} \frac{\tan 2y}{y} \quad [\tan(\pi + 2y) = \tan 2y] \\
 &= \lim_{y \rightarrow 0} \frac{\sin 2y}{y \cos 2y} \\
 &= \lim_{y \rightarrow 0} \left(\frac{\sin 2y}{2y} \times \frac{2}{\cos 2y} \right) \\
 &= \left(\lim_{y \rightarrow 0} \frac{\sin 2y}{2y} \right) \times \lim_{y \rightarrow 0} \left(\frac{2}{\cos 2y} \right) \quad [y \rightarrow 0 \Rightarrow 2y \rightarrow 0] \\
 &= 1 \times \frac{2}{\cos 0} \quad \left[\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right] \\
 &= 1 \times \frac{2}{1} \\
 &= 2
 \end{aligned}$$

Question 23: Find $\lim_{x \rightarrow 0} f(x)$ and $\lim_{x \rightarrow 1} f(x)$, where $f(x) = \begin{cases} 2x+3, & x \leq 0 \\ 3(x+1), & x > 0 \end{cases}$

Solution 23: The given function is

$$f(x) = \begin{cases} 2x + 3, & x \leq 0 \\ 3(x + 1), & x > 0 \end{cases}$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0} [2x + 3] = 2(0) + 3 = 3$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0} 3(x + 1) = 3(0 + 1) = 3$$

$$\therefore \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0} f(x) = 3$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1} (x + 1) = 3(1 + 1) = 6$$

$$\therefore \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1} f(x) = 6$$

Question 24: Find $\lim_{x \rightarrow 1} f(x)$, when $f(x) = \begin{cases} x^2 - 1, & x \leq 1 \\ -x - 1, & x > 1 \end{cases}$

Solution 24:

The given function is

$$f(x) = \begin{cases} x^2 - 1, & x \leq 1 \\ -x - 1, & x > 1 \end{cases}$$

$$\therefore \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1} [x^2 - 1] = 1^2 - 1 = 1 - 1 = 0$$

It is observed that $\lim_{x \rightarrow 1^-} f(x) \neq \lim_{x \rightarrow 1^+} f(x)$.

Hence, $\lim_{x \rightarrow 1} f(x)$ does not exist.

Question 25: Evaluate $\lim_{x \rightarrow 0} f(x)$, where $f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Solution 25: The given function is

$$f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \left[\frac{|x|}{x} \right]$$

$$= \lim_{x \rightarrow 0^-} \left(\frac{-x}{x} \right) \quad \text{[When } x \text{ is negative, } |x| = -x]$$

$$= \lim_{x \rightarrow 0} (-1)$$

$$= -1$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \left[\frac{|x|}{x} \right]$$

$$= \lim_{x \rightarrow 0} \left(\frac{x}{x} \right) \quad [\text{When } x \text{ is positive, } |x| = x]$$

$$= \lim_{x \rightarrow 0} (1)$$

$$= 1$$

It is observed that $\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$.

Hence, $\lim_{x \rightarrow 0} f(x)$ does not exist.

Question 26: Find $\lim_{x \rightarrow 0} f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Solution 26: The given function is

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \left[\frac{x}{|x|} \right]$$

$$= \lim_{x \rightarrow 0} \left(\frac{x}{-x} \right) \quad [\text{When } x < 0, |x| = -x]$$

$$= \lim_{x \rightarrow 0} (-1)$$

$$= -1$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \left[\frac{x}{|x|} \right]$$

$$= \lim_{x \rightarrow 0} \left(\frac{x}{x} \right) \quad [\text{When } x > 0, |x| = x]$$

$$= \lim_{x \rightarrow 0} (1)$$

$$= 1$$

It is observed that $\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$.

Hence, $\lim_{x \rightarrow 0} f(x)$ does not exist.

Question 27: Find $\lim_{x \rightarrow 5} f(x)$, where $f(x) = |x| - 5$

Solution 27: The given function is $f(x) = |x| - 5$

$$\lim_{x \rightarrow 5^-} f(x) = \lim_{x \rightarrow 5^-} (|x| - 5)$$

$$= \lim_{x \rightarrow 5} (x - 5) \quad [\text{When } x > 0, |x| = x]$$

$$= 5 - 5$$

$$= 0$$

$$\lim_{x \rightarrow 5^+} f(x) = \lim_{x \rightarrow 5^+} (|x| - 5)$$

$$= \lim_{x \rightarrow 5} (x - 5)$$

$$= 5 - 5$$

$$= 0$$

$$\lim_{x \rightarrow 5^-} f(x) = \lim_{x \rightarrow 5^+} f(x) = 0$$

$$\text{Hence, } \lim_{x \rightarrow 5} f(x) = 0$$

$$[\text{When } x > 0, |x| = x]$$

Question 28: Suppose $f(x) = \begin{cases} a + bx, & x < 0 \\ 4, & x = 1 \\ b - ax, & x > 1 \end{cases}$ and if $\lim_{x \rightarrow 1} f(x) = f(1)$ what are possible values of a and b?

Solution 28: The given function is

$$f(x) = \begin{cases} a + bx, & x < 0 \\ 4, & x = 1 \\ b - ax, & x > 1 \end{cases}$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (a + bx) = a + b$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (b - ax) = b - a$$

$$f(1) = 4$$

It is given that $\lim_{x \rightarrow 1} f(x) = f(1)$.

$$\therefore \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} f(x) = f(1)$$

$$\Rightarrow a + b = 4 \text{ and } b - a = 4$$

On solving these two equations, we obtain $a = 0$ and $b = 4$.

Thus, the respective possible values of a and b are 0 and 4.

Question 29: Let $a_1, a_2, \dots, a_n$ be fixed real numbers and define a function

$$f(x) = (x - a_1)(x - a_2) \dots (x - a_n)$$

What is $\lim_{x \rightarrow a_1} f(x)$? For some $a \neq a_1, a_2, \dots, a_n$. Compute $\lim_{x \rightarrow a} f(x)$.

Solution 29: The given function is $f(x) = (x - a_1)(x - a_2) \dots (x - a_n)$

$$\lim_{x \rightarrow a_1} f(x) = \lim_{x \rightarrow a_1} [(x - a_1)(x - a_2) \dots (x - a_n)]$$

$$= (a_1 - a_1)(a_1 - a_2) \dots (a_1 - a_n) = 0$$

$$\therefore \lim_{x \rightarrow a_1} f(x) = 0$$

$$\text{Now, } \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} [(x - a_1)(x - a_2) \dots (x - a_n)]$$

$$= (a - a_1)(a - a_2) \dots (a - a_n)$$

$$\therefore \lim_{x \rightarrow a} f(x) = (a - a_1)(a - a_2) \dots (a - a_n)$$

Question 30: If $f(x) = \begin{cases} |x| + 1, & x < 0 \\ 0, & x = 0 \\ |x| - 1, & x > 0 \end{cases}$

For what value (s) of does $\lim_{x \rightarrow a} f(x)$ exists?

Solution 30: The given function is

$$\text{If } f(x) = \begin{cases} |x| + 1, & x < 0 \\ 0, & x = 0 \\ |x| - 1, & x > 0 \end{cases}$$

When $a = 0$,

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (|x| + 1)$$

$$= \lim_{x \rightarrow 0^-} (-x + 1) \quad [\text{If } x < 0, |x| = -x]$$

$$= 0 + 1$$

$$= 1$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (|x| + 1)$$

$$= \lim_{x \rightarrow 0^+} (x - 1) \quad [\text{If } x > 0, |x| = x]$$

$$= 0 - 1$$

$$= -1$$

Here, it is observed that $\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$.

$\therefore \lim_{x \rightarrow 0} f(x)$ does not exist.

When $a < 0$,

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^-} (|x| + 1)$$

$$= \lim_{x \rightarrow a^-} (-x + 1) \quad [x < a < 0 \Rightarrow |x| = -x]$$

$$= -a + 1$$

$$\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^+} (|x| + 1)$$

$$= \lim_{x \rightarrow a^+} (-x + 1) \quad [a < x < 0 \Rightarrow |x| = -x]$$

$$= -a + 1$$

$$\therefore \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = -a + 1$$

Thus, limit of $f(x)$ exists at $x = a$, where $a < 0$.

When $a > 0$

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^-} (|x| + 1)$$

$$= \lim_{x \rightarrow a} (-x - 1) \quad [0 < x < a \Rightarrow |x| = x]$$

$$= a - 1$$

$$\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^+} (|x| - 1)$$

$$= \lim_{x \rightarrow a} (-x - 1) \quad [0 < x < a \Rightarrow |x| = x]$$

$$= a - 1$$

$$\therefore \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = a - 1$$

Thus, limit of $f(x)$ exists at $x = a$, where $a > 0$.

Thus, $\lim_{x \rightarrow a} f(x)$ exists for all $a \neq 0$.

Question 31: If the function $f(x)$ satisfies, $\lim_{x \rightarrow 1} \frac{f(x) - 2}{x^2 - 1} = \pi$, evaluate $\lim_{x \rightarrow 1} f(x)$.

Solution 31: $\lim_{x \rightarrow 1} \frac{f(x) - 2}{x^2 - 1} = \pi$

$$\Rightarrow \frac{\lim_{x \rightarrow 1} (f(x) - 2)}{\lim_{x \rightarrow 1} (x^2 - 1)} = \pi$$

$$\Rightarrow \lim_{x \rightarrow 1} (f(x) - 2) = \pi \lim_{x \rightarrow 1} (x^2 - 1)$$

$$\Rightarrow \lim_{x \rightarrow 1} (f(x) - 2) = \pi(1^2 - 1)$$

$$\Rightarrow \lim_{x \rightarrow 1} (f(x) - 2) = 0$$

$$\Rightarrow \lim_{x \rightarrow 1} f(x) - \lim_{x \rightarrow 1} 2 = 0$$

$$\Rightarrow \lim_{x \rightarrow 1} f(x) - 2 = 0$$

$$\therefore \lim_{x \rightarrow 1} f(x) = 2$$

Question 32: If $f(x) = \begin{cases} mx^2 + n, & x < 0 \\ nx + m, & 0 \leq x \leq 1. \\ nx^3 + m, & x > 1 \end{cases}$

For what integers m and n does $\lim_{x \rightarrow 0} f(x)$ and $\lim_{x \rightarrow 1} f(x)$ exists?

Solution 32: $f(x) = \begin{cases} mx^2 + n, & x < 0 \\ nx + m, & 0 \leq x \leq 1. \\ nx^3 + m, & x > 1 \end{cases}$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} (mx^2 + n)$$

$$= m(0)^2 + n$$

$$= n$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0} (nx + m)$$

$$= n(0) + m$$

$$= m$$

Thus, $\lim_{x \rightarrow 0^+} f(x)$ exists if $m = n$.

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1} (nx + m)$$

$$= n(1) + m$$

$$= m + n$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} (nx^3 + m)$$

$$= n(1)^3 + m$$

$$= m + n$$

$$\therefore \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} f(x).$$

Thus, $\lim_{x \rightarrow 1} f(x)$ exists for any internal value of m and n.

Exercise 13.2

Question 1: Find the derivative of $x^2 - 2$ at $x = 10$.

Solution 1: Let $f(x) = x^2 - 2$. Accordingly,

$$\begin{aligned}
 f'(10) &= \lim_{h \rightarrow 0} \frac{f(10+h) - f(10)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[(10+h)^2 - 2] - (10^2 - 2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{10^2 + 2 \cdot 10 \cdot h + h^2 - 2 - 10^2 + 2}{h} \\
 &= \lim_{h \rightarrow 0} \frac{20h + h^2}{h} \\
 &= \lim_{h \rightarrow 0} (20 + h) = 20 + 0 = 20
 \end{aligned}$$

Thus, the derivative of $x^2 - 2$ at $x = 10$ is 20.

Question 2: Find the derivative of $99x$ at $x = 100$.

Solution 2: Let $f(x) = 99x$. Accordingly,

$$\begin{aligned}
 f'(100) &= \lim_{h \rightarrow 0} \frac{f(100+h) - f(100)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{99(100+h) - 99(100)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{99 \times 100 + 99h - 99 \times 100}{h} \\
 &= \lim_{h \rightarrow 0} \frac{99h}{h} \\
 &= \lim_{h \rightarrow 0} (99) = 99
 \end{aligned}$$

Thus, the derivative of $99x$ at $x = 100$ is 99.

Question 3: Find the derivative of x at $x = 1$.

Solution 3: Let $f(x) = x$. Accordingly,

$$\begin{aligned} f'(1) &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(1+h) - 1}{h} \\ &= \lim_{h \rightarrow 0} \frac{h}{h} \\ &= \lim_{h \rightarrow 0} (1) = 1 \end{aligned}$$

Thus, the derivative of x at $x = 1$ is 1.

Question 4: Find the derivative of the following functions from first principle.

(i) $x^3 - 27$

(ii) $(x - 1)(x - 2)$

(iii) $\frac{1}{x^2}$

(iv) $\frac{x+1}{x-1}$

Solution 4: (i) Let $f(x) = x^3 - 27$. Accordingly, from the first principle,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[(x+h)^3 - 27] - (x^3 - 27)}{h} \\ &= \lim_{h \rightarrow 0} \frac{x^3 + h^3 + 3x^2h + 3xh^2 - x^3}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^3 + 3x^2h + 3xh^2}{h} \\ &= \lim_{h \rightarrow 0} (h^2 + 3x^2 + 3xh) \\ &= 0 + 3x^2 + 0 = 3x^2 \end{aligned}$$

(ii) Let $f(x) = (x - 1)(x - 2)$. Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x+h-1)(x+h-2) - (x-1)(x-2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x^2 + hx - 2x + hx + h^2 - 2h - x - h + 2) - (x^2 - 2x - x + 2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(hx + hx + h^2 - 2h - h)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2hx + h^2 - 3h}{h} \\
 &= \lim_{h \rightarrow 0} (2x + h - 3) \\
 &= 2x - 3
 \end{aligned}$$

(iii) Let $f(x) = \frac{1}{x^2}$. Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{x^2 - (x+h)^2}{x^2(x+h)^2} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{x^2 - x^2 - 2hx - h^2}{x^2(x+h)^2} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-h^2 - 2hx}{x^2(x+h)^2} \right] \\
 &= \lim_{h \rightarrow 0} \left[\frac{-h^2 - 2x}{x^2(x+h)^2} \right] \\
 &= \frac{0 - 2x}{x^2(x+0)^2} = \frac{-2}{x^3}
 \end{aligned}$$

(iv) Let $f(x) = \frac{x+1}{x-1}$. Accordingly, from the first principle,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\left(\frac{x+h+1}{x+h-1} - \frac{x+1}{x-1} \right)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{(x-1)(x+h+1) - (x+1)(x+h-1)}{(x-1)(x+h-1)} \right] \end{aligned}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2h}{(x-1)(x+h-1)} \right]$$

$$= \lim_{h \rightarrow 0} \left[\frac{-2}{(x-1)(x+h-1)} \right]$$

$$= \frac{-2}{(x-1)(x-1)} = \frac{-2}{(x-1)^2}$$

Question 5: For the function

$$F(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \cdots + \frac{x^2}{2} + x + 1$$

Prove that $f'(1) = 100 f'(0)$

Solution 5: The given function is

$$F(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \cdots + \frac{x^2}{2} + x + 1$$

$$\frac{d}{dx} f(x) = \frac{d}{dx} \left[\frac{x^{100}}{100} + \frac{x^{99}}{99} + \cdots + \frac{x^2}{2} + x + 1 \right]$$

$$\frac{d}{dx} f(x) = \frac{d}{dx} \left(\frac{x^{100}}{100} \right) + \frac{d}{dx} \left(\frac{x^{99}}{99} \right) + \cdots + \frac{d}{dx} \left(\frac{x^2}{2} \right) + \frac{d}{dx} (x) + \frac{d}{dx} (1)$$

On using theorem $\frac{d}{dx}(x^n) = nx^{n-1}$, we obtain

$$\frac{d}{dx} f(x) = \frac{100x^{99}}{100} + \frac{99x^{98}}{99} + \cdots + \frac{2x}{2} + 1 + 0$$

$$= x^{99} + x^{98} + \cdots + x + 1$$

$$\therefore f'(x) = x^{99} + x^{98} + \cdots + x + 1$$

At $x = 0$,

$$f'(0) = 1$$

At $x = 1$,

$$f'(1) = 1^{99} + 1^{98} + \cdots + 1 + 1 = [1 + 1 + \cdots + 1 + 1]_{100 \text{ terms}} = 1 \times 100 = 100$$

Thus, $f'(1) = 100 f'(0)$

Question 6: Find the derivative of $x^n + ax^{n-1} + a^2x^{n-2} + \cdots + a^{n-1}x + a^n$ for some fixed real number a .

Solution 6: Let $f(x) = x^n + ax^{n-1} + a^2x^{n-2} + \cdots + a^{n-1}x + a^n$

$$\frac{d}{dx} f(x) = \frac{d}{dx} (x^n + ax^{n-1} + a^2x^{n-2} + \cdots + a^{n-1}x + a^n)$$

$$= \frac{d}{dx}(x^n) + a \frac{d}{dx}(x^{n-1}) + a^2 \frac{d}{dx}(x^{n-2}) + \cdots + a^{n-1} \frac{d}{dx}(x) + a^n \frac{d}{dx}(1)$$

On using theorem $\frac{d}{dx}(x^n) = nx^{n-1}$, we obtain

$$f'(x) = nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + \cdots + a^{n-1} + a^n(0)$$

$$\therefore f'(x) = nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + \cdots + a^{n-1}$$

Question 7: For some constants a and b , find the derivative of

(i) $(x - a)(x - b)$

(ii) $(ax^2 + b)^2$

(iii) $\frac{x-a}{x-b}$

Solution 7: (i) Let $f(x) = (x - a)(x - b)$

$$\Rightarrow f(x) = x^2 - (a+b)x + ab$$

$$\therefore f'(x) = \frac{d}{dx}(x^2 - (a+b)x + ab)$$

$$= \frac{d}{dx}(x^2) - (a+b) \frac{d}{dx}(x) + \frac{d}{dx}(ab)$$

On using theorem $\frac{d}{dx}(x^n) = nx^{n-1}$, we obtain

$$f'(x) = 2x - (a+b) + 0$$

$$= 2x - a - b$$

(ii) Let $f(x) = (ax^2 + b)^2$

$$\Rightarrow f(x) = a^2x^4 + 2abx^2 + b^2$$

$$\therefore f'(x) = \frac{d}{dx}(a^2x^4 + 2abx^2 + b^2)$$

$$= a^2 \frac{d}{dx}(x^4) + 2ab \frac{d}{dx}(x^2) + \frac{d}{dx}b^2$$

On using theorem $\frac{d}{dx}(x^n) = nx^{n-1}$, we obtain

$$f'(x) = a^2(4x^3) + 2ab(2x) + b^2(0)$$

$$= 4a^2x^3 + 4abx$$

$$= 4ax(ax^2 + b)$$

(iii) Let $f(x) = \frac{x-a}{x-b}$

$$\Rightarrow f'(x) = \frac{d}{dx} \left(\frac{x-a}{x-b} \right)$$

By quotient rule,

$$f'(x) = \frac{(x-b) \frac{d}{dx}(x-a) - (x-a) \frac{d}{dx}(x-b)}{(x-b)^2}$$

$$= \frac{(x-b)(1) - (x-a)(1)}{(x-b)^2}$$

$$= \frac{x-b-x+a}{(x-b)^2}$$

$$= \frac{a-b}{(x-b)^2}$$

Question 8: Find the derivative of $\frac{x^n - a^n}{x - a}$ for some constant a.

Solution 8: Let $f(x) = \frac{x^n - a^n}{x - a}$

$$\Rightarrow f'(x) = \frac{d}{dx} \left(\frac{x^n - a^n}{x - a} \right)$$

By quotient rule,

$$f'(x) = \frac{(x-a) \frac{d}{dx} (x^n - a^n) - (x^n - a^n) \frac{d}{dx} (x-a)}{(x-a)^2}$$

$$= \frac{(x-a)(nx^{n-1} - 0) - (x^n - a^n)}{(x-a)^2}$$

$$= \frac{nx^n - anx^{n-1} - x^n + a^n}{(x-a)^2}$$

Question 9: Find the derivative of

(i) $2x - \frac{3}{4}$

(ii) $(5x^3 + 3x - 1)(x - 1)$

(iii) $x^{-3}(5 + 3x)$

(iv) $x^5(3 - 6x^{-9})$

(v) $x^{-4}(3 - 4x^{-5})$

(vi) $\frac{2}{x+1} - \frac{x^2}{3x-1}$

Solution 9: (i) Let $f(x) = 2x - \frac{3}{4}$

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left(2x - \frac{3}{4} \right) \\ &= 2 \frac{d}{dx} (x) - \frac{d}{dx} \left(\frac{3}{4} \right) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

(ii) Let $f(x) = (5x^3 + 3x - 1)(x - 1)$

By Leibnitz product rule,

$$\begin{aligned} f'(x) &= (5x^3 + 3x - 1) \frac{d}{dx} (x - 1) + (x - 1) \frac{d}{dx} (5x^3 + 3x - 1) \\ &= (5x^3 + 3x - 1)(1) + (x - 1)(15x^2 + 3 - 0) \\ &= (5x^3 + 3x - 1) + (x - 1)(15x^2 + 3) \\ &= 5x^3 + 3x - 1 + 15x^3 + 3x - 15x^2 - 3 \\ &= 20x^3 - 15x^2 + 6x - 4 \end{aligned}$$

(iii) Let $f(x) = x^{-3}(5 + 3x)$

By Leibnitz product rule,

$$\begin{aligned} f'(x) &= x^{-3} \frac{d}{dx} (5 + 3x) + (5 + 3x) \frac{d}{dx} (x^{-3}) \\ &= x^{-3} (0 + 3) + (5 + 3x)(3x^{-3-1}) \\ &= x^{-3} (3) + (5 + 3x)(3x^{-4}) \\ &= 3x^{-3} - 15x^{-4} - 9x^{-3} \\ &= -6x^{-3} - 15x^{-4} \\ &= -3x^{-3} \left(2 + \frac{5}{x} \right) \\ &= \frac{-3x^{-3}}{x} (2x + 5) \end{aligned}$$

$$= \frac{-3}{x^4} (5 + 2x)$$

(iv) Let $f(x) = x^5 (3 - 6x^{-9})$

By Leibnitz product rule,

$$\begin{aligned} f'(x) &= x^5 \frac{d}{dx} (3 - 6x^{-9}) + (3 - 6x^{-9}) \frac{d}{dx} (x^5) \\ &= x^5 \{0 - 6(-9)x^{-9-1}\} + (3 - 6x^{-9})(5x^4) \\ &= x^5 (54x^{-10}) + 15x^4 - 30x^{-5} \\ &= 54x^{-5} + 15x^4 - 30x^{-5} \\ &= 24x^{-5} + 15x^4 \\ &= 15x^4 + \frac{24}{x^5} \end{aligned}$$

(v) Let $f(x) = x^{-4} (3 - 4x^{-5})$

By Leibnitz product rule,

$$\begin{aligned} f'(x) &= x^{-4} \frac{d}{dx} (3 - 4x^{-5}) + (3 - 4x^{-5}) \frac{d}{dx} (x^{-4}) \\ &= x^{-4} \{0 - 4(-5)x^{-5-1}\} + (3 - 4x^{-5})(-4)x^{-4-1} \\ &= x^{-4} (20x^{-6}) + (3 - 4x^{-5})(-4x^{-5}) \\ &= 20x^{-10} - 12x^{-5} + 16x^{-10} \\ &= 36x^{-10} - 12x^{-5} \\ &= \frac{12}{x^{-5}} + \frac{36}{x^{10}} \end{aligned}$$

(vi) Let $f(x) = \frac{2}{x+1} - \frac{x^2}{3x-1}$

$$f'(x) = \frac{d}{dx} \left(\frac{2}{x+1} \right) - \frac{d}{dx} \left(\frac{x^2}{3x-1} \right)$$

By quotient rule,

$$\begin{aligned}
 f'(x) &= \left[\frac{(x+1) \frac{d}{dx}(2) - 2 \frac{d}{dx}(x+1)}{(x+1)^2} \right] - \left[\frac{(3x-1) \frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(3x-1)}{(3x-1)^2} \right] \\
 &= \left[\frac{(x+1)(0) - 2(1)}{(x+1)^2} \right] - \left[\frac{(3x-1)(2x) - x^2(3)}{(3x-1)^2} \right] \\
 &= \frac{-2}{(x+1)^2} - \left[\frac{6x^2 - 2x - 3x^2}{(3x-1)^2} \right] \\
 &= \frac{-2}{(x+1)^2} - \left[\frac{3x^2 - 2x^2}{(3x-1)^2} \right] \\
 &= \frac{-2}{(x+1)^2} - \frac{x(3x-2)}{(3x-1)^2}
 \end{aligned}$$

Question 10: Find the derivative of $\cos x$ from first principle.

Solution 10: Let $f(x) = \cos x$. Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \left[\frac{\cos(x+h) - \cos(x)}{h} \right] \\
 &= \lim_{h \rightarrow 0} \left[\frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \right] \\
 &= \lim_{h \rightarrow 0} \left[\frac{-\cos x(1 - \cos h) - \sin x \sin h}{h} \right] \\
 &= -\cos x \left[\lim_{h \rightarrow 0} \left(\frac{1 - \cos h}{h} \right) \right] - \sin x \left[\lim_{h \rightarrow 0} \left(\frac{\sin h}{h} \right) \right] \\
 &= -\cos x(0) - \sin x(1) \quad \left[\lim_{h \rightarrow 0} \frac{1 - \cos h}{h} = 0 \text{ and } \lim_{h \rightarrow 0} \frac{\sin h}{h} = 1 \right] \\
 \therefore f'(x) &= -\sin x
 \end{aligned}$$

Question 11: Find the derivative of the following functions:

(i) $\sin x \cos x$

(ii) $\sec x$

(iii) $5\sec x + 4\cos x$

(iv) $\operatorname{cosec} x$

(v) $3\cot x + 5\operatorname{cosec} x$

(vi) $5\sin x - 6\cos x + 7$

(vii) $2\tan x - 7\sec x$

Solution 11: (i) Let $f(x) = \sin x \cos x$. Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin(x+h)\cos(x+h) - \sin x \cos x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{2h} [2\sin(x+h)\cos(x+h) - 2\sin x \cos x] \\
 &= \lim_{h \rightarrow 0} \frac{1}{2h} [\sin 2(x+h) - \sin 2x] \\
 &= \lim_{h \rightarrow 0} \frac{1}{2h} \left[2\cos \frac{2x+2h+2x}{2} \cdot \sin \frac{2x+2h-2x}{2} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{2h} \left[2\cos \frac{4x+2h}{2} \cdot \sin \frac{2h}{2} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{2h} [\cos(2x+h) \sin h] \\
 &= \lim_{h \rightarrow 0} \cos(2x+h) \cdot \lim_{h \rightarrow 0} \frac{\sin h}{h} \\
 &= \cos(2x+h) \cdot 1 \\
 &= \cos 2x
 \end{aligned}$$

(ii) Let $f(x) = \sec x$. Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sec(x+h) - \sec x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\cos x - \cos(x+h)}{\cos x \cos(x+h)} \right] \\
 &= \frac{1}{\cos x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{x+x+h}{2}\right) \sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)} \right] \\
 &= \frac{1}{\cos x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{2x+h}{2}\right) \sin\left(\frac{-h}{2}\right)}{\cos(x+h)} \right] \\
 &= \frac{1}{\cos x} \cdot \lim_{h \rightarrow 0} \frac{1}{2h} \left[\frac{-2 \sin\left(\frac{2x+h}{2}\right) \sin\left(\frac{-h}{2}\right)}{\left(\frac{h}{2}\right) \cos(x+h)} \right] \\
 &= \frac{1}{\cos x} \cdot \lim_{h \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \cdot \lim_{h \rightarrow 0} \frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)} \\
 &= \frac{1}{\cos x} \cdot 1 \cdot \frac{\sin x}{\cos x} \\
 &= \sec x \tan x
 \end{aligned}$$

(iii) Let $f(x) = 5\sec x + 4\cos x$. Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{5\sec(x+h) + 4\cos(x+h) - [5\sec x + 4\cos x]}{h}
 \end{aligned}$$

$$\begin{aligned}
 &= 5 \lim_{h \rightarrow 0} \frac{[\sec(x+h) - \sec x]}{h} + 4 \lim_{h \rightarrow 0} \frac{[\cos(x+h) - \cos x]}{h} \\
 &= 5 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x} \right] + 4 \lim_{h \rightarrow 0} \frac{1}{h} [\cos(x+h) - \cos x] \\
 &= 5 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\cos x - \cos(x+h)}{\cos x \cos(x+h)} \right] + 4 \lim_{h \rightarrow 0} \frac{1}{h} [\cos x \cos h - \sin x \sin h - \cos x] \\
 &= \frac{5}{\cos x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{2x+h}{2}\right) \sin\left(\frac{-h}{2}\right)}{\cos(x+h)} \right] + 4 \left[-\cos x \lim_{h \rightarrow 0} \frac{(1 - \cos x)}{h} - \sin x \lim_{h \rightarrow 0} \frac{\sin h}{h} \right] \\
 &= \frac{5}{\cos x} \cdot \lim_{h \rightarrow 0} \left[\frac{\sin\left(\frac{2x+h}{2}\right) \sin\left(\frac{-h}{2}\right)}{\left(\frac{h}{2}\right) \cos(x+h)} \right] + 4 [-\cos x(0) - \sin x(1)] \\
 &= \frac{5}{\cos x} \cdot \left[\lim_{h \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \cdot \lim_{h \rightarrow 0} \frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)} \right] - 4 \sin x \\
 &= \frac{5}{\cos x} \cdot \frac{\sin x}{\cos x} \cdot 1 - 4 \sin x \\
 &= 5 \sec x \tan x - 4 \sin x
 \end{aligned}$$

(iv) Let $f(x) = \operatorname{cosec} x$. Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} [\operatorname{cosec}(x+h) - \operatorname{cosec} x] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{\sin(x+h)} - \frac{1}{\sin x} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin x - \sin(x+h)}{\sin x \sin(x+h)} \right]
 \end{aligned}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2 \cos\left(\frac{x+x+h}{2}\right) \cdot \sin\left(\frac{x-x-h}{2}\right)}{\sin x \sin(x+h)} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2 \cos\left(\frac{2x+h}{2}\right) \cdot \sin\left(\frac{-h}{2}\right)}{\sin x \sin(x+h)} \right]$$

$$= \lim_{h \rightarrow 0} \left[\frac{-\cos\left(\frac{2x+h}{2}\right) \frac{\sin\left(\frac{-h}{2}\right)}{\left(\frac{h}{2}\right)}}{\sin x \sin(x+h)} \right]$$

$$= \lim_{h \rightarrow 0} \left(\frac{-\cos\left(\frac{2x+h}{2}\right)}{\sin x \sin(x+h)} \right) \cdot \lim_{\frac{h}{2} \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$= \left(\frac{-\cos x}{\sin x \sin x} \right) \cdot 1$$

$$= -\operatorname{cosec} x \cot x$$

(v) Let $f(x) = 3\cot x + 5\operatorname{cosec} x$. Accordingly, from the first principle,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} [3\cot(x+h) + 5\operatorname{cosec}(x+h) - 3\cot x - 5\operatorname{cosec} x]$$

$$= 3 \lim_{h \rightarrow 0} \frac{1}{h} [\cot(x+h) - \cot x] + 5 \lim_{h \rightarrow 0} \frac{1}{h} [\operatorname{cosec}(x+h) - \operatorname{cosec} x] \quad \dots(1)$$

$$\text{Now, } \lim_{h \rightarrow 0} \frac{1}{h} [\cot(x+h) - \cot x] = \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos x}{\sin x} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\cos(x+h) \sin x - \cos x \sin(x+h)}{\sin x \sin(x+h)} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x-x-h)}{\sin x \sin(x+h)} \right]$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(-h)}{\sin x \sin(x+h)} \right] \\
 &= \lim_{h \rightarrow 0} \frac{\sin h}{h} \cdot \lim_{h \rightarrow 0} \left[\frac{1}{\sin x \sin(x+h)} \right] \\
 &= -1 \cdot \frac{1}{\sin x \sin(x+h)} = \frac{-1}{\sin^2 x} = -\operatorname{cosec}^2 x \quad \dots(2)
 \end{aligned}$$

$$\lim_{h \rightarrow 0} \frac{1}{h} [\operatorname{cosec}(x+h) - \operatorname{cosec} x] = \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{\sin(x+h)} - \frac{1}{\sin x} \right]$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin x - \sin(x+h)}{\sin x \sin(x+h)} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2 \cos \left(\frac{x+x+h}{2} \right) \cdot \sin \left(\frac{x-x-h}{2} \right)}{\sin x \sin(x+h)} \right]
 \end{aligned}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2 \cos \left(\frac{2x+h}{2} \right) \cdot \sin \left(\frac{-h}{2} \right)}{\sin x \sin(x+h)} \right]$$

$$= \lim_{h \rightarrow 0} \frac{-\cos \left(\frac{2x+h}{2} \right) \cdot \frac{\sin \left(\frac{-h}{2} \right)}{\left(\frac{h}{2} \right)}}{\sin x \sin(x+h)}$$

$$= \lim_{h \rightarrow 0} \left(\frac{-\cos \left(\frac{2x+h}{2} \right)}{\sin x \sin(x+h)} \right) \cdot \lim_{h \rightarrow 0} \frac{\sin \left(\frac{h}{2} \right)}{\left(\frac{h}{2} \right)}$$

$$= \left(\frac{-\cos x}{\sin x \sin x} \right) \cdot 1$$

$$= -\operatorname{cosec} x \cot x \quad \dots(3)$$

From (1), (2), and (3), we obtain

$$f'(x) = -3\operatorname{cosec}^2 x - 5\operatorname{cosec} x \cot x$$

(vi) Let $f(x) = 5\sin x - 6\cos x + 7$. Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} [5\sin(x+h) - 6\cos(x+h) + 7 - 5\sin x + 6\cos x - 7] \\
 &= 5 \lim_{h \rightarrow 0} \frac{1}{h} [\sin(x+h) - \sin x] - 6 \lim_{h \rightarrow 0} \frac{1}{h} [\cos(x+h) - \cos x] \\
 &= 5 \lim_{h \rightarrow 0} \frac{1}{h} \left[2 \cos\left(\frac{x+h+x}{2}\right) \cdot \sin\left(\frac{x+h-x}{2}\right) \right] - 6 \lim_{h \rightarrow 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\
 &= 5 \lim_{h \rightarrow 0} \frac{1}{h} \left[2 \cos\left(\frac{2x+h}{2}\right) \cdot \sin\left(\frac{h}{2}\right) \right] - 6 \lim_{h \rightarrow 0} \left[\frac{-\cos x(1 - \cos h) - \sin x \sin h}{h} \right] \\
 &= 5 \lim_{h \rightarrow 0} \frac{1}{h} \left[\cos\left(\frac{2x+h}{2}\right) \cdot \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \right] - 6 \lim_{h \rightarrow 0} \left[\frac{-\cos x(1 - \cos h)}{h} - \frac{\sin x \sin h}{h} \right] \\
 &= 5 \left[\lim_{h \rightarrow 0} \cos\left(\frac{2x+h}{2}\right) \right] \left[\lim_{h \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \right] - 6 \left[-\cos x \left(\lim_{h \rightarrow 0} \frac{1 - \cos h}{h} \right) - \sin x \left(\lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \right] \\
 &= 5 \cos x \cdot 1 - 6 [(-\cos x) \cdot (0) - \sin x \cdot 1] \\
 &= 5 \cos x + 6 \sin x
 \end{aligned}$$

(vii) Let $f(x) = 2 \tan x - 7 \sec x$. Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} [2\tan(x+h) - 7\sec(x+h) - 2\tan x + 7\sec x] \\
 &= 2 \lim_{h \rightarrow 0} \frac{1}{h} [\tan(x+h) - \tan x] - 7 \lim_{h \rightarrow 0} \frac{1}{h} [\sec(x+h) - \sec x] \\
 &= 2 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x} \right] - 7 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{1}{\operatorname{cosec}(x+h)} - \frac{1}{\operatorname{cosec} x} \right] \\
 &= 2 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\cos x \sin(x+h) - \sin x \cos(x+h)}{\cos x \cos(x+h)} \right] - 7 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\cos x - \cos(x+h)}{\cos x \cos(x+h)} \right]
 \end{aligned}$$

$$\begin{aligned}
 &= 2 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin x + h - x}{\cos x \cos(x+h)} \right] - 7 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{x+x+h}{2}\right) \sin\left(\frac{x-x-h}{2}\right)}{\cos x \cos(x+h)} \right] \\
 &= 2 \left[\lim_{h \rightarrow 0} \left(\frac{\sin h}{h} \right) \frac{1}{\cos x \cos(x+h)} \right] - 7 \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin\left(\frac{2x+h}{2}\right) \sin\left(\frac{-h}{2}\right)}{\cos x \cos(x+h)} \right] \\
 &= 2 \left(\lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \left[\lim_{h \rightarrow 0} \frac{1}{\cos x \cos(x+h)} \right] - 7 \left(\lim_{h \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \right) \left(\lim_{h \rightarrow 0} \frac{\sin\left(\frac{2x+h}{2}\right)}{\cos x \cos(x+h)} \right) \\
 &= 2 \cdot 1 \cdot \frac{1}{\cos x \cos x} - 7 \cdot 1 \left(\frac{\sin x}{\cos x \cos x} \right) \\
 &= 2 \sec^2 x - 7 \sec x \tan x
 \end{aligned}$$

Miscellaneous Exercise

Question 1: Find the derivative of the following functions from first principle:

- (i) $-x$
- (ii) $(-x)^{-1}$
- (iii) $\sin(x+1)$
- (iv) $\cos\left(x - \frac{\pi}{8}\right)$

Solution 1: (i) Let $f(x) = -x$. Accordingly, $f(x+h) = -(x+h)$

By first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-(x+h) - (-x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-x-h+x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-h}{h}
 \end{aligned}$$

$$= \lim_{h \rightarrow 0} (-1) = -1$$

(ii) Let $f(x) = (-x)^{-1} = \frac{1}{-x} = \frac{-1}{x}$. Accordingly, $f(x+h) = \frac{-1}{(x+h)}$

By first principle,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-1}{(x+h)} - \left(\frac{-1}{x} \right) \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-x + (x+h)}{x(x+h)} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{h}{x(x+h)} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{x(x+h)}$$

$$= \frac{1}{x \cdot x} = \frac{1}{x^2}$$

(iii) Let $f(x) = \sin(x+1)$. Accordingly, $f(x+h) = \sin(x+h+1)$

By first principle,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} [\sin(x+h+1) - \sin(x+1)]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[2 \cos \left(\frac{x+h+1+x+1}{2} \right) \sin \left(\frac{x+h+1-x-1}{2} \right) \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[2 \cos \left(\frac{2x+h+2}{2} \right) \sin \left(\frac{h}{2} \right) \right]$$

$$= \lim_{h \rightarrow 0} \left[\cos \left(\frac{2x+h+2}{2} \right) \cdot \frac{\sin \left(\frac{h}{2} \right)}{\left(\frac{h}{2} \right)} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \cos\left(\frac{2x+h+2}{2}\right) \cdot \lim_{\frac{h}{2} \rightarrow 0} \frac{1}{h} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \quad \left[\text{As } h \rightarrow 0 \Rightarrow \frac{h}{2} \rightarrow 0 \right]$$

$$= \cos\left(\frac{2x+0+2}{2}\right) \cdot 1 \quad \left[\lim_{h \rightarrow 0} \frac{\sin x}{x} = 1 \right]$$

$$= \cos(x+1)$$

(iv) Let $f(x) = \cos\left(x - \frac{\pi}{8}\right)$. Accordingly, $f(x+h) = \cos\left(x+h - \frac{\pi}{8}\right)$

By first principle,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\cos\left(x+h - \frac{\pi}{8}\right) - \cos\left(x - \frac{\pi}{8}\right) \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[-2 \sin\left(\frac{x+h - \frac{\pi}{8} + x - \frac{\pi}{8}}{2}\right) \sin\left(\frac{x+h - \frac{\pi}{8} - x + \frac{\pi}{8}}{2}\right) \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[-2 \sin\left(\frac{2x+h - \frac{\pi}{4}}{2}\right) \sin\left(\frac{h}{2}\right) \right]$$

$$= \lim_{h \rightarrow 0} \left[-\sin\left(\frac{2x+h - \frac{\pi}{4}}{2}\right) \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \right]$$

$$= \lim_{h \rightarrow 0} \left[-\sin\left(\frac{2x+h - \frac{\pi}{4}}{2}\right) \right] \cdot \lim_{\frac{h}{2} \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \quad \left[\text{As } h \rightarrow 0 \Rightarrow \frac{h}{2} \rightarrow 0 \right]$$

$$= -\sin\left(\frac{2x+0 - \frac{\pi}{4}}{2}\right) \cdot 1$$

$$= -\sin\left(x - \frac{\pi}{8}\right)$$

Question 2: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(x + a)$

Solution 2: Let $f(x) = x + a$. Accordingly, $f(x + h) = x + h + a$

By first principle,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{x + h + a - x - a}{h}$$

$$= \lim_{h \rightarrow 0} \left(\frac{h}{h}\right)$$

$$= \lim_{h \rightarrow 0} (1)$$

$$= 1$$

Question 3: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(px + q)\left(\frac{r}{x} + s\right)$

Solution 3: Let $f(x) = (px + q)\left(\frac{r}{x} + s\right)$

By Leibnitz product rule,

$$f'(x) = (px + q)\left(\frac{r}{x} + s\right)' + \left(\frac{r}{x} + s\right)(px + q)'$$

$$= (px + q)(rx^{-1} + s)' + \left(\frac{r}{x} + s\right)(p)$$

$$= (px + q)(-rx^{-2}) + \left(\frac{r}{x} + s\right)p$$

$$\begin{aligned}
 &= (px + q) \left(\frac{-r}{x^2} \right) + \left(\frac{r}{x} + s \right) p \\
 &= \frac{-px}{x} - \frac{qr}{x^2} + \frac{pr}{x} + ps \\
 &= ps - \frac{qr}{x^2}
 \end{aligned}$$

Question 4: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(ax + b)(cx + d)^2$

Solution 4: Let $f(x) = (ax + b)(cx + d)^2$

By Leibnitz product rule,

$$\begin{aligned}
 f'(x) &= (ax + b) \frac{d}{dx} (cx + d)^2 + (cx + d)^2 \frac{d}{dx} (ax + b) \\
 &= (ax + b) \frac{d}{dx} (c^2x^2 + 2cdx) + (cx + d)^2 \frac{d}{dx} (ax + b) \\
 &= (ax + b) \left[\frac{d}{dx} (c^2x^2) + \frac{d}{dx} (2cdx) + \frac{d}{dx} d^2 \right] + (cx + d)^2 \left[\frac{d}{dx} ax + \frac{d}{dx} b \right] \\
 &= (ax + b)(2c^2x + 2cd) + (cx + d)^2 a \\
 &= 2c(ax + b)(cx + d) + a(cx + d)^2
 \end{aligned}$$

Question 5: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non zero constants and m and n are integers): $\frac{ax + b}{cx + d}$

Solution 5: Let $f(x) = \frac{ax + b}{cx + d}$

By quotient rule,

$$\begin{aligned}
 f'(x) &= \frac{(cx + d) \frac{d}{dx} (ax + b) - (ax + b) \frac{d}{dx} (cx + d)}{(cx + d)^2} \\
 &= \frac{(cx + d)(a) - (ax + d)(c)}{(cx + d)^2}
 \end{aligned}$$

$$= \frac{acx + ad - acx - bc}{(cx + d)^2}$$

$$= \frac{ad - bc}{(cx + d)^2}$$

Question 6: Find the derivative of the following functions (it is to be understood that a, b, c,

d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{1 + \frac{1}{x}}{1 - \frac{1}{x}}$

Solution 6: Let $f(x) = \frac{1 + \frac{1}{x}}{1 - \frac{1}{x}} = \frac{\frac{x+1}{x}}{\frac{x-1}{x}} = \frac{x+1}{x-1}$, where $x \neq 0$

By quotient rule,

$$f'(x) = \frac{(x-1) \frac{d}{dx}(x-1) - (x+1) \frac{d}{dx}(x-1)}{(x-1)^2}, x \neq 0, 1$$

$$= \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2}, x \neq 0, 1$$

$$= \frac{x-1-x-1}{(x-1)^2}, x \neq 0, 1$$

$$= \frac{-2}{(x-1)^2}, x \neq 0, 1$$

Question 7: Find the derivative of the following functions (it is to be understood that a, b, c,

d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{1}{ax^2 + bx + c}$

Solution 7: Let $f(x) = \frac{1}{ax^2 + bx + c}$

By quotient rule,

$$\begin{aligned}
 f'(x) &= \frac{(ax^2 + bx + c) \frac{d}{dx}(1) - \frac{d}{dx}(ax^2 + bx + c)}{(ax^2 + bx + c)^2} \\
 &= \frac{(ax^2 + bx + c)(0) - (2ax + b)}{(ax^2 + bx + c)^2} \\
 &= \frac{-(2ax + b)}{(ax^2 + bx + c)^2}
 \end{aligned}$$

Question 8: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{ax+b}{px^2+qx+r}$

Solution 8: Let $f(x) = \frac{ax+b}{px^2+qx+r}$

By quotient rule,

$$\begin{aligned}
 f'(x) &= \frac{(px^2 + qx + r) \frac{d}{dx}(ax + b) - (ax + b) \frac{d}{dx}(px^2 + qx + r)}{(px^2 + qx + r)^2} \\
 &= \frac{(px^2 + qx + r)(a) - (ax + b)(2px + q)}{(px^2 + qx + r)^2} \\
 &= \frac{apx^2 + aqx + ar - aqx - 2bpx + bq}{(px^2 + qx + r)^2} \\
 &= \frac{-apx^2 + 2bpx + ar - bq}{(px^2 + qx + r)^2}
 \end{aligned}$$

Question 9: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{px^2+qx+r}{ax+b}$

Solution 9: Let $f(x) = \frac{px^2+qx+r}{ax+b}$

By quotient rule,

$$\begin{aligned}
 f'(x) &= \frac{(ax+b) \frac{d}{dx}(px^2+qx+r) - (px^2+qx+r) \frac{d}{dx}(ax+b)}{(ax+b)^2} \\
 &= \frac{(ax+b)(2px+q) - (px^2+qx+r)(a)}{(ax+b)^2} \\
 &= \frac{2apx^2 + aqx + 2bpx + bq - apx^2 - apx - ar}{(ax+b)^2} \\
 &= \frac{apx^2 + 2bpx + bq - ar}{(ax+b)^2}
 \end{aligned}$$

Question 10: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{a}{x^4} - \frac{b}{x^2} + \cos x$

Solution 10: Let $f(x) = \frac{a}{x^4} - \frac{b}{x^2} + \cos x$

$$\begin{aligned}
 f'(x) &= \frac{d}{dx} \left(\frac{a}{x^4} \right) - \frac{d}{dx} \left(\frac{b}{x^2} \right) + \frac{d}{dx} (\cos x) \\
 &= a \frac{d}{dx} (x^{-4}) - b \frac{d}{dx} (x^{-2}) + \frac{d}{dx} (\cos x) \\
 &= a(-4x^{-5}) - b(-2x^{-3}) + (-\sin x) \quad \left[\frac{d}{dx} (x^n) = nx^{n-1} \text{ and } \frac{d}{dx} (\cos x) = -\sin x \right] \\
 &= \frac{-4a}{x^5} + \frac{2b}{x^3} - \sin x
 \end{aligned}$$

Question 11: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non zero constants and m and n are integers): $4\sqrt{x} - 2$

Solution 11: Let $f(x) = 4\sqrt{x} - 2$

$$f'(x) = \frac{d}{dx} (4\sqrt{x} - 2) = \frac{d}{dx} (4\sqrt{x}) - \frac{d}{dx} (2)$$

$$= 4 \frac{d}{dx} (x^{\frac{1}{2}}) - 0 = 4 \left(\frac{1}{2} x^{\frac{1}{2}-1} \right)$$

$$= \left(2x^{-\frac{1}{2}} \right) = \frac{2}{\sqrt{x}}$$

Question 12: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(ax + b)^n$

Solution 12: Let $f(x) = (ax + b)^n$. Accordingly, $f(x + h) = \{a(x + h) + b\}^n = (ax + ah + b)^n$

By first principle,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(ax + ah + b)^n - (ax + b)^n}{h} \\ &= \lim_{h \rightarrow 0} \frac{(ax + b)^n \left(1 + \frac{ah}{ax + b} \right)^n - (ax + b)^n}{h} \\ &= (ax + b)^n \lim_{h \rightarrow 0} \frac{1}{h} \left[\left\{ 1 + n \left(\frac{ah}{ax + b} \right) + \frac{n(n-1)}{2} \left(\frac{ah}{ax + b} \right)^2 + \dots \right\} - 1 \right] \quad (\text{using binomial theorem}) \\ &= (ax + b)^n \lim_{h \rightarrow 0} \frac{1}{h} \left[n \left(\frac{ah}{ax + b} \right) + \frac{n(n-1)a^2h^2}{2(ax + b)^2} + \dots (\text{Terms containing higher degrees of } h) \right] \\ &= (ax + b)^n \lim_{h \rightarrow 0} \left[\frac{na}{(ax + b)} + \frac{n(n-1)a^2h^2}{2(ax + b)^2} + \dots \right] \\ &= (ax + b)^n \left[\frac{na}{(ax + b)} + 0 \right] \\ &= na \frac{(ax + b)^n}{ax + b} \\ &= na(ax + b)^{n-1} \end{aligned}$$

Question 13: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(ax + b)^n (cx + d)^m$

Solution 13: Let $f(x) = (ax + b)^n (cx + d)^m$

By Leibnitz product rule,

$$f'(x) = (ax + b)^n \frac{d}{dx} (cx + d)^m + (cx + d)^m \frac{d}{dx} (ax + b)^n \quad \dots (1)$$

Now let $f_1(x) = (cx + d)^m$

$$f_1(x + h) = (cx + ch + d)^m$$

$$f_1'(x) = \lim_{h \rightarrow 0} \frac{f_1(x + h) - f_1(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(cx + ch + d)^m - (cx + d)^m}{h}$$

$$= (cx + d)^m \lim_{h \rightarrow 0} \frac{1}{h} \left[\left(1 + \frac{ch}{cx + d} \right)^m - 1 \right]$$

$$= (cx + d)^m \lim_{h \rightarrow 0} \frac{1}{h} \left[\left(1 + \frac{mch}{(cx + d)} + \frac{m(m-1)}{2} \frac{c^2 h^2}{(cx + d)^2} + \dots \right)^m - 1 \right]$$

$$= (cx + d)^m \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{mch}{(cx + d)} + \frac{m(m-1)c^2 h^2}{2(cx + d)^2} + \dots (\text{Terms containing higher degree of } h) \right]$$

$$= (cx + d)^m \lim_{h \rightarrow 0} \left[\frac{mc}{(cx + d)} + \frac{m(m-1)c^2 h}{2(cx + d)^2} + \dots \right]$$

$$= (cx + d)^m \left[\frac{mch}{(cx + d)} + 0 \right]$$

$$= \frac{mc(cx + d)^m}{(cx + d)}$$

$$= mc(cx + d)^{m-1}$$

$$\frac{d}{dx} (cx + d)^m = mc(cx + d)^{m-1} \quad \dots (2)$$

Similarly, $\frac{d}{dx}(ax+b)^n = na(ax+b)^{n-1} \quad \dots (3)$

Therefore, from (1), (2), and (3), we obtain

$$\begin{aligned} f'(x) &= (ax+b)^n \{mc(cx+d)^{m-1}\} + (cx+d)^m \{na(ax+b)^{n-1}\} \\ &= (ax+b)^{n-1}(cx+d)^{m-1} [mc(ax+b) + na(cx+d)] \end{aligned}$$

Question 14: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\sin(x+a)$

Solution 14: Let, $f(x) = \sin(x+a)$

$$f(x+h) = \sin(x+h+a)$$

By first principle,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin(x+h+a) - \sin(x+a)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[2 \cos \left(\frac{x+h+a+x+a}{2} \right) \sin \left(\frac{x+h+a-x-a}{2} \right) \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[2 \cos \left(\frac{2x+2a+h}{2} \right) \sin \left(\frac{h}{2} \right) \right] \\ &= \lim_{h \rightarrow 0} \left[\cos \left(\frac{2x+2a+h}{2} \right) \left[\frac{\sin \left(\frac{h}{2} \right)}{\left(\frac{h}{2} \right)} \right] \right] \\ &= \lim_{h \rightarrow 0} \cos \left(\frac{2x+2a+h}{2} \right) \cdot \lim_{\frac{h}{2} \rightarrow 0} \left[\frac{\sin \left(\frac{h}{2} \right)}{\left(\frac{h}{2} \right)} \right] \quad \left[\text{As } h \rightarrow 0 \Rightarrow \frac{h}{2} \rightarrow 0 \right] \\ &= \cos \left(\frac{2x+2a}{2} \right) \times 1 \quad \left[\lim_{h \rightarrow 0} \frac{\sin x}{x} = 1 \right] \\ &= \cos(x+a) \end{aligned}$$

Question 15: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): cosec x cot x

Solution 15: Let $f(x) = \text{cosec } x \cot x$

By Leibnitz product rule,

$$f'(x) = \text{cosec } x (\cot x)' + \cot x (\text{cosec } x)' \dots (1)$$

Let $f_1(x) = \cot x$. Accordingly, $f_1(x + h) = \cot(x + h)$

By first principle,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\cot(x+h) - \cot(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos(x)}{\sin x} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{\sin x \cos(x+h) - \cos x \sin(x+h)}{\sin x \sin(x+h)} \right) \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{\sin(x-x+h)}{\sin x \sin(x+h)} \right) \\ &= \frac{1}{\sin x} \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(-h)}{\sin(x+h)} \right] \\ &= \frac{-1}{\sin x} \left(\lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \left(\lim_{h \rightarrow 0} \frac{1}{\sin(x+h)} \right) \\ &= \frac{-1}{\sin x} \cdot 1 \cdot \left(\lim_{h \rightarrow 0} \frac{1}{\sin(x+0)} \right) \\ &= \frac{-1}{\sin^2 x} \\ &= -\text{cosec}^2 x \\ \therefore (\cot x)' &= -\text{cosec}^2 x \quad \dots (2) \end{aligned}$$

Now, let $f_2(x) = \text{cosec } x$. Accordingly, $f_2(x + h) = \text{cosec}(x + h)$

By first principle,

$$\begin{aligned}
 f_2'(x) &= \lim_{h \rightarrow 0} \frac{f_2(x+h) - f_2(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} [\operatorname{cosec}(x+h) - \operatorname{cosec}(x)] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1}{\sin(x+h)} - \frac{1}{\sin x} \right) \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{\sin x - \sin(x+h)}{\sin x \sin(x+h)} \right) \\
 &= \frac{1}{\sin x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2 \cos \left(\frac{x+x+h}{2} \right) \sin \left(\frac{x-x-h}{2} \right)}{\sin(x+h)} \right] \\
 &= \frac{1}{\sin x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{2 \cos \left(\frac{2x+h}{2} \right) \sin \left(\frac{-h}{2} \right)}{\sin(x+h)} \right] \\
 &= \frac{1}{\sin x} \cdot \lim_{h \rightarrow 0} \left[\frac{-\sin \left(\frac{h}{2} \right)}{\left(\frac{h}{2} \right)} \cdot \frac{\cos \left(\frac{2x+h}{2} \right)}{\sin(x+h)} \right] \\
 &= \frac{-1}{\sin x} \cdot \lim_{h \rightarrow 0} \frac{\sin \left(\frac{h}{2} \right)}{\left(\frac{h}{2} \right)} \cdot \lim_{h \rightarrow 0} \frac{\cos \left(\frac{2x+h}{2} \right)}{\sin(x+h)} \\
 &= \frac{-1}{\sin x} \cdot 1 \cdot \frac{\cos \left(\frac{2x+h}{2} \right)}{\sin(x+0)} \\
 &= \frac{-1}{\sin x} \cdot \frac{\cos x}{\sin x} \\
 &= -\operatorname{cosec} x \cdot \cot x \\
 \therefore (\operatorname{cosec} x)' &= -\operatorname{cosec} x \cdot \cot x
 \end{aligned}$$

From (1), (2), and (3), we obtain

$$f'(x) = \operatorname{cosec} x (-\operatorname{cosec}^2 x) + \cot x (-\operatorname{cosec} x \cot x)$$

$$= -\operatorname{cosec}^3 x - \cot^2 x \operatorname{cosec} x$$

Question 16: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{\cos x}{1 + \sin x}$

Solution 16: Let $f(x) = \frac{\cos x}{1 + \sin x}$

By quotient rule,

$$\begin{aligned} f'(x) &= \frac{(1 + \sin x) \frac{d}{dx}(\cos x) - (\cos x) \frac{d}{dx}(1 + \sin x)}{(1 + \sin x)^2} \\ &= \frac{(1 + \sin x)(-\sin x) - (\cos x)(\cos x)}{(1 + \sin x)^2} \\ &= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2} \\ &= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1 + \sin x)^2} \\ &= \frac{-\sin x - 1}{(1 + \sin x)^2} \\ &= \frac{-(1 + \sin x)}{(1 + \sin x)^2} \\ &= \frac{-1}{(1 + \sin x)} \end{aligned}$$

Question 17: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non zero constants and m and n are integers): $\frac{\sin x + \cos x}{\sin x - \cos x}$

Solution 17: Let $f(x) = \frac{\sin x + \cos x}{\sin x - \cos x}$

By quotient rule,

$$\begin{aligned}
 f'(x) &= \frac{(\sin x - \cos x) \frac{d}{dx}(\sin x + \cos x) - (\sin x + \cos x) \frac{d}{dx}(\sin x - \cos x)}{(\sin x + \cos x)^2} \\
 &= \frac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x + \cos x)^2} \\
 &= \frac{-(\sin x - \cos x)^2 - (\sin x + \cos x)^2}{(\sin x + \cos x)^2} \\
 &= \frac{-[\sin^2 x + \cos^2 x - 2 \sin x \cos x + \sin^2 x + \cos^2 x + 2 \sin x \cos x]}{(\sin x + \cos x)^2} \\
 &= \frac{-[1+1]}{(\sin x - \cos x)^2} \\
 &= \frac{-2}{(\sin x - \cos x)^2}
 \end{aligned}$$

Question 18: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{\sec x - 1}{\sec x + 1}$

Solution 18: Let $f(x) = \frac{\sec x - 1}{\sec x + 1}$

$$f(x) = \frac{\frac{1}{\cos x} - 1}{\frac{1}{\cos x} + 1} = \frac{1 - \cos x}{1 + \cos x}$$

By quotient rule,

$$\begin{aligned}
 f'(x) &= \frac{(1 + \cos x) \frac{d}{dx}(1 - \cos x) - (1 - \cos x) \frac{d}{dx}(1 + \cos x)}{(1 + \cos x)^2} \\
 &= \frac{(1 + \cos x)(\sin x) - (1 - \cos x)(-\sin x)}{(1 + \cos x)^2} \\
 &= \frac{\sin x + \cos x \sin x + \sin x - \sin x \cos x}{(1 + \cos x)^2}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{2 \sin x}{(1 + \cos x)^2} \\
 &= \frac{2 \sin x}{\left(1 + \frac{1}{\sec x}\right)^2} = \frac{2 \sin x}{\frac{(\sec x + 1)^2}{\sec^2 x}} \\
 &= \frac{2 \sin x \sec^2 x}{(\sec x + 1)^2} \\
 &= \frac{\frac{2 \sin x}{\cos x} \sec x}{(\sec x + 1)^2} \\
 &= \frac{2 \sec x \tan x}{(\sec x + 1)^2}
 \end{aligned}$$

Question 19: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\sin^n x$

Solution 19: Let $y = \sin^n x$.

Accordingly, for $n = 1$, $y = \sin x$.

$$\therefore \frac{dy}{dx} = \cos x, \text{ i.e., } \frac{d}{dx} \sin x = \cos x$$

For $n = 2$, $y = \sin^2 x$.

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (\sin x \sin x)$$

$$= (\sin x)'(\sin x) + \sin x(\sin x)' \quad [\text{By Leibnitz product rule}]$$

$$= \cos x \sin x + \sin x \cos x$$

$$= 2 \sin x \cos x \quad \dots (1)$$

For $n = 3$, $y = \sin^3 x$.

$$\therefore \frac{dy}{dx} = \frac{d}{dx} (\sin x \sin^2 x)$$

$$= (\sin x)' \sin^2 x + \sin x (\sin^2 x)' \quad [\text{By Leibnitz product rule}]$$

$$= \cos x \sin^2 x + \sin x (2 \sin x \cos x) \quad [\text{Using (1)}]$$

$$= \cos x \sin^2 x + \sin^2 x \cos x$$

$$= 3\sin^2 x \cos x$$

We assert that $\frac{d}{dx}(\sin^n x) = n \sin^{(n-1)} x \cos x$

Let our assertion be true for $n = k$.

$$\text{i.e., } \frac{d}{dx}(\sin^k x) = k \sin^{(k-1)} x \cos x \quad \dots (2)$$

Consider

$$\begin{aligned} \frac{d}{dx}(\sin^{k+1} x) &= \frac{d}{dx}(\sin x \sin^k x) \\ &= (\sin x)' \sin^k x + \sin x (\sin^k x)' && \text{[By Leibnitz product rule]} \\ &= \cos x \sin^k x + \sin x (k \sin^{k-1} \cos x) && \text{[Using (2)]} \\ &= \cos x \sin^k x + 2 \sin^k x \cos x \\ &= (k + 1) \sin^k x \cos x \end{aligned}$$

Thus, our assertion is true for $n = k + 1$.

Hence, by mathematical induction, $\frac{d}{dx}(\sin^n x) = n \sin^{(n-1)} x \cos x$

Question 20: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{a + b \sin x}{c + d \cos x}$

Solution 20: Let $f(x) = \frac{a + b \sin x}{c + d \cos x}$

By quotient rule,

$$\begin{aligned} f'(x) &= \frac{(c + d \cos x) \frac{d}{dx}(a + b \sin x) - (a + b \sin x) \frac{d}{dx}(c + d \cos x)}{(c + d \cos x)^2} \\ &= \frac{(c + d \cos x)(b \cos x) - (a + b \sin x)(-d \sin x)}{(c + d \cos x)^2} \\ &= \frac{cb \cos x + bd \cos^2 x + ad \sin x + bd \sin^2 x}{(c + d \cos x)^2} \end{aligned}$$

$$= \frac{bc \cos x + ad \sin x + bd(\cos^2 x + \sin^2 x)}{(c + d \cos x)^2}$$

$$= \frac{bc \cos x + ad \sin x + bd}{(c + d \cos x)^2}$$

Question 21: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{\sin(x+a)}{\cos x}$

Solution 21: Let $f(x) = \frac{\sin(x+a)}{\cos x}$

By quotient rule,

$$f'(x) = \frac{\cos x \frac{d}{dx} [\sin(x+a)] - \sin(x+a) \frac{d}{dx} \cos x}{\cos^2 x}$$

$$f'(x) = \frac{\cos x \frac{d}{dx} [\sin(x+a)] - \sin(x+a) \frac{d}{dx} (-\sin x)}{\cos^2 x} \quad \dots (i)$$

Let $g(x) = \sin(x+a)$. Accordingly, $g(x+h) = \sin(x+h+a)$

By first principle,

$$g'(x) = \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} [\sin(x+h+a) - \sin(x+a)]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[2 \cos \left(\frac{x+h+a+x+a}{2} \right) \sin \left(\frac{x+h+a-x-a}{2} \right) \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[2 \cos \left(\frac{2x+2a+h}{2} \right) \sin \left(\frac{h}{2} \right) \right]$$

$$= \lim_{h \rightarrow 0} \left[\cos \left(\frac{2x+2a+h}{2} \right) \left\{ \frac{\sin \left(\frac{h}{2} \right)}{\left(\frac{h}{2} \right)} \right\} \right]$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \cos\left(\frac{2x+2a+h}{h}\right) \cdot \lim_{h \rightarrow 0} \left\{ \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \right\} \quad \left[\text{As } h \rightarrow 0 \Rightarrow \frac{h}{2} \rightarrow 0 \right] \\
 &= \left(\cos \frac{2x+2a}{2} \right) \times 1 \quad \left[\lim_{h \rightarrow 0} \frac{\sin h}{h} = 1 \right] \\
 &= \cos(x+a) \quad \dots \text{ (ii)}
 \end{aligned}$$

From (i) and (ii), we obtain

$$\begin{aligned}
 f'(x) &= \frac{\cos x \cos(x+a) + \sin x \sin(x+a)}{\cos^2 x} \\
 &= \frac{\cos(x+a-x)}{\cos^2 x} \\
 &= \frac{\cos a}{\cos^2 x}
 \end{aligned}$$

Question 22: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $x^4 (5 \sin x - 3 \cos x)$

Solution 22: Let $f(x) = x^4 (5 \sin x - 3 \cos x)$

By product rule,

$$\begin{aligned}
 f'(x) &= x^4 \frac{d}{dx} (5 \sin x - 3 \cos x) + (5 \sin x - 3 \cos x) \frac{d}{dx} (x^4) \\
 &= x^4 \left[5 \frac{d}{dx} (\sin x) - 3 \frac{d}{dx} (\cos x) \right] + (5 \sin x - 3 \cos x) \frac{d}{dx} (x^4) \\
 &= x^4 [5 \cos x - 3(-\sin x)] + (5 \sin x - 3 \cos x)(4x^3) \\
 &= x^3 [5x \cos x + 3x \sin x + 20 \sin x - 12 \cos x]
 \end{aligned}$$

Question 23: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(x^2 + 1) \cos x$

Solution 23: Let $f(x) = (x^2 + 1) \cos x$

By product rule,

$$f'(x) = (x^2 + 1) \frac{d}{dx} (\cos x) + \cos x \frac{d}{dx} (x^2 + 1)$$

$$= (x^2 + 1)(-\sin x) + \cos x(2x)$$

$$= -x^2 \sin x - \sin x + 2x \cos x$$

Question 24: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(ax^2 + \sin x)(p + q \cos x)$

Solution 24: Let $f(x) = (ax^2 + \sin x)(p + q \cos x)$

By product rule,

$$f'(x) = (ax^2 + \sin x) \frac{d}{dx} (p + q \cos x) + (p + q \cos x) \frac{d}{dx} (ax^2 + \sin x)$$

$$= (ax^2 + \sin x)(-q \sin x) + (p + q \cos x)(2ax + \cos x)$$

$$= -q \sin x(ax^2 + \sin x) + (p + q \cos x)(2ax + \cos x)$$

Question 25: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(x + \cos x)(x - \tan x)$

Solution 25: Let $f(x) = (x + \cos x)(x - \tan x)$

By product rule,

$$f'(x) = (x + \cos x) \frac{d}{dx} (x - \tan x) + (x - \tan x) \frac{d}{dx} (x + \cos x)$$

$$= (x + \cos x) \left[\frac{d}{dx} (x) - \frac{d}{dx} (\tan x) \right] + (x - \tan x)(1 - \sin x)$$

$$= (x + \cos x) \left[1 - \frac{d}{dx} (\tan x) \right] + (x - \tan x)(1 - \sin x) \quad \dots (i)$$

Let $g(x) = \tan x$. Accordingly, $g(x + h) = \tan(x + h)$

By first principle,

$$\begin{aligned}
 g'(x) &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\tan(x+h) - \tan(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h)\cos x - \sin x \cos(x+h)}{\cos x \cos(x+h)} \right] \\
 &= \frac{1}{\cos x} \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h-x)}{\cos(x+h)} \right] \\
 &= \frac{1}{\cos x} \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin h}{\cos(x+h)} \right] \\
 &= \frac{1}{\cos x} \left(\lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \left(\lim_{h \rightarrow 0} \frac{1}{\cos(x+h)} \right) \\
 &= \frac{1}{\cos x} \cdot 1 \cdot \left(\frac{1}{\cos(x+0)} \right) \\
 &= \frac{1}{\cos^2 x} \\
 &= \sec^2 x \quad \dots \text{ (ii)}
 \end{aligned}$$

Therefore, from (i) and (ii), We obtain

$$\begin{aligned}
 f'(x) &= (x + \cos x)(1 - \sec^2 x) + (x - \tan x)(1 - \sin x) \\
 &= (x + \cos x)(-\tan^2 x) + (x - \tan x)(1 - \sin x) \\
 &= -\tan^2 x(x + \cos x) + (x - \tan x)(1 - \sin x)
 \end{aligned}$$

Question 26: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{4x + 5 \sin x}{3x + 7 \cos x}$

Solution 26: Let $f(x) = \frac{4x + 5\sin x}{3x + 7\cos x}$

By quotient rule,

$$\begin{aligned}
 f'(x) &= \frac{(3x + 7\cos x) \frac{d}{dx}(4x + 5\sin x) - (4x + 5\sin x) \frac{d}{dx}(3x + 7\cos x)}{(3x + 7\cos x)^2} \\
 &= \frac{(3x + 7\cos x) \left[4 \frac{d}{dx}(x) + 5 \frac{d}{dx}(\sin x) \right] - (4x + 5\sin x) \left[3 \frac{d}{dx}(x) + 7 \frac{d}{dx}(\cos x) \right]}{(3x + 7\cos x)^2} \\
 &= \frac{(3x + 7\cos x)[4x + 5\cos x] - (4x + 5\sin x)[3 - 7\sin x]}{(3x + 7\cos x)^2} \\
 &= \frac{12x + 15x\cos x + 28x\cos x + 35\cos^2 x - 12x + 28x\sin x - 15\sin x + 35(\cos^2 x + \sin^2 x)}{(3x + 7\cos x)^2} \\
 &= \frac{15x\cos x + 28\cos x + 28x\sin x - 15\sin x + 35(\cos^2 x + \sin^2 x)}{(3x + 7\cos x)^2} \\
 &= \frac{35 + 15x\cos x + 28\cos x + 28x\sin x - 15\sin x}{(3x + 7\cos x)^2}
 \end{aligned}$$

Question 27: Find the derivative of the following functions (it is to be understood that a, b, c,

d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{x^2 \cos\left(\frac{\pi}{4}\right)}{\sin x}$

Solution 27: Let $f(x) = \frac{x^2 \cos\left(\frac{\pi}{4}\right)}{\sin x}$

By quotient rule,

$$f'(x) = \cos\left(\frac{\pi}{4}\right) \left[\frac{\sin x \frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(\sin x)}{\sin^2 x} \right]$$

$$= \cos\left(\frac{\pi}{4}\right) \left[\frac{\sin x(2x) - x^2(\cos x)}{\sin^2 x} \right]$$

$$= \frac{x \cos \frac{\pi}{4} [2 \sin x - x \cos x]}{\sin^2 x}$$

Question 28: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{x}{1 + \tan x}$

Solution 28: Let $f(x) = \frac{x}{1 + \tan x}$

$$f'(x) = \frac{(1 + \tan x) \frac{d}{dx}(x) - (x) \frac{d}{dx}(1 + \tan x)}{(1 + \tan x)^2}$$

$$= f'(x) = \frac{(1 + \tan x) - x \frac{d}{dx}(1 + \tan x)}{(1 + \tan x)^2} \quad \dots(i)$$

Let $g(x) = 1 + \tan x$. Accordingly, $g(x + h) = 1 + \tan(x + h)$.

By first principle,

$$g'(x) = \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \rightarrow 0} \left[\frac{1 + \tan(x+h) - 1 - \tan(x)}{h} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h) \cos x - \sin x \cos(x+h)}{\cos(x+h) \cos x} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h-x)}{\cos(x+h) \cos x} \right]$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sinh}{\cos(x+h) \cos x} \right] \\
 &= \left(\lim_{h \rightarrow 0} \frac{\sinh}{h} \right) \cdot \left(\lim_{h \rightarrow 0} \frac{1}{\cos(x+h) \cos x} \right) \\
 &= 1 \times \frac{1}{\cos^2 x} = \sec^2 x \\
 &\Rightarrow \frac{d}{dx} (1 + \tan^2 x) = \sec^2 x \quad \dots \text{(ii)}
 \end{aligned}$$

From (i) and (ii), we obtain

$$f'(x) = \frac{1 + \tan x - x \sec^2 x}{(1 + \tan x)^2}$$

Question 29: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $(x + \sec x)(x - \tan x)$

Solution 29: Let $f(x) = (x + \sec x)(x - \tan x)$

By product rule,

$$\begin{aligned}
 f(x) &= (x + \sec x) \frac{d}{dx} (x - \tan x) + (x - \tan x) \frac{d}{dx} (x + \sec x) \\
 &= (x + \sec x) \left[\frac{d}{dx} (x) - \frac{d}{dx} \tan x \right] + (x - \tan x) \left[\frac{d}{dx} (x) + \frac{d}{dx} \sec x \right] \\
 &= f(x + \sec x) \left[1 - \frac{d}{dx} \tan x \right] + (x - \tan x) \left[1 + \frac{d}{dx} \sec x \right] \quad \dots \text{(i)}
 \end{aligned}$$

Let $f_1(x) = \tan x$, $f_2(x) = \sec x$

Accordingly, $f_1(x+h) = \tan(x+h)$ and $f_2(x+h) = \sec(x+h)$

$$\begin{aligned}
 f_1'(x) &= \lim_{h \rightarrow 0} \left(\frac{f_1(x+h) - f_1(x)}{h} \right) \\
 &= \lim_{h \rightarrow 0} \left[\frac{\tan(x+h) - \tan(x)}{h} \right]
 \end{aligned}$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h) \cos x - \sin x \cos(x+h)}{\cos(x+h) \cos x} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin(x+h-x)}{\cos(x+h) \cos x} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin h}{\cos(x+h) \cos x} \right] \\
 &= \left(\lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \cdot \left(\lim_{h \rightarrow 0} \frac{1}{\cos(x+h) \cos x} \right) \\
 &= 1 \times \frac{1}{\cos^2 x} = \sec^2 x \\
 &\Rightarrow \frac{d}{dx} (1 + \tan^2 x) = \sec^2 x \quad \dots \text{(ii)}
 \end{aligned}$$

$$\begin{aligned}
 f'_2(x) &= \lim_{h \rightarrow 0} \left(\frac{f_2(x+h) - f_2(x)}{h} \right) \\
 &= \lim_{h \rightarrow 0} \left(\frac{\sec(x+h) - \sec(x)}{h} \right) \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1}{\cos(x+h)} - \frac{1}{\cos x} \right) \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{\cos x - \cos(x+h)}{\cos(x+h) \cos x} \right) \\
 &= \frac{1}{\cos x} \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin \left(\frac{x+x+h}{2} \right) \cdot \sin \left(\frac{x-x-h}{2} \right)}{\cos(x+h)} \right] \\
 &= \frac{1}{\cos x} \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{-2 \sin \left(\frac{2x+h}{2} \right) \cdot \sin \left(\frac{-h}{2} \right)}{\cos(x+h)} \right]
 \end{aligned}$$

$$= \frac{1}{\cos x} \lim_{h \rightarrow 0} \frac{1}{h} \left[\frac{\sin\left(\frac{2x+h}{2}\right) \left\{ \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \right\}}{\cos(x+h)} \right]$$

$$= \sec x \cdot \frac{\left\{ \lim_{h \rightarrow 0} \sin\left(\frac{2x+h}{2}\right) \right\} \left\{ \lim_{\frac{h}{2} \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \right\}}{\lim_{h \rightarrow 0} \cos(x+h)}$$

$$= \sec x \cdot \frac{\sin x \cdot 1}{\cos x}$$

$$\Rightarrow \frac{d}{dx} \sec x = \sec x \tan x$$

From (i), (ii), and (iii), we obtain

$$f'(x) = (x + \sec x)(1 - \sec^2 x) + (x - \tan x)(1 + \sec x \tan x)$$

Question 30: Find the derivative of the following functions (it is to be understood that a, b, c, d, p, q, r and s are fixed non-zero constants and m and n are integers): $\frac{x}{\sin^n x}$

Solution 30: Let $f(x) = \frac{x}{\sin^n x}$

By quotient rule,

$$f'(x) = \frac{\sin^n x \frac{d}{dx} x - x \frac{d}{dx} \sin^n x}{\sin^{2n} x}$$

It can be easily shown that $\frac{d}{dx} \sin^n x = n \sin^{n-1} x \cos x$

Therefore,

$$\begin{aligned}f'(x) &= \frac{\sin^n x \frac{d}{dx} x - x \frac{d}{dx} \sin^n x}{\sin^{2n} x} \\&= \frac{\sin^n x \cdot 1 - x(n \sin^{n-1} x \cos x)}{\sin^{2n} x} \\&= \frac{\sin^{n-1} x (\sin x - nx \cos x)}{\sin^{2n} x} \\&= \frac{\sin x - nx \cos x}{\sin^{n+1} x}\end{aligned}$$
